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L^2-Approximations of power and logarithmic functions with applications to numerical conformal mapping

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Dedicated to Nick Papamichael in honor of his sixtieth birthday and in recognition of his substantial contributions to the subject of this paper.

Summary. For a bounded Jordan domain G with quasiconformal boundary L, two-sided estimates are obtained for the error in best $L^2(G)$ polynomial approximation to functions of the form $(z-\tau)^\beta$, $\beta > -1$, and $(z-\tau)^m \log^l(z-\tau)$, $m > -1$, $l \neq 0$, where $\tau \in L$. Furthermore, Andrievskii's lemma that provides an upper bound for the $L^\infty(G)$ norm of a polynomial p_n in terms of the $L^2(G)$ norm of p_n' is extended to the case when a finite linear combination (independent of n) of functions of the above form is added to p_n. For the case when the boundary of G is piecewise analytic without cusps, the results are used to analyze the improvement in rate of convergence achieved by using augmented, rather than classical, Bieberbach polynomial approximants of the Riemann mapping function of G onto a disk. Finally, numerical results are presented that illustrate the theoretical results obtained.

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1 Introduction and notations

If G a bounded Jordan domain whose boundary L is piecewise analytic without cusps and f_0 is a normalized conformal mapping of G onto a disk, then Bieberbach polynomials π_n can be used to approximate f_0 (see Sect. 4). In [9], [10], and [11], D. Gaier obtained estimates of the form

$$(1.1) \qquad \|f_0 - \pi_n\|_{L^\infty(G)} = \mathcal{O}\left(\frac{\log n}{n^s}\right) \qquad (n \to \infty)$$

for the uniform norm of the error in approximation, where s is explicitly determined from the interior angles $\alpha_j \pi$, $0 < \alpha_j < 2$, at the corners τ_j of L (see [11, Theorem 2]).

To obtain (1.1), two essential ingredients are the Lehman formulas [13] for the asymptotic expansion of f_0 near the corners τ_j, and Andrievskii's lemma [2] which provides an estimate for the $L^\infty(G)$ norm of a polynomial in terms of the $L^2(G)$ norm of its derivative. Since the Lehman formulas involve power functions of the form

$$(1.2) \qquad f_{\beta,\tau_j}(z) = (z - \tau_j)^\beta$$

and logarithmic functions of the form

$$(1.3) \qquad g_{m,l,\tau_j}(z) = (z - \tau_j)^m \log^l (z - \tau_j)\,,$$

it is natural to expect an improvement in the convergence rate (1.1) if the ordinary Bieberbach polynomials are replaced by "augmented Bieberbach polynomials" that include suitable singular functions of the above power and logarithmic type. This was first observed by Levin, Papamichael and Sideridis [14] and subsequently used by Papamichael, Kokkinos, Hough and Warby for improving the convergence rates of certain orthonormalization methods associated with the mapping of interior, exterior and doubly-connected domains; see e.g. [16], [15], [19] and [18].

One goal of the present paper is to obtain sharp estimates for the improvement gained in using such augmented Bieberbach polynomials. For this purpose, upper and lower bounds are derived for the error in the best $L^2(G)$ n-th degree polynomial approximation to functions of the form (1.2) and (1.3). These estimates (cf. Corollary 2.2) are, in fact, obtained in the more general setting when the boundary L is a quasiconformal curve.

In Lemma 2.3 and Corollary 2.5 we present extensions of Andrievskii's lemma to the case when one or several singular functions of the form (1.2), (1.3) are adjoined to ordinary polynomials to form "augmented polynomials".

In Sect. 3 we apply the above results to obtain upper and lower estimates for the error in approximating f_0 by augmented Bieberbach polynomials (see Theorems 3.1 and 3.2). The lower estimates provide new sharpness results, even in the case of classical Bieberbach polynomials (cf. (3.19)). Finally, in Sect. 4, we present numerical computations that illustrate our theoretical results.

For convenient reference we provide here a listing of the main notations used throughout the paper.

2 Approximation of power and logarithmic functions

In what follows we denote by $C, c, C_1, \ldots$ constants whose values either are absolute or depend on parameters not essential for arguments; at least, they are independent of n.

For quantities $A > 0$, $B > 0$, which depend on some parameters, we use the notation $A \preceq B$ (inequality with respect to the order) if $A \leq CB$; the expression $A \asymp B$ means that $A \preceq B$ and $B \preceq A$ simultaneously.

Unless otherwise specified, we assume throughout that G is a bounded Jordan domain with quasiconformal boundary L, and $z_0 \in G$. Denote by $y(\zeta)$ a quasiconformal reflection with respect to L, i.e., an orientation-changing quasiconformal mapping of the extended plane $\overline{\mathbb{C}}$ onto itself that carries G into its complement $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ and conversely, leaves the points of the curve L fixed, and satisfies $y(z_0) = \infty$, $y(\infty) = z_0$ ([1, Chapter IV, §D]).

Let $\Phi(z)$ denote the Riemann function that conformally and univalently maps Ω onto the complement of the unit disk $\overline{\mathbb{D}}$ normalized by the conditions $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. This function can be extended to a homeomorphism between closed domains, and we keep the previous notation for the extension.

We extend $\Phi(z)$ to a quasiconformal map of the plane onto itself by setting for $z \in G$

$$\Phi(z) = \begin{cases} 1/\overline{\Phi\left[y(z)\right]}, & z \neq z_0, \\ 0, & z = z_0. \end{cases}$$

Denote by L_r, $r > 0$, the r-th level curve of the function $\Phi(z)$, i.e., $L_r := \{\zeta : |\Phi(\zeta)| = r\}$. Also, let $G_r := \operatorname{Int} L_r$ and $\Omega_r := \overline{\mathbb{C}} \setminus \overline{G_r}$.

For $z \in \mathbb{C}$ and $u > 0$ we define

$$d_u(z) := \max_{0 \leq \varphi < 2\pi} \left| z - \Psi\left[\Phi(z) + u e^{i\varphi}\right] \right|.$$

Remark 2.1 Recall that any quasiconformal map F of $\mathbb{C}$ onto itself satisfies the so-called $\mathbf{D}$-property, i.e.,

$$\text{(2.1)} \quad \min_{\zeta:|\zeta-z|=r} |F(\zeta) - F(z)| \asymp \max_{\zeta:|\zeta-z|=r} |F(\zeta) - F(z)|, \quad z \in \mathbb{C}.$$

This implies that for any $z \in L_{1-u}$, $0 < u < 1$,

$$\text{dist}\,(z, L) \asymp d_u(z) \tag{2.2}$$

and, for any fixed $0 < \varepsilon < 1$, $z \in G_{1+\varepsilon/n} \backslash G_{1-\varepsilon/n}$, the quantity $d_{1/n}\,(z)$ has the same order (as $n \to \infty$) as the traditional quantity in the approximation theory of functions of complex variable - the distance from z to $L_{1+1/n}$ (see [5]). In particular, it follows from Warschawski's results in [21] that if L consists, near $\tau \in L$, of two analytic arcs meeting at an interior angle $\alpha\pi$, $0 < \alpha < 2$, at τ, then

$$d_{1/n}\,(\tau) \asymp n^{\alpha-2}. \tag{2.3}$$

For $z \in \mathbb{C}$, $r > 0$, denote by $D(z, r) := \{\zeta : |\zeta - z| < r\}$ the open disk centered at z with the radius r.

Let $\omega := \Phi(\tau)$, $\gamma_\tau := \Psi(\{w : \arg w = \arg \omega, |w| \geq 1\})$. For noninteger $\beta > -1$ and arbitrary $m > -1$, $l \neq 0$ denote by $f_{\beta,\tau}(z)$ and $g_{m,l,\tau}(z)$ branches of $(z - \tau)^\beta$ and $(z - \tau)^m \log^l(z - \tau)$, respectively, which are analytic in $\mathbb{C} \setminus \gamma_\tau$. Given m, l we also define

$$l^* = l^*(m, l) = \begin{cases} l - 1, & \text{if } m \geq 0,\ l \geq 1 \text{ are integer;} \\ l, & \text{otherwise.} \end{cases} \tag{2.4}$$

In this paper we obtain two-sided estimates for the error in the best L^2-approximation

$$E_{n,2}(f, G) := \min_{p:\deg p \leq n} \|f - p\|_{L^2(G)}$$

of $f_{\beta,\tau}(z)$ and $g_{m,l,\tau}(z)$ by polynomials of degree at most n, $n = 1, 2, \ldots$, and apply these estimates to the problem of approximation of the Riemann function $f_0(z)$ that conformally and univalently maps the domain G onto the disk $D\,(0, r_0)$ with $f_0\,(z_0) = 0$, $f_0'\,(z_0) = 1$ (r_0 is the conformal radius of G with respect to z_0).

In our arguments we need the following two auxiliary results. The first lemma describes the properties of $d_{1/n}\,(z)$ (cf. [5, Lemma 2]).

Lemma 2.1 *There exists a constant $c = c(G) > 0$ such that for arbitrary points $z \in \overline{G}$ and $\zeta \in \Omega_{1-1/n}$:*

1) *if $|\zeta - z| \preceq d_{1/n}\,(z)$, then*

$$d_{1/n}\,(\zeta) \asymp d_{1/n}\,(z)\,; \tag{2.5}$$

2) *if $|\zeta - z| \succeq d_{1/n}\,(z)$, then*

$$\left|\frac{d_{1/n}\,(z)}{\zeta - z}\right|^{1/c} \preceq \frac{d_{1/n}\,(\zeta)}{|\zeta - z|} \preceq \left|\frac{d_{1/n}\,(z)}{\zeta - z}\right|^{c}; \tag{2.6}$$

3) *for any $0 < v < u < 1$,*

$$(u/v)^c \preceq d_u(z)/d_v(z) \preceq (u/v)^{1/c}. \tag{2.7}$$

The next lemma is a modification of the Tamrazov's result [12, Theorem 1].

Lemma 2.2 *Suppose that for a polynomial p_n of degree at most n, $n = 1, 2, \ldots$, some positive constants M, ρ, and a point $\tau \in L$ the inequality*

$$|p_n(z)| \le M\left(1 + \left|\frac{z-\tau}{\rho}\right|^{C_1}\right), \quad C_1 > 0,$$

is satisfied for all points $z \in L$. Then, for any fixed positive constant C_2 and all $z \in D(\tau, \rho) \cap \operatorname{Int} L_{1+C_2/n}$, there holds

$$|p_n(z)| \le C_3 M, \quad C_3 = C_3(C_1, C_2) \ge 1.$$

We remark that this result holds more generally for any bounded continuum with connected complement (cf. [4, Theorem 6.1]).

Let $\omega(\delta)$ be continuous, monotonic and positive on $(0, \infty)$. Suppose there exist constants c_1, $C_1 > 0$, $c_2 \in [0, 1)$ and $C_2 \ge 0$ such that for all $\delta > 0$ and $t \ge 1$

$$c_1 t^{-c_2}\omega(\delta) \le \omega(t\delta) \le C_1 t^{C_2}\omega(\delta). \tag{2.8}$$

Note that (2.8) is clearly satisfied (with $C_1 = c_1 = 1$) if for $x > 0$ small enough the functions $\omega(x)/x^{C_2}$ and $\omega(x)x^{c_2}$ are decreasing and increasing, respectively. In particular, if we specify the monotonicity of $\omega(\delta)$, one of the inequalities in (2.8) becomes obvious (say, for increasing functions - the left one). Note also that the left-hand side of (2.8) implies

$$\int_0^1 \omega(x)dx \le \frac{1}{c_1}\int_0^1 \frac{\omega(1)}{x^{c_2}}dx < \infty. \tag{2.9}$$

Theorem 2.1 *Let $g(z)$ be an analytic and single-valued branch in $\mathbb{C} \setminus \gamma_\tau$ of some multi-valued analytic function having branch points at $\tau \in L$ and ∞. Denote by $g^\pm(\zeta)$, $\zeta \in \gamma_\tau$, the boundary values of $g(z)$ on γ_τ. Suppose the functions $g^\pm(\zeta)$, $\zeta \in \gamma_\tau$, $|\zeta - \tau|$ small enough, satisfy the inequality*

$$|g^+(\zeta) - g^-(\zeta)| \le \omega(|\zeta - \tau|). \tag{2.10}$$

Then

$$E_{n,2}(g, G) \preceq d_{1/n}(\tau)\,\omega\left(d_{1/n}(\tau)\right). \tag{2.11}$$

Proof. Fix $R > 1$ such that (2.10) holds true on the subarc $\gamma := \gamma_\tau \cap G_R$ of γ_τ. It was shown in [3] that γ is a quasi-smooth arc (i.e., for any subarc, its length and the distance between endpoints have the same order) and, for any $\zeta \in \gamma$, $z \in G$

$$\operatorname{dist}(\zeta, L) \succeq |\zeta - \tau|, \qquad \operatorname{dist}(z, \gamma) \succeq |z - \tau| \tag{2.12}$$

and, hence,

$$|\zeta - z| \succeq |\zeta - \tau| + |z - \tau|. \tag{2.13}$$

Because of (2.9), it is easy to verify that (2.10) implies the validity of the Cauchy formula for the function $g(z)$ in the domain $G_R \setminus \gamma$, i.e., under suitable orientation of all the arcs,

$$g(z) = \frac{1}{2\pi i}\left(\int\limits_{\gamma} \frac{g^+(\zeta) - g^-(\zeta)}{\zeta - z} d\zeta + \int\limits_{L_R} \frac{g(\zeta)}{\zeta - z} d\zeta\right).$$

The last integral represents a function that is analytic up to the R-th level line of the domain G and, therefore, can be approximated by polynomials geometrically fast (with the rate $\preceq R_1^{-n}$ for any fixed $R_1 \in (1, R)$). So, we can restrict ourselves to the approximation of the Cauchy-type integral along γ. Set

$$f(z) := \int\limits_{\gamma} \frac{g^+(\zeta) - g^-(\zeta)}{\zeta - z} d\zeta.$$

In order to estimate $E_{n,2}(f, G)$ we use Dzjadyk's polynomial kernels to approximate the Cauchy kernel $1/(\zeta - z)$. By [5, Lemma 3], for every fixed $m > 0$ and $R > 1$ and for all $n = 1, 2, \ldots$ there exists a polynomial kernel $K_{n,m}(\zeta, z)$ of degree (in z) at most n with the following property: for any $z \in G$ and $\zeta \in G_R \setminus G$

$$\left|\frac{1}{\zeta - z} - K_{n,m}(\zeta, z)\right| \preceq \frac{1}{|\zeta - z|}\left(\frac{d_{1/n}(z)}{|\zeta - z| + d_{1/n}(z)}\right)^m. \tag{2.14}$$

We define approximating polynomials p_n by the formula

$$p_n(z) := \int\limits_{\gamma} \left(g^+ - g^-\right)(\zeta) K_{n,m}(\zeta, z) d\zeta.$$

Set $r := |z - \tau|$ and $d_n := d_{1/n}(\tau)$.

First, suppose that $r \le d_n$, and denote $\widetilde{\gamma} := \gamma \cap D(\tau, d_n)$. Then, by using (2.14), (2.13), (2.5) and taking into account the quasi-smoothness of γ, we get

$$\begin{aligned} |f(z) - p_n(z)| &\preceq \int\limits_{\widetilde{\gamma}} \frac{\omega(|\zeta - \tau|)}{|\zeta - \tau| + |z - \tau|} |d\zeta| + \left(d_{1/n}(z)\right)^m \\ &\quad \times \int\limits_{\gamma \setminus \widetilde{\gamma}} \frac{\omega(|\zeta - \tau|)}{|\zeta - \tau|^{m+1}} |d\zeta| \\ &\preceq \int_0^{d_n} \frac{\omega(t)}{t + r} dt + (d_n)^m \int_{d_n}^{\infty} \frac{\omega(t)}{t^{m+1}} dt = I_1 + (d_n)^m I_2. \end{aligned} \tag{2.15}$$

For the first integral we have

$$I_1 = \left(\int_0^r + \int_r^{d_n}\right) \frac{\omega(t)}{t+r}dt \le \frac{1}{r}\int_0^r \omega(t)dt + \int_r^{d_n} \frac{\omega(t)}{t+r}dt$$

$$\preceq \frac{\omega(r)}{r^{1-c_2}}\int_0^r \frac{dt}{t^{c_2}} + \max\{\omega(r), \omega(d_n)\}\int_r^{d_n}\frac{dt}{t+r}$$

$$\preceq \omega(r) + (\omega(r) + \omega(d_n))\log\frac{d_n}{r}. \tag{2.16}$$

To estimate I_2 we choose $m > C_2$, where the constant C_2 is taken from (2.8). Then

$$\frac{\omega(t)}{t^{m+1}} = \frac{\omega((t/d_n)\,d_n)}{t^{m+1}} \preceq \frac{\omega(d_n)}{(d_n)^{C_2}}t^{C_2-m-1}$$

and

$$I_2 \preceq \frac{\omega(d_n)}{(d_n)^{C_2}}\int_{d_n}^{\infty} t^{C_2-m-1}dt \preceq \frac{\omega(d_n)}{(d_n)^m}. \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15) we get

$$|f(z) - p_n(z)| \preceq (\omega(r) + \omega(d_n))\log\frac{ed_n}{r}. \tag{2.18}$$

Suppose now that $r > d_n$, and set $\widetilde{\gamma} := \gamma \cap D(\tau, r)$. Using (2.6), in a manner similar to the previous case we obtain

$$|f(z) - p_n(z)|$$

$$\preceq \left(d_{1/n}(z)\right)^m \left(r^{-(m+1)}\int_{\widetilde{\gamma}} \omega(|\zeta-\tau|)\,|d\zeta| + \int_{\gamma\setminus\widetilde{\gamma}} \frac{\omega(|\zeta-\tau|)}{|\zeta-\tau|^{m+1}}|d\zeta|\right)$$

$$\preceq \left(d_{1/n}(z)\right)^m \left(r^{-(m+1)}\int_0^r \omega(t)dt + \int_r^{\infty}\frac{\omega(t)}{t^{m+1}}dt\right)$$

$$\preceq \omega(r)\left(\frac{d_{1/n}(z)}{r}\right)^m \preceq \omega(r)\left(\frac{d_n}{r}\right)^{cm}. \tag{2.19}$$

Then from (2.18) and (2.19) we conclude

$$\begin{aligned}
&\|f-p_n\|^2_{L^2(G)}\\
&\preceq \iint\limits_{D(\tau,d_n)} (\omega(r)+\omega(d_n))^2 \log^2 \frac{ed_n}{r} dxdy\\
&\quad + \iint\limits_{\mathbb{C}\setminus D(\tau,d_n)} \omega^2(r) \left(\frac{d_n}{r}\right)^{2mc} dxdy\\
&\preceq \int_0^{d_n} r\omega^2(r)\log^2\frac{ed_n}{r}dr + \omega(d_n)\int_0^{d_n} r\omega(r)\log^2\frac{ed_n}{r}dr\\
&\quad +\omega^2(d_n)\int_0^{d_n} r\log^2\frac{ed_n}{r}dr + (d_n)^{2mc}\int_{d_n}^{\infty}\frac{\omega^2(r)}{r^{2mc-1}}dr\\
&= J_1+\omega(d_n)J_2+\omega^2(d_n)J_3+(d_n)^{2mc}J_4. \qquad (2.20)
\end{aligned}$$

According to (2.8), for all $r \in (0, d_n)$, there holds

$$\omega(r) \preceq \omega(d_n)\left(\frac{d_n}{r}\right)^{c_2}.$$

So, using integration by parts, we easily get

$$J_1 \preceq \omega^2(d_n)(d_n)^{2c_2}\int_0^{d_n} r^{1-2c_2}\log^2\frac{ed_n}{r}dr \preceq (d_n\omega(d_n))^2.$$

Similarly,

$$J_2 \preceq (d_n)^2\omega(d_n) \qquad \text{and} \qquad J_3 \preceq (d_n)^2.$$

Furthermore, to estimate J_4 we apply the right-hand side of (2.8) to deduce that

$$J_4 \preceq \frac{\omega^2(d_n)}{(d_n)^{2C_2}}\int_{d_n}^{\infty}\frac{dr}{r^{2(mc-C_2)-1}} \preceq (d_n)^{2(1-mc)}\omega^2(d_n),$$

provided $m > (C_2+1)/c$.

Combining these estimates we finally get

$$\|f-p_n\|^2_{L^2(G)} \preceq (d_n\omega(d_n))^2,$$

and (2.11) follows. □

Since for all $\zeta \in \gamma_\tau$, $|\zeta - \tau|$ small enough,

$$\begin{aligned}
&\left|f^+_{\beta,\tau}(\zeta) - f^-_{\beta,\tau}(\zeta)\right| \\
&\quad \le \left|f^+_{\beta,\tau}(\zeta)\right| + \left|f^-_{\beta,\tau}(\zeta)\right| \preceq |\zeta - \tau|^\beta , \\
&\left|g^+_{m,l,\tau}(\zeta) - g^-_{m,l,\tau}(\zeta)\right| \\
&\quad \le \begin{cases} |\zeta-\tau|^m \left|\left(\log^+(\zeta-\tau)\right)^l - \left(\log^-(\zeta-\tau)\right)^l\right|, & m,l \in \mathbb{N}; \\ \left|g^+_{m,l,\tau}(\zeta)\right| + \left|g^-_{m,l,\tau}(\zeta)\right|, & \text{otherwise} \end{cases} \\
&\quad \preceq |\zeta-\tau|^m \log^{l^*} \frac{1}{|\zeta-\tau|},
\end{aligned}$$

applying Theorem 2.1 we get

Corollary 2.1 *For all* $n \ge 1$ *large enough there holds*

$$E_{n,2}\left(f_{\beta,\tau}, G\right) \preceq \left(d_{1/n}(\tau)\right)^{\beta+1}, \tag{2.21}$$

$$E_{n,2}\left(g_{m,l,\tau}, G\right) \preceq \left(d_{1/n}(\tau)\right)^{m+1} \log^{l^*} \frac{1}{d_{1/n}(\tau)}. \tag{2.22}$$

Theorem 2.2 *For all* $n \ge 1$ *large enough there holds*

$$E_{n,2}\left(f_{\beta,\tau}, G\right) \succeq \left(d_{1/n}(\tau)\right)^{\beta+1}, \tag{2.23}$$

$$E_{n,2}\left(g_{m,l,\tau}, G\right) \succeq \left(d_{1/n}(\tau)\right)^{m+1} \log^{l^*} \frac{1}{d_{1/n}(\tau)}. \tag{2.24}$$

Proof. Suppose that

$$E_{n,2} := E_{n,2}\left(f_{\beta,\tau}, G\right) \le \left(d_{1/n}(\tau)\right)^{\beta+1}.$$

Denote by $\widetilde{p}_n(z)$, $\deg \widetilde{p}_n \le n$, the polynomial of best L_2-approximation to the function $f_{\beta,\tau}(z)$. Then, for any fixed $k \in \mathbb{N}$ and any point $z \in L_{1-1/n}$, applying Lemma 1 of [8, p. 4] and taking into account (2.2), we get

$$\left|\left(f_{\beta,\tau} - \widetilde{p}_n\right)^{(k)}(z)\right| \preceq k!\sqrt{k+1}\frac{E_{n,2}}{\left(\text{dist}\,(z,L)\right)^{k+1}} \preceq \frac{E_{n,2}}{\left(d_{1/n}(z)\right)^{k+1}}. \tag{2.25}$$

Let $\tau_n := \Psi\left[(1-1/n)\,\Phi(\tau)\right] \in L_{1-1/n}$. It follows from (2.1) that

$$d_{1/n}(\tau_n) \preceq |\tau - \tau_n| \le d_{1/n}(\tau_n), \tag{2.26}$$

and (2.5) implies

$$d_{1/n}(\tau_n) \asymp d_{1/n}(\tau). \tag{2.27}$$

Denote $D_n := D\left(\tau_n, 2d_{1/n}\left(\tau_n\right)\right)$. Setting $k := [\beta]+1$, for $z \in L_{1-1/n}$ we conclude:

1. if $z \in D_n$, it follows from (2.5), (2.25), (2.26) and (2.27) that

$$\left|\left(f_{\beta,\tau} - \widetilde{p}_n\right)^{(k)}(z)\right| \preceq \frac{E_{n,2}}{\left(d_{1/n}\left(\tau_n\right)\right)^{k+1}} \preceq \left(d_{1/n}\left(\tau\right)\right)^{\beta-k}$$
$$\preceq \left(d_{1/n}\left(\tau\right)\right)^{\{\beta\}-1},$$

where $\{\beta\} := \beta - [\beta]$. Since for $z \in D_n \cap L_{1-1/n}$, n large enough,

$$|z-\tau| \geq d_{1/n}(z) \succeq d_{1/n}\left(\tau_n\right) \succeq d_{1/n}\left(\tau\right), \tag{2.28}$$

we have

$$\left|\widetilde{p}_n^{(k)}(z)\right| \leq \left|f_{\beta,\tau}^{(k)}(z)\right| + \left|\left(f_{\beta,\tau} - \widetilde{p}_n\right)^{(k)}(z)\right| \preceq \left(d_{1/n}\left(\tau\right)\right)^{\{\beta\}-1}; \tag{2.29}$$

2. if $z \notin D_n$ then (2.5), (2.6) and (2.25) imply

$$\left|\left(f_{\beta,\tau} - \widetilde{p}_n\right)^{(k)}(z)\right| \preceq \frac{E_{n,2}}{\left(d_{1/n}\left(\tau_n\right)\right)^{k}} \left(\frac{d_{1/n}\left(\tau_n\right)}{d_{1/n}(z)}\right)^k$$
$$\preceq \left(d_{1/n}\left(\tau\right)\right)^{\{\beta\}-1} \left(\frac{|z-\tau_n|}{d_{1/n}(z)}\right)^k$$
$$\preceq \left(d_{1/n}\left(\tau\right)\right)^{\{\beta\}-1} \left|\frac{z-\tau_n}{d_{1/n}\left(\tau_n\right)}\right|^{k/c}$$

and, therefore,

$$\left|\widetilde{p}_n^{(k)}(z)\right| \leq |z-\tau|^{\{\beta\}-1} + \left(d_{1/n}\left(\tau\right)\right)^{\{\beta\}-1} \left|\frac{z-\tau_n}{d_{1/n}\left(\tau_n\right)}\right|^{k/c}$$
$$\preceq |z-\tau_n|^{\{\beta\}-1} + \left(d_{1/n}\left(\tau\right)\right)^{\{\beta\}-1} \left|\frac{z-\tau_n}{d_{1/n}\left(\tau_n\right)}\right|^{k/c}$$
$$\preceq \left(d_{1/n}\left(\tau\right)\right)^{\{\beta\}-1} \left(1 + \left|\frac{z-\tau_n}{d_{1/n}\left(\tau_n\right)}\right|^{k/c}\right). \tag{2.30}$$

Combining the estimates (2.29) and (2.30) we get for all $z \in L_{1-1/n}$

$$\left|\widetilde{p}_n^{(k)}(z)\right| \preceq \left(d_{1/n}\left(\tau\right)\right)^{\{\beta\}-1} \left(1 + \left|\frac{z-\tau_n}{d_{1/n}\left(\tau_n\right)}\right|^{k/c}\right).$$

Note that we can choose the constant C_2 such that D_n lies inside the $(1 + C_2/n)$-th level line for domain $G_{1-1/n}$ (see the proof of [5, Lemma 1]). Applying Lemma 2.2 to the polynomial $\widetilde{p}_n^{(k)}$, considered in $G_{1-1/n}$, we get

$$\left|\widetilde{p}_n^{(k)}(z)\right| \leq C_4 \left(d_{1/n}\left(\tau\right)\right)^{\{\beta\}-1}, \qquad z \in D\left(\tau_n, d_{1/n}\left(\tau_n\right)\right), \tag{2.31}$$

with some constant $C_4 = C_4(\beta, G)$.

On the other hand, for $z \in G$

$$\left| f_{\beta,\tau}^{(k)}(z) \right| = (\{\beta\})_k |z - \tau|^{\{\beta\}-1}, \tag{2.32}$$

where $(\{\beta\})_k := \beta\,(\beta-1)\cdots\{\beta\}$.

For $\varepsilon > 0$ small enough take

$$z_\varepsilon \in \partial D\left(\tau, \varepsilon d_{1/n}(\tau)\right) \cap \Psi\left([\Phi(\tau), \Phi(\tau_n)]\right). \tag{2.33}$$

As $\Phi(z)$ is a quasiconformal mapping, the arc $\Psi\left([\Phi(\tau), \Phi(\tau_n)]\right)$ satisfies an inequality similar to the first one in (2.12), and we have

$$\operatorname{dist}(z_\varepsilon, L) \geq c_1 \varepsilon d_{1/n}(\tau). \tag{2.34}$$

Since $-1 < \{\beta\} - 1 < 0$, (2.31) and (2.32) allow us to choose $\varepsilon > 0$ (take, for instance, $\varepsilon = \left((\{\beta\})_k / (2C_4)\right)^{1/(1-\{\beta\})}$) such that

$$\left| (f_{\beta,\tau} - \widetilde{p}_n)^{(k)}(z_\varepsilon) \right| \geq \frac{1}{2} (\{\beta\})_k \left(\varepsilon d_{1/n}(\tau)\right)^{\{\beta\}-1}. \tag{2.35}$$

At the same time, by Lemma 1 of [8, p. 4] applied in the disk $D(z_\varepsilon, \operatorname{dist}(z_\varepsilon, L))$

$$E_{n,2} = \|f_{\beta,\tau} - \widetilde{p}_n\|_{L_2(G)} \succeq \frac{\left| (f_{\beta,\tau} - \widetilde{p}_n)^{(k)}(z_\varepsilon) \right|}{k!\sqrt{k+1}} \left(\operatorname{dist}(z_\varepsilon, L)\right)^{k+1}.$$

This inequality together with (2.34) and (2.35) implies

$$E_{n,2} \succeq \frac{(\{\beta\})_k\, c_1^{k+1}}{k!\sqrt{k+1}} \left(\varepsilon d_{1/n}(\tau)\right)^{\{\beta\}+k} = c_2(\beta, G) \left(d_{1/n}(\tau)\right)^{\beta+1},$$

which completes the proof of (2.23).

In (2.24), the case when $l^* = l$ can be treated similarly. So, we assume that $l^* = l - 1$, i.e., $m \geq 0$ and $l \geq 1$ are both integers. Suppose that

$$E_{n,2} := E_{n,2}(g_{m,l,\tau}, G) \leq \left(d_{1/n}(\tau)\right)^{m+1} \log^{l^*} \frac{1}{d_{1/n}(\tau)},$$

and let $\widetilde{p}_n(z)$, $\deg \widetilde{p}_n \leq n$, be the polynomial of best L_2-approximation to the function $g_{m,l,\tau}(z)$. Then, similarly to (2.25), for any point $z \in L_{1-1/n}$ we have

$$\left| (g_{m,l,\tau} - \widetilde{p}_n)^{(m+1)}(z) \right| \preceq \frac{E_{n,2}}{\left(d_{1/n}(z)\right)^{m+2}}. \tag{2.36}$$

In the previous notations, we get for $z \in L_{1-1/n}$:

1. if $z \in D_n$, it follows from (2.5), (2.26), (2.27) and (2.36) that

$$\left|(g_{m,l,\tau} - \widetilde{p}_n)^{m+1}(z)\right| \preceq \frac{E_{n,2}}{\left(d_{1/n}(\tau_n)\right)^{m+2}} \preceq \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau)}.$$

Note that $g^{(m)}_{m,l,\tau}(z)$ is, in fact, a polynomial on $\log(z-\tau)$ of degree l, i.e.,

$$g^{(m)}_{m,l,\tau}(z) = q_l(\log(z-\tau)). \tag{2.37}$$

Therefore, using (2.28) we get

$$\left|g^{(m+1)}_{m,l,\tau}(z)\right| \preceq \frac{\log^{l^*}(1/|z-\tau|)}{|z-\tau|} \preceq \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau)}$$

and

$$\begin{aligned}\left|\widetilde{p}^{(m+1)}_n(z)\right| &\le \left|g^{(m+1)}_{m,l,\tau}(z)\right| + \left|(g_{m,l,\tau} - \widetilde{p}_n)^{(m+1)}(z)\right| \\ &\preceq \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau)};\end{aligned} \tag{2.38}$$

2. if $z \notin D_n$, then (2.5), (2.6), and (2.36) imply

$$\begin{aligned}&\left|(g_{m,l,\tau} - \widetilde{p}_n)^{m+1}(z)\right| \\ &\preceq \frac{E_{n,2}}{\left(d_{1/n}(\tau_n)\right)^{(m+2)}}\left(\frac{d_{1/n}(\tau_n)}{d_{1/n}(z)}\right)^{(m+2)} \\ &\preceq \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau_n)}\left(\frac{d_{1/n}(\tau)}{d_{1/n}(\tau_n)}\right)^{(m+1)}\left(\frac{|z-\tau_n|}{d_{1/n}(z)}\right)^{(m+2)} \\ &\preceq \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau)}\left|\frac{z-\tau_n}{d_{1/n}(\tau_n)}\right|^{(m+2)/c}.\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\widetilde{p}^{(m+1)}_n(z)\right| \\ &\le \left|\frac{q_l'(\log(z-\tau))}{z-\tau}\right| + \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau)}\left|\frac{z-\tau_n}{d_{1/n}(\tau_n)}\right|^{(m+2)/c} \\ &\preceq \frac{\log^{l^*}\left(C/d_{1/n}(\tau_n)\right)}{d_{1/n}(\tau_n)} + \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau)}\left|\frac{z-\tau_n}{d_{1/n}(\tau_n)}\right|^{(m+2)/c} \\ &\preceq \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau)}\left(1+\left|\frac{z-\tau_n}{d_{1/n}(\tau_n)}\right|^{C_1}\right),\end{aligned} \tag{2.39}$$

where $C_1 = (m+2)/c$.

Estimates (2.38) and (2.39) imply

$$\left|\widetilde{p}_n^{(m+1)}(z)\right| \preceq \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau)}\left(1+\left|\frac{z-\tau_n}{d_{1/n}(\tau_n)}\right|^{C_1}\right), \quad z \in L_{1-1/n},$$

and, consequently,

$$\left|\widetilde{p}_n^{(m+1)}(z)\right| \preceq \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{d_{1/n}(\tau)}, \quad z \in D\left(\tau_n, d_{1/n}(\tau_n)\right). \tag{2.40}$$

On the other hand, because of (2.37), for $|z-\tau|$ small enough there holds

$$\left|g_{m,l,\tau}^{(m+1)}(z)\right| \succeq \frac{\log^{l^*}(1/|z-\tau|)}{|z-\tau|}. \tag{2.41}$$

This inequality and (2.40) allow us to choose $\varepsilon > 0$ such that at the point z_ε, defined by (2.33), we have

$$\left|\left(g_{m,l,\tau}-\widetilde{p}_n\right)^{m+1}(z_\varepsilon)\right| \succeq \frac{\log^{l^*}\left(1/d_{1/n}(\tau)\right)}{\varepsilon d_{1/n}(\tau)}. \tag{2.42}$$

Now applying Lemma 1 of [8, p. 4] we get

$$E_{n,2} = \|g_{m,l,\tau}-\widetilde{p}_n\|_{L_2(G)} \succeq \frac{\left(g_{m,l,\tau}-\widetilde{p}_n\right)^{m+1}(z_\varepsilon)}{(m+1)!\sqrt{m+2}}\left(\text{dist}\,(z_\varepsilon, L)\right)^{m+2},$$

which together with (2.34) and (2.42) implies

$$E_{n,2} \succeq \left(\varepsilon d_{1/n}(\tau)\right)^{m+1}\log^{l^*}\frac{1}{d_{1/n}(\tau)}.$$

□

From Corollary 2.1 and Theorem 2.2 we immediately deduce

Corollary 2.2 *If G is a quasidisk, $\tau \in L$, then*

$$\begin{aligned} E_{n,2}\left(f_{\beta,\tau}, G\right) &\asymp \left(d_{1/n}(\tau)\right)^{\beta+1}, \\ E_{n,2}\left(g_{m,l,\tau}, G\right) &\asymp \left(d_{1/n}(\tau)\right)^{m+1}\log^{l^*}\frac{1}{d_{1/n}(\tau)} \end{aligned} \tag{2.43}$$

as $n \to \infty$. In particular, if (near τ) L consists of two analytic arcs forming an interior angle $\alpha\pi$, then (see (2.3))

$$E_{n,2}\left(f_{\beta,\tau}, G\right) \asymp n^{(\alpha-2)(\beta+1)},$$
$$E_{n,2}\left(g_{m,l,\tau}, G\right) \asymp n^{(\alpha-2)(m+1)}\log^{l^*} n,$$

where l^* is defined by (2.4).

We can prove even more.

Theorem 2.3 *Let $\tau \in L$, $\{a_{n,j}\}_{j=1}^{s}$, $n = 1, 2, \ldots$, be arbitrary complex numbers, and, for each j, let h_j denote either $f_{\beta_j,\tau}$, $\beta_j > -1$ noninteger, or $g_{m_j,l_j,\tau}$, $m_j > -1$. Suppose also that $h_j \neq h_k$ for $j \neq k$. Then for the function*

$$h(z) = h(n, z) := \sum_{j=1}^{s} a_{n,j} h_j \tag{2.44}$$

the inequalities

$$E_{n,2}(h, G) \asymp \sum_{j=1}^{s} |a_{n,j}| \, E_{n,2}(h_j, G) \qquad n = 1, 2, \ldots, \tag{2.45}$$

are satisfied.

Proof. The upper estimate in (2.45) is trivial. Proving the lower one, we restrict ourselves, for simplicity, to the case $s = 2$, i.e., we have to show that for

$$\begin{aligned} &E_{n,2}(a_{n,1}h_1 + a_{n,2}h_2, G) \\ &\qquad \succeq |a_{n,1}| \, E_{n,2}(h_1, G) + |a_{n,2}| \, E_{n,2}(h_2, G). \end{aligned} \tag{2.46}$$

If either $a_{n,1} = 0$ or $a_{n,2} = 0$, then (2.46) is trivially satisfied. So assume that neither of $a_{n,1}$, $a_{n,2}$ is zero. Taking into account (2.43) we can further assume that

$$E_{n,2}(h_2, G) = o(E_{n,2}(h_1, G)) \qquad \text{as} \qquad n \to \infty.$$

Without loss of generality, we take $a_{n,2} = 1$. If

$$|a_{n,1}| < \frac{1}{2} \frac{E_{n,2}(h_2, G)}{E_{n,2}(h_1, G)} \qquad \text{or} \qquad |a_{n,1}| > 2 \frac{E_{n,2}(h_2, G)}{E_{n,2}(h_1, G)},$$

then (2.46) holds with $c = 1/4$. For instance, in the first case, the triangle inequality yields

$$\begin{aligned} E_{n,2}(a_{n,1}h_1 + h_2, G) &\geq E_{n,2}(h_2, G) - |a_{n,1}| \, E_{n,2}(h_1, G) \\ &\geq \frac{1}{2} E_{n,2}(h_2, G) \\ &\geq \frac{1}{4} \left(|a_{n,1}| \, E_{n,2}(h_1, G) + E_{n,2}(h_2, G) \right). \end{aligned}$$

Suppose now that

$$\frac{1}{2} \frac{E_{n,2}(h_2, G)}{E_{n,2}(h_1, G)} \leq |a_{n,1}| \leq 2 \frac{E_{n,2}(h_2, G)}{E_{n,2}(h_1, G)}. \tag{2.47}$$

In this case, using (2.43), we can apply the arguments, given in the proof of Theorem 2.2, to the function $h = a_{n,1}h_1 + h_2$ to get an estimate similar to either (2.31), if $h_2 = f_{\beta,\tau}$, or (2.40), if $h_2 = g_{m,l,\tau}$. Further, instead of (2.32) (or (2.41)) we have

$$\left|h^{(k)}(z)\right| = \left|a_{n,1}h_1^{(k)}(z) + h_2^{(k)}(z)\right| = \left|h_2^{(k)}(z)\right| \left|a_{n,1}\frac{h_1^{(k)}(z)}{h_2^{(k)}(z)} + 1\right|,$$

where k is chosen (see proof of Theorem 2.2) for h_2. If we show that, for every $r > 0$ small enough, one can find a point $\zeta \in G$ satisfying $|\zeta - \tau| = r$, the first inequality in (2.12), and such that

$$\left|h^{(k)}(z)\right| \geq c\,(h_1, h_2, G)\left|h_2^{(k)}(z)\right|, \tag{2.48}$$

then we can again follow the proof of Theorem 2.2 to obtain a lower estimate for $E_{n,2}\,(h, G)$ in terms of $d_{1/n}\,(\tau)$, which is similar to that for $E_{n,2}\,(h_2, G)$. Then, in view of (2.43), we get

$$E_{n,2}\,(h, G) \succeq E_{n,2}\,(h_2, G) \geq \frac{1}{4}\left(|a_{n,1}|\, E_{n,2}\,(h_1, G) + E_{n,2}\,(h_2, G)\right)$$

because of the assumption (2.47).

To prove (2.48), one can proceed as follows. Let $\omega := f_0(\tau)$,

$$\varDelta := f_0^{-1}\left(\{w : \, |\omega - w| < 5/4, |\text{arg}\;(\omega - w)| < \pi/6\}\right).$$

Since G is a quasidisk, so is $\varDelta$ (with the coefficient of quasiconformality depending only on that of G). Moreover, for any point $\zeta \in \overline{\varDelta}$ the first inequality in (2.12) holds. The quasiconformality of $\partial\varDelta$ also implies that, for $r < r_0$ small enough, for the circular arc

$$l_r = \left\{z \in \overline{\varDelta} : \, |z - \tau| = r, \, \theta_1(r) \leq \text{arg}\,(z - \tau) \leq \theta_2(r)\right\},$$

separating in $\overline{\varDelta}$ points τ and z_0, we have

$$\theta_2(r) - \theta_1(r) \geq \delta = \delta(L).$$

Further arguments depend on the explicit forms of h_1 and h_2, and are left to the reader.

Note that, for each particular pair of functions h_1, h_2, the above arguments can be substantially simplified. For example, if h_1 and h_2 are both power functions, i.e., $h_j = f_{\beta_j,\tau}$, $j = 1, 2$, $\beta_2 > \beta_1$, (the case we are mostly interested in), then using the left inequality in (2.47), (2.43), and applying (2.32) to h_1 and h_2 with $k = [\beta_2] + 1$, we get

$$\left|a_{n,1}\frac{h_1^{(k)}\,(z)}{h_2^{(k)}\,(z)}\right| \succeq c\,(\beta_1, \beta_2)\left|\frac{d_{1/n}\,(\tau)}{z - \tau}\right|^{\beta_2 - \beta_1} \geq 2,$$

if $|z-\tau| = \varepsilon d_{1/n}(\tau)$ with any $\varepsilon < \varepsilon_0(\beta_1, \beta_2, G)$. Hence, for the point z_ε defined by (2.33), we have

$$\left|h^{(k)}(z_\varepsilon)\right| \geq \left|h_2^{(k)}(z_\varepsilon)\right|$$

for any $\varepsilon < \varepsilon_0$, which is sufficient to then follow the proof of Theorem 2.2. □

Corollary 2.3 *Let h be defined by* (2.44). *Then, for any fixed $R > 0$,*

$$E_{n,2}(h, G \cap D(\tau, R)) \succeq E_{n,2}(h, G).$$

Proof. In view of (2.45), it is sufficient to get the desired inequality in the case when h is either $f_{\beta,\tau}$ or $g_{m,l,\tau}$. Let $r > 0$ be chosen in such a way that $D(\Phi(\tau), r) \subset \Phi(D(\tau, R))$. Denote $\widetilde{G} = \Psi(D(\Phi(\tau), r) \cap \mathbb{D})$. Then $\widetilde{G}$ is a quasidisk and $\widetilde{G} \subset G \cap D(\tau, R)$. Obviously,

$$E_{n,2}(h, G \cap D(\tau, R)) \geq E_{n,2}\left(h, \widetilde{G}\right). \tag{2.49}$$

The domains G and $\widetilde{G}$ have the same local structure near the point τ. So, it is not difficult to verify that

$$\widetilde{d}_{1/n}(\tau) \asymp d_{1/n}(\tau) \qquad \text{as} \qquad n \to \infty. \tag{2.50}$$

Thus, applying lower estimate in (2.43) to the function h in $\widetilde{G}$, using (2.50) and upper estimate in (2.43) for G, we easily get

$$E_{n,2}\left(h, \widetilde{G}\right) \asymp E_{n,2}(h, G), \tag{2.51}$$

which together with (2.49) gives the required. □

Corollary 2.4 *Suppose $f(z)$ is analytic in G and, for some $R > 0$, on the set $G \cap D(\tau, R)$, $\tau \in L$, can be represented in the form*

$$f(z) = h(z) + g(z),$$

where g is analytic on $\overline{D(\tau, R)}$, $\|g\|_{L^\infty\left(\overline{D(\tau,R)}\right)} \leq 1$, and h is of the form (2.44). *Then*

$$E_{n,2}(f, G) \geq cE_{n,2}(h, G),$$

where c is a constant independent of n and g.

Proof. We follow the previous proof and construct, for $R/2$ instead of R, the domain $\widetilde{G}$, for which (2.49) and (2.50) are satisfied. Observe now that

$$\begin{aligned} E_{n,2}\left(g, \widetilde{G}\right) &\leq E_{n,2}\left(g, D(\tau, R/2)\right) \\ &\leq R\,\|g\|_{L^\infty(D(\tau,R))}\, 2^{-n} \leq 2^{-n} R. \end{aligned} \tag{2.52}$$

It follows from (2.7) that, for any $\tau \in \overline{G}$,

$$d_{1/n}(\tau) \succeq n^{-1/c}. \tag{2.53}$$

Hence, (2.45) and (2.43) imply

$$E_{n,2}(h, G) \succeq n^{-C_1}. \tag{2.54}$$

Similar estimates hold for $\widetilde{G}$. Using the triangle inequality, (2.52), and (2.51) we get

$$E_{n,2}(f, G) \geq E_{n,2}\left(h, \widetilde{G}\right) - E_{n,2}\left(g, \widetilde{G}\right) \geq E_{n,2}\left(h, \widetilde{G}\right) \asymp E_{n,2}\left(h, \widetilde{G}\right).$$

This completes the proof. □

The following result is an analog of Andrievskii's lemma ([2]).

Lemma 2.3 *Let $z_0 \in G$ and $h(z)$ be analytic on G, continuous on $\overline{G}$ and such that $h' \in L^2(G)$. Suppose that for some positive integer constant C_0 and every $k \in \mathbb{N}$*

$$E_{kC_0,2}\left(h', G\right) \leq \frac{1}{2} E_{k,2}\left(h', G\right). \tag{2.55}$$

If for some $p \in \mathbb{P}_{n+1}$ $(n \geq 2)$ and constants $c_n \in \mathbb{C}$, $M > 0$,

$$\left\|p' + c_n h'\right\|_{L^2(G)} \leq M \tag{2.56}$$

and $p(z_0) + c_n h(z_0) = 0$, then

$$\|p + c_n h\|_{L^\infty(G)} \leq CM\sqrt{\log n},$$

where C is independent of c_n, M, and n.

Proof. If $c_n = 0$, the assertion reduces to Andrievskii's lemma, so we assume that $c_n \neq 0$. Let Q_k', $\deg Q_k' \leq k$, $k = 1, 2, \ldots$, be the best L^2-approximants to h'. Then from (2.56) we have

$$E_{n,2}\left(h', G\right) = \left\|h' - Q_n'\right\|_{L^2(G)} \leq \left\|h' + \frac{p'}{c_n}\right\|_{L^2(G)} \leq \frac{M}{|c_n|}, \tag{2.57}$$

and so

$$\left\|p' + c_n Q_n'\right\|_{L^2(G)} \leq M + |c_n| \, E_{n,2}\left(h', G\right) \leq 2M.$$

Selecting Q_k so that $p\left(z_0\right) + c_n Q_k\left(z_0\right) = 0$, i.e., $Q_k\left(z_0\right) = h\left(z_0\right)$ for all k, Andrievskii's lemma yields

$$\|p + c_n Q_n\|_{L^\infty(G)} \leq C_1 M \sqrt{\log n}. \tag{2.58}$$

Next we claim that

$$\|h - Q_n\|_{L^\infty(G)} \leq C_2 \sqrt{\log n} \frac{M}{|c_n|}. \tag{2.59}$$

Indeed, following a well-known scheme (see e.g. [2]) we choose k satisfying $C_0^k \leq n < C_0^{k+1}$. Since

$$\left\|Q'_{C_0^{j+1}} - Q'_{C_0^j}\right\|_{L^2(G)} \leq 2E_{C_0^j,2}\left(h', G\right) \tag{2.60}$$

and, thanks to (2.55), the series $\sum_j E_{C_0^j,2}\left(h', G\right)$ converges, we get for $z \in G$

$$\left(h - Q_n\right)(z) = \left(Q_{C_0^{k+1}} - Q_n\right)(z) + \sum_{j=k+1}^{\infty} \left(Q_{C_0^{j+1}} - Q_{C_0^j}\right)(z).$$

From (2.60) we conclude (via Andrievskii's lemma) that

$$\left\|Q_{C_0^{j+1}} - Q_{C_0^j}\right\|_{L^\infty(G)} \leq C_1 \sqrt{\log C_0^{j+1}} E_{C_0^j,2}\left(h', G\right).$$

In the same way,

$$\left\|Q_{C_0^{k+1}} - Q_n\right\|_{L^\infty(G)} \leq C_1 \sqrt{\log C_0^{k+1}} E_{n,2}\left(h', G\right).$$

Hence

$$\begin{aligned}
&\|h - Q_n\|_{L^\infty(G)} \\
&\leq C_1 \left(\sqrt{\log C_0^{k+1}} E_{n,2}\left(h', G\right) + \sum_{j=k+1}^{\infty} \sqrt{\log C_0^{j+1}} E_{C_0^j,2}\left(h', G\right) \right) \\
&\leq C_3 \Bigg(\sqrt{\log C_0^{k+1}} E_{n,2}\left(h', G\right) + E_{C_0^{k+1},2}\left(h', G\right) \\
&\qquad \times \sum_{j=k+1}^{\infty} \sqrt{\log C_0^{j+1}} 2^{k+1-j} \Bigg)
\end{aligned}$$

$$\le C_4 \sqrt{\log n} E_{n,2}\left(h', G\right) \left(1 + \sum_{j=k+1}^{\infty} \sqrt{\frac{j+1}{k+1}} 2^{k+1-j}\right)$$
$$\le C_4 \sqrt{\log n} E_{n,2}\left(h', G\right) \left(1 + 2 \sum_{m=1}^{\infty} \sqrt{m} 2^{-m}\right)$$
$$\le C_5 \sqrt{\log n} E_{n,2}\left(h', G\right) \le C_5 \sqrt{\log n} \frac{M}{|c_n|},$$

which proves the claim.

Finally, from (2.58) and (2.59) we obtain

$$\|p + c_n h\|_{L^\infty(G)} \le \|p + c_n Q_n\|_{L^\infty(G)} + |c_n| \, \|h - Q_n\|_{L^\infty(G)}$$
$$\le C_1 M \sqrt{\log n} + C_2 |c_n| \sqrt{\log n} \frac{M}{|c_n|} = CM\sqrt{\log n}.$$

This completes the proof. □

We will need the following application of Lemma 2.3.

Corollary 2.5 *Let* $\tau_j \in L$, $j = \overline{1, r}$, *and, for each* j *and* $k = \overline{1, k_j}$, *let* $h_{j,k}$ *denote either* $f_{\beta_{j,k},\tau_j}$, $\beta_{j,k} > 0$ *noninteger, or* $g_{m_{j,k},l_{j,k},\tau_j}$, $m_{j,k} > 0$. *Suppose that for some constants* $c_{n,j,k}$, $k = \overline{1, k_j}$, $j = \overline{1, r}$, *and a polynomial* $p \in \mathbb{P}_{n+1}$ $(n \ge 2)$ *the inequality*

$$\left\| p' + \sum_{j=1}^{r} \sum_{k=1}^{k_j} c_{n,j,k} h'_{j,k} \right\|_{L^2(G)} \le M$$

holds. If

$$p\left(z_0\right) + \sum_{j=1}^{r} \sum_{k=1}^{k_j} c_{n,j,k} h_{j,k}\left(z_0\right) = 0,$$

then

$$\left\| p + \sum_{j=1}^{r} \sum_{k=1}^{k_j} c_{n,j,k} h_{j,k} \right\|_{L^\infty(G)} \le CM\sqrt{\log n},$$

where C *is a constant independent of* n *and* $\left\{ \{c_{n,j,k}\}_{k=1}^{k_j} \right\}_{j=1}^{r}$.

Proof. Let $c_n := \max_{k,j} |c_{n,j,k}|$. For all j, k denote $\widetilde{c}_{n,j,k} := c_{n,j,k}/c_n$, and set

$$h := \sum_{j=1}^{r} \sum_{k=1}^{k_j} \widetilde{c}_{n,j,k} h_{j,k}.$$

It is sufficient to show that the function h satisfies (2.55). Since

$$h'_{j,k} = \begin{cases} \beta_{j,k} f_{\beta_{j,k}-1,\tau_j} \quad \text{if} \quad h_{j,k} = f_{\beta_{j,k},\tau_j}, \\ m_{j,k} g_{m_{j,k}-1,l_{j,k},\tau_j} + l_{j,k} g_{m_{j,k}-1,l_{j,k}-1,\tau_j} \\ \qquad \text{if} \quad h_{j,k} = g_{m_{j,k},l_{j,k},\tau_j}, \end{cases}$$

we have

$$h' = \sum_{j=1}^{r} \sum_{k} b_{n,j,k} \widetilde{h}_{j,k} =: \sum_{j=1}^{r} \widetilde{h}_j,$$

where $\widetilde{h}_{j,k}$ is one of the functions $f_{\beta_{j,k}-1,\tau_j}$, $g_{m_{j,k}-1,l_{j,k},\tau_j}$, and $g_{m_{j,k}-1,l_{j,k}-1,\tau_j}$. Hence, $\widetilde{h}_j$ is of the form (2.44). Note that the set of all $b_{n,j,k}$'s is bounded by a constant independent of n. Clearly,

$$E_{n,2}\left(h', G\right) \leq \sum_{k=1}^{r} \sum_{k} \left|b_{n,j,k}\right| E_{n,2}\left(\widetilde{h}_{j,k}, G\right). \tag{2.61}$$

On the other hand, by (2.12), there is an R, depending only on the set $\{\tau_j\}_{j=1}^{r}$ and the coefficient of quasiconformality of L, such that for $j = \overline{1,r}$ the function $\tilde{h}_j$ is analytic on $\overline{D(\tau_k, R)}$, for $k \neq j$. Therefore, by Corollary 2.4 and Theorem 2.3, we get

$$\begin{aligned} E_{n,2}\left(h', G\right) &\geq c_1 \max_{1\leq j\leq r} \left\{E_{n,2}\left(\tilde{h}_j, G\right)\right\} \\ &\geq c_2 \sum_{j=1}^{r} E_{n,2}\left(\tilde{h}_j, G\right) \\ &\geq c_3 \sum_{k=1}^{r} \sum_{k} \left|b_{n,j,k}\right| E_{n,2}\left(\widetilde{h}_{j,k}, G\right). \end{aligned} \tag{2.62}$$

If $\widetilde{h}_{j,k} = f_{\beta_{j,k}-1,\tau_j}$, then by Corollary 2.2 and (2.7) we have

$$\begin{aligned} E_{nC_0,2}\left(\widetilde{h}_{j,k}, G\right) &\leq C_1 \left(d_{1/(nC_0)}\left(\tau_j\right)\right)^{\beta_{j,k}} \\ &\leq C_2 \left(C_0^{-c} d_{1/n}\left(\tau_j\right)\right)^{\beta_{j,k}} \\ &\leq C_3 C_0^{-c\beta_{j,k}} E_{n,2}\left(\widetilde{h}_{j,k}, G\right). \end{aligned}$$

Therefore, taking $C_{j,k} = \left[(2C_3/c_3)^{1/\left(c\beta_{j,k}\right)}\right] + 1$, where c_3 is the constant from (2.62), we get

$$E_{nC_0,2}\left(\widetilde{h}_{j,k}\right) \leq \frac{c_3}{2} E_{n,2}\left(\widetilde{h}_{j,k}, G\right).$$

A similar inequality holds if $\widetilde{h}_{j,k}$ is any of the other possible functions. With $C_0 := \max_{j,k} C_{j,k}$, (2.61) and (2.62) yield

$$E_{nC_0,2}\left(h',G\right) \le \sum_{k=1}^{r}\sum_{k}|b_{n,j,k}|\, E_{nC_0,2}\left(\widetilde{h}_{j,k},G\right)$$
$$\le \frac{1}{2}c_3\sum_{k=1}^{r}\sum_{k}|b_{n,j,k}|\, E_{nC_0,2}\left(\widetilde{h}_{j,k},G\right) \le \frac{1}{2}E_{n,2}\left(h',G\right),$$

which establishes (2.55). The result now follows from Lemma 2.3. □

3 Approximation of the Riemann mapping

Now we apply the results obtained to the approximation of Riemann mapping function $f_0(z)$.

Suppose that the boundary L of a domain G is a piecewise analytic curve without cusps, i.e., L is composed of a finite number of analytic arcs meeting at corners τ_j and forming there interior angles $\alpha_j\pi$, $0<\alpha_j<2$, $j=\overline{1,M}$. Obviously, G is a quasidisk.

With $z_0 \in G$, let $w = f_0(z)$ denote the conformal mapping of G onto the disk $D(0,r_0)$, normalized so that $f_0(z_0) = 0$ and $f_0'(z_0) = 1$, where $r_0 := r_0(G,z_0)$ is the conformal radius of G with respect to z_0. The behavior of f_0 at an analytic corner has been considered in [13] and applied to the problem of approximation of the Riemann mapping function in [11].

We shall assume throughout this section that *no logarithmic terms occur in the asymptotic expansions of f_0 near the corners τ_j*, $j=\overline{1,M}$, where M is the total number of corners of the boundary L. This would be the case if, for every j, $j=\overline{1,M}$, either the corner τ_j is formed by two straight-line segments or two circular arcs, or if α_j is irrational; see [13, Theorem 2], [7, p. 170] and [19, pp. 169–170]. Suitable modifications of the analysis for the cases when logarithmic terms appear in the asymptotic expansions of f_0 near some corner τ_j are left to the reader.

Let m denote the number of corners for which α_j is not of the form $1/N$, $N\in\mathbb{N}$, and assume $m\ge 1$. For convenience, such corners τ_j will be indexed by $j=\overline{1,m}$. (That is, if $j>m$, then the mapping function f_0 has an analytic continuation in some neighborhood of the corner τ_j.)

For each $k=\overline{1,m}$ denote by $\left\{\gamma_j^{(k)}\right\}_{j=1}^{\infty}$ the increasing arrangement of the possible powers $p+q/\alpha_k$ ($p\in\mathbb{N}_0$, $q\in\mathbb{N}$) of $(z-\tau_k)$ that appear in the asymptotic expansion of $f_0(z)$ near τ_k. In particular, if τ_k is formed by two straight-line segments, then

$$\gamma_j^{(k)} = j/\alpha_k, \quad j=1,2,\ldots.$$

Also, if α_k is irrational, or the corner τ_k is formed by two circular arcs, then

$$\gamma_1^{(k)} = 1/\alpha_k;$$
$$\gamma_2^{(k)} = 1/\alpha_k + \min(1/\alpha_k, 1);$$
$$\gamma_3^{(k)} = \begin{cases} 1/\alpha_k + 2, & 0 < \alpha_k < 1/2, \\ 2/\alpha_k, & 1/2 < \alpha_k < 1, \\ 1/\alpha_k + 1, & 1 < \alpha_k < 2 \end{cases};$$
$$\vdots$$

The mentioned asymptotic expansion near τ_k can thus be written in the form

$$f_0(z) = \sum_{j=0}^{\infty} a_j^{(k)} (z - \tau_k)^{\gamma_j^{(k)}} = \sum_{j=0}^{\infty} a_j^{(k)} f_{\gamma_j^{(k)}, \tau_k}(z), \tag{3.1}$$

where

$$\gamma_0^{(k)} := 0, \quad f_{0,\tau_k}(z) \equiv 1, \quad \text{and} \quad a_1^{(k)} \neq 0. \tag{3.2}$$

In (3.1) we remark that $\gamma_1^{(k)} = 1/\alpha_k > 1/2$ and that in the case when α_k is rational, it is possible that $\gamma_j^{(k)} \in \mathbb{N}$ for indices $j \geq 2$, so that $f_{\gamma_j^{(k)}, \tau_k}$ is analytic at τ_k.

For each $k = \overline{1, m}$ choose a number $p_k \in \mathbb{N}_0$ and denote

$$\nu_k := \min \left\{ j > p_k \mid \gamma_j^{(k)} \notin \mathbb{N},\, a_j^{(k)} \neq 0 \right\}.$$

In what follows, we assume that at least one of ν_k's is finite; otherwise, results become trivial.

Consider the function

$$f(z) := f_0(z) - \sum_{k=1}^{m} \sum_{j=0}^{\nu_k} a_j^{(k)} f_{\gamma_j^{(k)}, \tau_k}(z) + H(z),$$

where, for $m > 1$, $H(z)$ is the polynomial interpolating each of the functions

$$\sum_{\substack{k=1 \\ k \neq l}}^{m} \sum_{j=0}^{\nu_k} a_j^{(k)} f_{\gamma_j^{(k)}, \tau_k}(z)$$

and its derivatives up to and including the order $\left[\gamma_{\nu_l+1}^{(l)}\right]$ at the point τ_l, $l = \overline{1, m}$. For $m = 1$, we take $H(z) \equiv 0$. Clearly,

$$\deg H \leq \sum_{l=1}^{m} \left[\gamma_{\nu_l+1}^{(l)}\right] + m - 1$$

and the function

$$f'(z) = f_0'(z) - \sum_{k=1}^{m}\sum_{j=1}^{\nu_k} a_j^{(k)} f'_{\gamma_j^{(k)},\tau_k}(z) + H'(z)$$

has, near each τ_k, the asymptotic expansion

$$f'(z) = \gamma_{\nu_k+1}^{(k)} a_{\nu_k+1}^{(k)} (z-\tau_k)^{\gamma_{\nu_k+1}^{(k)}-1} + \dots .$$

Proceeding as in [11], we denote

$$g(z) := f'(z)\prod_{k=1}^{m}(z-\tau_k)$$

and conclude that $h := g\circ\Psi \in \Lambda(\widetilde{s})$ with

$$\widetilde{s} := \min_{1\le k\le m}\left\{(2-\alpha_k)\,\gamma_{\nu_k+1}^{(k)}\right\}.$$

This means that for $\widetilde{s} = p+\gamma$, $p\in\mathbb{N}_0$, $0<\gamma\le 1$, $h^{(p)}\in \mathrm{Lip}\,\gamma$, if $\gamma<1$, and $h^{(p)}$ belongs to the Zygmund class, if $\gamma=1$, on $\partial\mathbb{D}$. Since G is a Faber domain, it immediately follows that

$$E_{n,\infty}(g,G) := \min_{p\in\mathbb{P}_n}\|g-p\|_{L^\infty(G)} \preceq n^{-\widetilde{s}}, \quad n=1,2,\dots . \tag{3.3}$$

Then (see [10, Theorem 2]) there is a polynomial sequence $\{Q_n\}_{n>m}$, $Q_n\in\mathbb{P}_n$, such that

$$\left\|f'-Q_n\right\|_{L^2(G)} \preceq n^{-\widetilde{s}}\sqrt{\log n}.$$

That is,

$$\left\|f_0' - \sum_{k=1}^{m}\sum_{j=1}^{\nu_k} a_j^{(k)} f'_{\gamma_j^{(k)},\tau_k} + H' - Q_n\right\|_{L^2(G)} \preceq n^{-\widetilde{s}}\sqrt{\log n}. \tag{3.4}$$

Note now that $f'_{\beta,\tau}(z) = \beta f_{\beta-1,\tau}(z)$. So, using Corollary 2.2 and (2.3) we construct polynomials $P_{k,n}\in\mathbb{P}_n$, $k=\overline{1,m}$, such that

$$\left\|f'_{\gamma_{\nu_k}^{(k)},\tau_k} - P_{k,n}\right\|_{L^2(G)} \preceq n^{-(2-\alpha_k)\gamma_{\nu_k}^{(k)}}. \tag{3.5}$$

Then for the polynomials

$$\begin{aligned}\hat{P}_n(z) &= \hat{P}_n(p_1,\dots,p_m;z)\\ &:= \sum_{k=1}^{m}\sum_{j=p_k+1}^{\nu_k-1} a_j^{(k)} f'_{\gamma_j^{(k)},\tau_k}(z) + \sum_{k=1}^{m} a_{\nu_k}^{(k)} P_{k,n}(z) - H'(z) + Q_n(z)\end{aligned}$$

we get from (3.4) and (3.5)

$$\left\| f_0' - \sum_{k=1}^{m} \sum_{j=1}^{p_k} a_j^{(k)} f'_{\gamma_j^{(k)},\tau_k} - \hat{P}_n \right\|_{L^2(G)}$$

(3.6) $$\preceq n^{-\widetilde{s}} \sqrt{\log n} + \sum_{k=1}^{m} n^{-(2-\alpha_k)\gamma_{\nu_k}^{(k)}} \preceq n^{-s^*},$$

where

(3.7) $$s^* = s^* (p_1, \dots, p_m) := \min_{1 \le k \le m} \left\{ (2 - \alpha_k) \gamma_{\nu_k}^{(k)} \right\},$$

and an empty sum has value zero.

Remark 3.1 Taking, for instance, all the $p_k = 0$, $k = \overline{1, m}$, and setting $\check{P}_n(z) := \hat{P}_n(0, \dots, 0; z)$ we get from (3.6):

$$\left\| f_0' - \check{P}_n \right\|_{L^2(G)} \preceq n^{-s},$$

where

$$s := s^*(0, \dots, 0) = \min_{1 \le k \le m} \left\{ (2 - \alpha_k) \gamma_1^{(k)} \right\} = \min_{1 \le k \le m} \left\{ \frac{2 - \alpha_k}{\alpha_k} \right\}.$$

(3.8)

Because of the minimal property of Bieberbach polynomials $\pi_n(z)$, it follows that

(3.9) $$\left\| f_0' - \pi_n' \right\|_{L^2(G)} = \mathcal{O}\left(n^{-s} \right)$$

and, consequently,

(3.10) $$\| f_0 - \pi_n \|_{L^\infty(G)} = \mathcal{O}\left(\frac{\sqrt{\log n}}{n^s} \right) \qquad \text{as} \quad n \to \infty.$$

This rate of convergence is an improvement of [10, Theorem 1] and, particularly, [11, Theorem 2], where the corresponding results contain the factor $\log n$ instead of its square root. Such improvement was also shown in [6] for domains with piecewise quasianalytic boundary.

In view of the asymptotic expansion of f_0 near the corners τ_k, $k = \overline{1, m}$, and the estimate (3.6), it is reasonable to extend the power system $\{z^n\}$, $n \in \mathbb{N}_0$, by adjoining the functions $f'_{\gamma_j^{(k)},\tau_k}(z)$, $j = \overline{1, p_k}$, $k = \overline{1, m}$ (cf. [14]). For this purpose, put

$$r_0 := 0, \qquad r_l := \sum_{k=1}^{l} p_k, \quad l = \overline{1, m},$$

and consider the system $\{\eta_j\}_1^\infty$ defined by

$$\eta_j(z) := f'_{\gamma^{(l)}_{j-r_{l-1}},\tau_l}(z), \qquad j=\overline{r_{l-1}+1, r_l}, \quad l=\overline{1,m},$$

$$\eta_{r_m+1}(z) := 1, \quad \eta_{r_m+2}(z) := 2z, \quad \dots \quad \eta_{r_m+n}(z) := nz^{n-1}, \quad \dots .$$

If some α_k is rational, it is possible that some η_j, $1 \le j \le r_m$, is a polynomial, in which case we avoid redundancy in the basis by omitting such η_j. For convenience in exposition, we assume that this situation does not arise.

Set

$$\mu_j(z) := \int_{z_0}^{z} \eta_j(\zeta) d\zeta, \quad z \in \overline{G}. \tag{3.11}$$

Next we orthonormalize the system $\{\eta_k\}_1^\infty$ by means of the Gram-Schmidt process to get $\{\eta_k^*\}_1^\infty$. The functions η_k^* have the representation

$$\eta_k^*(z) = \sum_{j=1}^{k} b_{k,j}\eta_j(z),$$

where $b_{k,k} > 0$, $k = 1, 2, \dots$.

Let

$$\begin{aligned}\mathbb{P}^A_{n-1} &:= \left\{ p^A : p^A(z) = \sum_{k=1}^{r_m+n} t_k\eta_k(z), \quad t_k \in \mathbb{C} \right\} \\ &= \left\{ t_1 f'_{\gamma_1^{(1)},\tau_1}(z) + \dots + t_{r_m} f'_{\gamma_{p_m}^{(m)},\tau_m}(z) + t_{r_m+1} + 2t_{r_m+2}z \right. \\ &\qquad \left. + \dots + nt_{r_m+n}z^{n-1} \right\}.\end{aligned}$$

Also, for $z_0 \in G$, let $K(z, z_0)$ denote the *Bergman kernel function of* G, which has the reproducing property

$$g(z_0) = \iint_G g(z)\overline{K(z, z_0)}dxdy, \quad \text{for any } g \in L^2(G). \tag{3.12}$$

Then (see [8, p. 34])

$$f_0'(z) = \frac{K(z, z_0)}{K(z_0, z_0)},$$

$$f_0(z) = \frac{1}{K(z_0, z_0)} \int_{z_0}^{z} K(\zeta, z_0)\, d\zeta. \tag{3.13}$$

We form the partial Fourier sum for $K(z,z_0)$:

$$\begin{aligned}\widetilde{K}_n(z,z_0) &= \sum_{j=1}^{r_m+n} \left(K(\cdot,z_0),\eta_j^*\right)\eta_j^*(z)\\ &= \sum_{j=1}^{r_m+n} \overline{\eta_j^*(z_0)}\eta_j^*(z) = \sum_{j=1}^{r_m+n} h_{n,j}\eta_j(z)\\ &= h_{n,1}f'_{\gamma_1^{(1)},\tau_1}(z)+\ldots+h_{n,r_m}f'_{\gamma_{p_m}^{(m)},\tau_m}(z)+h_{n,r_m+1}\\ &\quad +\ldots+nh_{n,r_m+n}z^{n-1}.\end{aligned}$$

Obviously, augmented polynomials $\widetilde{K}_n(z,z_0)$ are the best approximants to $K(z,z_0)$ in $L^2(G)$ out of the space $\mathbb{P}_{n-1}^A$, i.e.,

$$\left\|K(\cdot,z_0)-\widetilde{K}_n(\cdot,z_0)\right\|_{L^2(G)} \le \left\|K(\cdot,z_0)-p^A\right\|_{L^2(G)} \qquad \text{for any } p^A\in\mathbb{P}_{n-1}^A.$$

Following (3.13) we approximate $f_0'(z)$ and $f_0(z)$ respectively by

$$\widetilde{\pi}_n'(z) := \frac{\widetilde{K}_n(z,z_0)}{\widetilde{K}_n(z_0,z_0)} = \frac{1}{\widetilde{K}_n(z_0,z_0)}\sum_{j=1}^{r_m+n} h_{n,j}\eta_j(z) \tag{3.14}$$

and

$$\begin{aligned}\widetilde{\pi}_n(z) &:= \frac{1}{\widetilde{K}_n(z_0,z_0)}\int_{z_0}^{z}\widetilde{K}_n(\zeta,z_0)\,d\zeta\\ &= \frac{1}{\widetilde{K}_n(z_0,z_0)}\sum_{j=1}^{r_m+n} h_{n,j}\mu_j(z)\\ &= \frac{1}{\widetilde{K}_n(z_0,z_0)}\left[\sum_{l=1}^{m}\sum_{j=r_{l-1}+1}^{r_l} h_{n,j}\left(f_{\gamma_{j-r_{l-1}}^{(l)},\tau_\ell}(z)-f_{\gamma_{j-r_{l-1}}^{(l)},\tau_\ell}(z_0)\right)\right.\\ &\quad \left.+\sum_{j=1}^{n} h_{n,r_m+j}\left(z^j-z_0^j\right)\right]. \end{aligned}\tag{3.15}$$

Clearly, $\widetilde{\pi}_n(z_0)=0$, $\widetilde{\pi}_n'(z_0)=1$, $n=1,2,\ldots$. It is natural to call the functions $\widetilde{\pi}_n$ the *augmented Bieberbach polynomials* over the system $\{\mu_k\}_1^\infty$.

Hence, from (3.6) and the minimum property of the Fourier sum we get

$$\left\|f_0'-\widetilde{\pi}_n'\right\|_{L^2(G)} \preceq n^{-s^*}. \tag{3.16}$$

Now we follow the method of Andrievskii [2]. Applying Corollary 2.5 we derive an estimate for the uniform norm of $\widetilde{\pi}_{2^k} - \widetilde{\pi}_{2^{k-1}}$, $k = 1, 2, \ldots$, from an estimate of the L^2-norm of $(\widetilde{\pi}_{2^k} - \widetilde{\pi}_{2^{k-1}})'$ and get

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(G)} \preceq n^{-s^*}\sqrt{\log n}.$$

Thus, we have established

Theorem 3.1 *Suppose $L = \partial G$ is a piecewise analytic curve with interior angles $\alpha_j\pi$, $0 < \alpha_j < 2$, at the corners τ_j, $j = \overline{1, M}$, where we assume that no logarithmic terms occur in the asymptotic expansions of f_0 near τ_j. With m (≥ 1) described as at the beginning of this section, we have: For any fixed numbers $p_j \in \mathbb{N}_0$, $j = \overline{1, m}$, the augmented Bieberbach polynomials $\widetilde{\pi}_n$, defined by* (3.15), *approximate $f_0(z)$ with the estimate*

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(G)} = \mathcal{O}\left(\frac{\sqrt{\log n}}{n^{s^*}}\right) \qquad \text{as} \quad n \to \infty, \tag{3.17}$$

where $s^* = s^*(p_1, \ldots, p_m)$ is defined by (3.7).

Let $E^A_{n,\infty}(f_0, G)$ denote the error in best uniform approximation to $f_0(z)$ out of the space $\widetilde{\mathbb{P}}^A_n$ spanned by the functions $\{\mu_j\}_1^{r_m+n}$ (cf. (3.11)).

Theorem 3.2 *Let G and s^* be as in Theorem* 3.1, *and let κ be the index for which the minimal value in* (3.7) *is attained. Then*

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(G)} \geq E^A_{n,\infty}(f_0, G) \asymp n^{-s^*}. \tag{3.18}$$

In particular (*since $a_1^{(\kappa)} \neq 0$ and $\gamma_1^{(\kappa)} = 1/\alpha_\kappa \notin \mathbb{N}$*), *for the classical Bieberbach polynomials,*

$$\left\|f_0' - \pi_n'\right\|_{L^2(G)} \asymp \frac{1}{n^s} \text{ and } \frac{1}{n^s} \preceq \|f_0 - \pi_n\|_{L^\infty(G)} \preceq \frac{\sqrt{\log n}}{n^s}, \tag{3.19}$$

where s is given by (3.8).

Proof. Since G is a Faber domain, the upper estimate in (3.18) follows from the fact that for the function

$$h(z) := f_0(z) - \sum_{k=1}^{m}\sum_{j=0}^{p_k} a_j^{(k)} f_{\gamma_j^{(k)},\tau_k}(z)$$

we have $h \circ \Psi \in \Lambda(s^*)$ on $\partial\mathbb{D}$.

Let $q_n^A \in \widetilde{\mathbb{P}}_n^A$, $n \geq \nu_\kappa$, be an arbitrary augmented polynomial. Then, for r small enough, using the expansion (3.1) of $f_0(z)$ in $G^r := G \cap D(\tau_\kappa, r)$ we get

$$
\begin{aligned}
f_0(z) - q_n^A(z) &= \sum_{j=0}^{p_\kappa} c_{n,j} f_{\gamma_j^{(\kappa)},\tau_\kappa}(z) + a_{\nu_\kappa}^{(\kappa)} f_{\gamma_{\nu_\kappa}^{(\kappa)},\tau_\kappa}(z) \\
&\quad + \sum_{j=\nu_\kappa+1}^{\infty} c_j f_{\gamma_j^{(\kappa)},\tau_\kappa}(z) - h(z) - q_n(z),
\end{aligned}
\tag{3.20}
$$

where the function h is analytic in $D(\tau_\kappa, 2r)$, $q_n \in \mathbb{P}_n$ is an algebraic polynomial of degree at most n. By arguments similar to those used for the estimate (3.3) (see also [11, Sec. 1.3]), the second sum in (3.20) can be approximated with the rate $n^{(\alpha_\kappa-2)\gamma_{\nu_\kappa+1}^{(\kappa)}}$. Also, h can be approximated on G^r geometrically fast. Therefore, it is sufficient to show that for any constants $\{C_{n,j}\}_{j=0}^{p_\kappa}$ the function

$$
g(z) := \sum_{j=0}^{p_\kappa} C_{n,j} f_{\gamma_j^{(\kappa)},\tau_\kappa}(z) + f_{\gamma_{\nu_\kappa}^{(\kappa)},\tau_\kappa}(z)
$$

cannot be uniformly approximated on G^r by algebraic polynomials essentially faster than $f_{\gamma_{\nu_\kappa}^{(\kappa)},\tau_\kappa}$ for which the rate, according to the Corollary 2.2, is n^{-s^*}. Estimating the uniform norm from below by the L^2-norm and applying Theorem 2.3, we arrive at the desired lower estimate in (3.18).

The assertions of (3.19) follow from (3.9) by applying a similar argument to get a lower bound for $\|f_0' - \pi_n'\|_{L^2(G)}$. □

We remark that we can also apply Gaier's method ([11, p. 39]) to get the pointwise estimate

$$
|\eta_n^*(z_0)| = \mathcal{O}\left(\frac{1}{n^{s^*}}\right), \quad n \to \infty \quad (z_0 \in G),
\tag{3.21}
$$

for the augmented orthonormal polynomials η_n^*. For the nonaugmented case, i.e. when $\eta_n^* = P_{n-1}$ is the ordinary Bergman orthonormal polynomial, Gaier [11] raised the question of finding more precise estimates for $|P_n(z_0)|$. In our case, (3.21) gives

$$
|P_n(z_0)| = \mathcal{O}\left(\frac{1}{n^{s}}\right), \quad (n \to \infty)
\tag{3.22}
$$

with s given by (3.8). On the other hand, from (3.19) and Lemma 4.4 of [17],

$$\sum_{k=n+1}^{\infty} |P_k(z_0)|^2 = \|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)}^2 \asymp \left\|f_0' - \pi_n'\right\|_{L^2(G)}^2 \asymp n^{-2s},$$

from which it follows that for each $\varepsilon > 0$, there exists a subsequence $\Lambda_\varepsilon \subset \mathbb{N}$ such that

$$|P_n(z_0)| \succeq \frac{1}{n^{s+\frac{1}{2}+\varepsilon}}, \quad n \in \Lambda_\varepsilon. \tag{3.23}$$

Numerical results in Sect. 4 indicate that for certain regions G, the precise rate of decrease is indeed

$$|P_n(z_0)| \approx \frac{1}{n^{s+\frac{1}{2}}}.$$

4 Numerical experiments

4.1 The test regions

Let G_α denote the circular sector of radius 2 with interior angle $\alpha\pi$:

$$G_\alpha := \{z : |z| < 2, -\alpha\pi/2 < \arg z < \alpha\pi/2\}, \quad 0 < \alpha < 2;$$

let f_0 be the conformal map $G_\alpha \to D(0, r_0)$, normalized by the conditions

$$f_0(1) = 0 \quad \text{and} \quad f_0'(1) = 1;$$

(recall that r_0 is the conformal radius of G_α with respect to $z_0 = 1$); and, as in Sect. 3, let $K(z, 1)$ denote the Bergman kernel function of G_α with respect to $z_0 = 1$. Also, let $\pi_n(z)$ denote the Bieberbach polynomial obtained, as indicated in Sect. 3, by integrating the best approximant to $K(z, 1)$ in $L^2(G_\alpha)$ out of the space of polynomials of degree $n - 1$, and similarly, let $\widetilde{\pi}_n(z)$ denote the augmented Bieberbach polynomial obtained by integrating the best approximant to $K(z, 1)$ in $L^2(G_\alpha)$ out of the space

$$\mathbb{P}_{n-1}^A = \left\{ t_1 z^{1/\alpha - 1} + t_2 + t_3 z + \cdots + t_{n+1} z^{n-1}, \; t_j \in \mathbb{C},\ j = \overline{1, n+1} \right\}; \tag{4.1}$$

see (3.15). Finally, for noninteger $\beta > -1$ and $m \in \mathbb{N}$, let

$$f_\beta(z) = z^\beta \quad \text{and} \quad g_m(z) = z^m \log z. \tag{4.2}$$

In this section we present numerical results illustrating the orders of approximation predicted by the theory, regarding the following six errors:

$$E_{n,2}\left(f_\beta, G_\alpha\right) := \min_{p\in\mathbb{P}_n} \|f_\beta - p\|_{L^2(G_\alpha)}\,, \tag{4.3}$$

$$E_{n,2}\left(g_m, G_\alpha\right) := \min_{p\in\mathbb{P}_n} \|g_m - p\|_{L^2(G_\alpha)} \tag{4.4}$$

and

$$\begin{aligned} \mathcal{E}_{n,2}\left(f_0, G_\alpha\right) &:= \|f_0' - \pi_n'\|_{L^2(G_\alpha)}\,, \\ \mathcal{E}_{n,\infty}\left(f_0, G_\alpha\right) &:= \|f_0 - \pi_n\|_{L^\infty(G_\alpha)}\,, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \widetilde{\mathcal{E}}_{n,2}\left(f_0, G_\alpha\right) &:= \|f_0' - \widetilde{\pi}_n'\|_{L^2(G_\alpha)}\,, \\ \widetilde{\mathcal{E}}_{n,\infty}\left(f_0, G_\alpha\right) &:= \|f_0 - \widetilde{\pi}_n\|_{L^\infty(G_\alpha)}\,. \end{aligned} \tag{4.6}$$

In addition, we present numerical results illustrating the rate of decrease of the Bergman orthonormal polynomials $P_n(z)$ of G_α, as $n \to \infty$. We do these by considering: (a) an Orthonormalization Method (ONM) for constructing both $P_n(z)$ and the polynomials that realize the minimum in (4.3) and (4.4), see e.g. [8, Chapter I]; (b) the application of the Bergman Kernel Method (BKM) for computing approximations to the conformal map f_0, with respect to the errors in (4.5) and (4.6), see e.g. [8, Chapter I] and [14].

For each value of the parameter α, the mapping $w = f_0(z)$ can be computed by means of the transformation

$$f_0(z) = \left(\frac{2\alpha(4^{1/\alpha}-1)}{4^{1/\alpha}+1}\right)\frac{t-d}{td-1}, \tag{4.7}$$

where

$$t = \left(\frac{iz^{1/\alpha}+2^{1/\alpha}}{iz^{1/\alpha}-2^{1/\alpha}}\right)^2 \quad \text{and} \quad d = \left(\frac{i+2^{1/\alpha}}{i-2^{1/\alpha}}\right)^2, \tag{4.8}$$

Thus, the value of the conformal radius r_0 is given by

$$r_0 = f_0(2) = \frac{2\alpha(4^{1/\alpha}-1)}{4^{1/\alpha}+1}.$$

Since the kernel function $K(z,1)$ is related to $f_0(z)$ by

$$K(z,1) = \frac{1}{\pi r_0^2} f_0'(z); \tag{4.9}$$

see e.g. [8, p. 34], it follows, in particular, that

$$K(1,1) = \frac{1}{\pi r_0^2} = \frac{1}{\pi}\left(\frac{4^{1/\alpha}+1}{2\alpha(4^{1/\alpha}-1)}\right)^2. \tag{4.10}$$

We consider now the asymptotic behavior of the map f_0 near the three corners $\tau_1 = 0$, $\tau_2 = 2\mathrm{e}^{-\mathrm{i}\alpha\pi/2}$ and $\tau_3 = 2\mathrm{e}^{\mathrm{i}\alpha\pi/2}$ of G_α. The two arms forming the corner at the origin are both straight lines, and therefore from the Schwarz-Christoffel transformation, as $z \to 0$,

$$f_0(z) = f_0(0) + \sum_{j=1}^{\infty} a_j z^{j/\alpha}, \quad a_1 \neq 0; \tag{4.11}$$

see e.g. [19, pp. 169–170]. We note that the coefficients a_j in (4.11) can be computed explicitly in terms of α from (4.7)–(4.8). In particular,

$$a_1 = 2r_0\,(1 - 4^{-1/\alpha}) \quad \text{and} \quad a_2 = -2r_0\,(1 - 4^{1/\alpha})^2\, 4^{-2/\alpha}. \tag{4.12}$$

Each of the two other corners τ_j, $j = 2, 3$, has interior angle $\pi/2$ and is formed by a straight line and a circular arc; hence, as remarked in Sect. 3, f_0 is regular at τ_j, $j = 2, 3$. Furthermore, f_0 has a branch point singularity at 0 whenever $1/\alpha \notin \mathbb{N}$. In other words, the only singularity of f_0 on the boundary L of G_α occurs at the origin, in cases when $1/\alpha$ is not an integer and, in such cases, the dominant term of the asymptotic expansion of f_0 at 0 is $z^{1/\alpha}$. This observation, and the fact that in the BKM one constructs best approximations with respect to the kernel function (4.9) explains the particular choice of the space (4.1); see also [14], [16].

4.2 Computational details

Let $\{\eta_j\}_{j=1}^{\infty}$ be a complete set of functions in $L^2\,(G_\alpha)$. In both ONM and BKM the set $\{\eta_j\}_{j=1}^{l}$ is orthonormalized by means of the Gram-Schmidt process to produce the orthonormal set $\left\{\eta_j^*\right\}_{j=1}^{l}$. This, in particular, requires the computation of the inner products

$$(\eta_k, \eta_j) = \iint_{G_\alpha} \eta_k(z)\overline{\eta_j(z)}\, dxdy, \quad k = \overline{1,l}, \quad j = \overline{1,l}. \tag{4.13}$$

For the application of the ONM we use the computational convenient monomial set

$$\eta_j(z) = z^{j-1}, \quad j = \overline{1, n+1}.$$

Once the orthonormal system $\left\{\eta_j^*\right\}_{j=1}^{n+1}$ has been constructed, the values of the Bergman polynomials are obtained from

$$P_{j-1}(z) = \eta_j^*(z), \quad j = \overline{1, n+1}.$$

Also, the two errors (4.3) and (4.4) can be computed from the Fourier coefficients of the functions f_β and g_m, since the minimum property of finite Fourier sums implies

$$E^2_{n,2}\left(f_\beta, G_\alpha\right) = \|f_\beta\|^2_{L^2(G_\alpha)} - \sum_{j=1}^{n+1} \left|\left(f_\beta, \eta^*_j\right)\right|^2, \tag{4.14}$$

and

$$E^2_{n,2}(g_m, G_\alpha) = \|g_m\|^2_{L^2(G_\alpha)} - \sum_{j=1}^{n+1} \left|\left(g_m, \eta^*_j\right)\right|^2. \tag{4.15}$$

For our purposes here it is important to note that each one of the inner products involved in (4.13), (4.14), and (4.15) is given explicitly in terms of sines and cosines of the opening angle $\alpha\pi$.

In the BKM, the approximation to f_0 is obtained from (4.9) after first approximating the kernel $K(z,1)$ by a finite Fourier sum. The reason for doing so is that, due to the reproducing property (3.12), the Fourier coefficients of $K(z,1)$ can be computed without requiring the explicit knowledge of $K(z,1)$. Therefore, in order to construct the approximations $\pi_n(z)$ and $\widetilde{\pi}_n(z)$ we orthonormalize, respectively, the monomial set (MB),

$$\eta_j(z) = z^{j-1}, \quad j = \overline{1,n}, \tag{4.16}$$

and the augmented set (AB)

$$\widetilde{\eta}_1(z) = z^{1/\alpha - 1}, \quad \widetilde{\eta}_j(z) = z^{j-2}, \quad j = \overline{2, n+1}. \tag{4.17}$$

As with the ONM, the inner products in (4.13) are given explicitly in terms of sines and cosines of the opening angle $\alpha\pi$.

The details of the BKM are as follows: Let $\{\eta^*_j\}_{j=1}^n$ and $\{\widetilde{\eta}^*_j\}_{j=1}^{n+1}$ denote respectively the two orthonormal systems obtained from (4.16) and (4.17). Then, because of (3.12),

$$K_n(z,1) = \sum_{j=1}^{n} \overline{\eta^*_j(1)}\eta^*_j(z), \tag{4.18}$$

and

$$\widetilde{K}_n(z,1) = \sum_{j=1}^{n+1} \overline{\widetilde{\eta}^*_j(1)}\widetilde{\eta}^*_j(z), \tag{4.19}$$

are, respectively, the n-th BKM/MB and the n-th BKM/AB approximation to $K(z,1)$, and from (3.15) we set

$$\pi_n(z) = \frac{1}{K_n(1,1)} \int_1^z K_n(\zeta, 1)\, d\zeta, \tag{4.20}$$

$$\widetilde{\pi}_n(z) = \frac{1}{\widetilde{K}_n(1,1)} \int_1^z \widetilde{K}_n(\zeta,1) d\zeta. \tag{4.21}$$

(Note that $\pi_n(z)$ and $\widetilde{\pi}_n(z)$ are normalized so that $\pi_n(1) = \widetilde{\pi}_n(1) = 0$ and $\pi_n'(1) = \widetilde{\pi}_n'(1) = 1$.)

Regarding the four errors (4.5)–(4.6) we observe the following:

The order of approximation in each of the two errors $\|f_0' - \pi_n'\|_{L^2(G_\alpha)}$ and $\|f_0' - \widetilde{\pi}_n'\|_{L^2(G_\alpha)}$, can be computed with the orthonormal functions, using (4.10), (4.18) and (4.19). This follows by noting that:

i)

$$\left\|f_0' - \pi_n'\right\|_{L^2(G_\alpha)} \asymp \|K(\cdot,1) - K_n(\cdot,1)\|_{L^2(G_\alpha)}$$

and

$$\left\|f_0' - \widetilde{\pi}_n'\right\|_{L^2(G_\alpha)} \asymp \left\|K(\cdot,1) - \widetilde{K}_n(\cdot,1)\right\|_{L^2(G_\alpha)};$$

see [17, Lemma 4.4].

ii) The minimum property of finite Fourier sums and Parseval's identity imply that

$$\|K(\cdot,1) - K_n(\cdot,1)\|^2_{L^2(G_\alpha)} = K(1,1) - K_n(1,1), \tag{4.22}$$

and

$$\left\|K(\cdot,1) - \widetilde{K}_n(\cdot,1)\right\|^2_{L^2(G_\alpha)} = K(1,1) - \widetilde{K}_n(1,1); \tag{4.23}$$

see e.g. [8, p. 25].

Estimates for the two other errors $\|f_0 - \pi_n\|_{L^\infty(G_\alpha)}$ and $\|f_0 - \widetilde{\pi}_n\|_{L^\infty(G_\alpha)}$ can be obtained from (4.7)–(4.8) and (4.18)–(4.21), by using a number of test points on the boundary L.

4.3 Numerical results

The results of Corollary 2.2 indicate that for any α, $0 < \alpha < 2$,

$$E_{n,2}(f_\beta, G_\alpha) \asymp \frac{1}{n^{\lambda(\beta+1)}}, \quad E_{n,2}(g_m, G_\alpha) \asymp \frac{1}{n^{\lambda(m+1)}}, \tag{4.24}$$

with $\lambda = 2 - \alpha$, for the polynomial approximations to the special functions (4.2). Furthermore, since for any $0 < \alpha < 2$, the coefficients a_1 and a_2 in the asymptotic expansion (4.11) of f_0 near $\tau_1 = 0$ are different from 0 (cf. (4.12)), we have from Theorems 3.1, 3.2 and (3.16) that, for $1/\alpha \notin \mathbb{N}$,

$$\text{(4.25)}\quad \mathcal{E}_{n,2}(f_0, G_\alpha) \asymp \frac{1}{n^{\lambda/\alpha}}, \quad \frac{1}{n^{\lambda/\alpha}} \preceq \mathcal{E}_{n,\infty}(f_0, G_\alpha) \preceq \frac{\sqrt{\log n}}{n^{\lambda/\alpha}},$$

$$\text{(4.26)}\quad \widetilde{\mathcal{E}}_{n,2}(f_0, G_\alpha) \asymp \frac{1}{n^{2\lambda/\alpha}}, \quad \frac{1}{n^{2\lambda/\alpha}} \preceq \widetilde{\mathcal{E}}_{n,\infty}(f_0, G_\alpha) \preceq \frac{\sqrt{\log n}}{n^{2\lambda/\alpha}}.$$

Finally, regarding the Bergman polynomials $\{P_n\}_{n=0}^{\infty}$, for any α, $0 < \alpha < 2$, with $1/\alpha \notin \mathbb{N}$, and any $\zeta \in G_\alpha$, we have from (3.22) and (3.23):

– for all $n \in \mathbb{N}$,

$$\text{(4.27)}\qquad |P_n(\zeta)| \preceq \frac{1}{n^{\lambda/\alpha}};$$

– for any $\varepsilon > 0$, there exist infinitely many n such that,

$$\text{(4.28)}\qquad |P_n(\zeta)| \succeq \frac{1}{n^{\lambda/\alpha + \frac{1}{2} + \varepsilon}}.$$

For the remainder of this section we present numerical results that illustrate the laws in the above errors and rates. All the results were obtained with Maple V, using the systems facility for 128-digit floating point arithmetic, on an IBM RS/6000. We have chosen to perform numerical work using this high accuracy, in order to postpone the breakdown of the Gram-Schmidt process. This was essential for our purposes here, because we needed to use a large number of basis functions, typically up to 100, to observe a distinct behavior for the orders of approximation in (4.25) and (4.26). (We note in passing that numerical experiments with G_α, using the double precision Fortran conformal mapping package BKMPACK of Warby [20], failed to produce precise conclusions for the orders in (4.25)–(4.26); see also [18, Example 5.3].)

We recall that the Gram-Schmidt process is required by the application of both ONM and BKM for the construction of the orthonormal system. See [18] for a comprehensive study regarding the stability properties of Bergman Kernel Methods and a characterization of the level of instability in the Gram-Schmidt process, in terms of the geometry of the domain under consideration. In particular, [18, Theorem 3.1] implies that the level of instability for the domain G_α, increases with decreasing α. Also, see [9, Sect. 6] for a report on numerical experiments regarding the errors $\|f_0' - \pi_n'\|_{L^2(G)}$ and $\|f_0 - \pi_n\|_{L^\infty(G)}$, when G is the image of $\{t : |t-1| < 1\}$ under the mapping $z = t^\alpha$, $0 < \alpha < 2$.

In presenting the numerical results we use the following notations:

– σ: This denotes the order of approximation (the exponent of $1/n$) in the errors (4.3)–(4.6), or the rate of decrease of $|P_n(\zeta)|$, as they are predicted by the theory of Sects. 2 and 3; see (4.24)–(4.28).

– σ_n: This denotes the estimate of σ corresponding to the use of n basis functions and is determined as follows: With E_n denoting any of the errors $E_{n,2}(f_\beta, G_\alpha)$, $E_{n,2}(g_m, G_\alpha)$, $\mathcal{E}_{n,2}(f_0, G_\alpha)$, $\widetilde{\mathcal{E}}_{n,2}(f_0, G_\alpha)$, or the value of $|P_n(\zeta)|$, we assume that

$$E_n \approx C\frac{1}{n^\sigma}, \tag{4.29}$$

and seek to estimate σ by means of the formula

$$\sigma_n = \log\left(\frac{E_{n-20}}{E_n}\right) / \log\left(\frac{n}{n-20}\right). \tag{4.30}$$

If E_n denotes either of the uniform errors $\mathcal{E}_{n,\infty}(f_0, G_\alpha)$ or $\widetilde{\mathcal{E}}_{n,\infty}(f_0, G_\alpha)$, then we assume that

$$E_n \approx C\sqrt{\log n}\,\frac{1}{n^\sigma}, \tag{4.31}$$

and seek to estimate σ by means of the formula

$$\sigma_n = \left(\log\left(\frac{E_{n-20}}{E_n}\right) - \frac{1}{2}\log\left(\frac{\log(n-20)}{\log n}\right)\right) / \log\left(\frac{n}{n-20}\right). \tag{4.32}$$

– σ_n^*: With E_n denoting either of the uniform errors $\mathcal{E}_{n,\infty}(f_0, G_\alpha)$ or $\widetilde{\mathcal{E}}_{n,\infty}(f_0, G_\alpha)$, we also test the law

$$E_n \approx C\frac{1}{n^\sigma}; \tag{4.33}$$

thereby estimating σ by means of

$$\sigma_n^* = \log\left(\frac{E_{n-20}}{E_n}\right) / \log\left(\frac{n}{n-20}\right). \tag{4.34}$$

L^2-approximations to special functions.

The numerical results for the values 0.5 and 1.5 of the parameter α, and for $n = 20(20)100$, are given in Tables 4.1–4.4. More precisely, Table 4.1 and Table 4.2 contain the results for the orders in the L^2-approximations to z^β, corresponding to the values $\beta = -0.5$, $\beta = 0.5$ and $\beta = 1.5$. Table 4.3 and Table 4.4 contain the results for the orders in the L^2-approximations to $z^m \log z$, corresponding to the values $m = 1$, $m = 2$ and $m = 3$.

The presented results indicate clearly a close agreement between the theoretical and the computed order of approximation, thus providing experimental confirmation of the results in Sect. 2.

Table 4.1. L^2 polynomial approximations to z^β: Case $\alpha = 0.5$

	$\beta = -0.5$ $\sigma = 0.75$		$\beta = 0.5$ $\sigma = 2.25$		$\beta = 1.5$ $\sigma = 3.75$	
n	$E_{n,2}(f_\beta, G_\alpha)$	σ_n	$E_{n,2}(f_\beta, G_\alpha)$	σ_n	$E_{n,2}(f_\beta, G_\alpha)$	σ_n
20	9.8e-02	-	8.6e-04	-	4.1e-05	-
40	6.0e-02	0.73	1.8e-04	2.22	3.2e-06	3.71
60	4.4e-02	0.74	7.5e-05	2.23	7.1e-07	3.72
80	3.5e-02	0.75	3.9e-05	2.24	2.4e-07	3.73
100	3.0e-02	0.75	2.4e-05	2.24	1.1e-07	3.73

Table 4.2. L^2 polynomial approximations to z^β: Case $\alpha = 1.5$

	$\beta = -0.5$ $\sigma = 0.25$		$\beta = 0.5$ $\sigma = 0.75$		$\beta = 1.5$ $\sigma = 1.25$	
n	$E_{n,2}(f_\beta, G_\alpha)$	σ_n	$E_{n,2}(f_\beta, G_\alpha)$	σ_n	$E_{n,2}(f_\beta, G_\alpha)$	σ_n
20	1.1e-00	-	1.7e-01	-	7.1e-02	-
40	9.6e-01	0.24	1.0e-01	0.74	3.0e-02	1.24
60	8.7e-01	0.24	7.4e-02	0.74	1.8e-02	1.24
80	8.1e-01	0.25	6.0e-02	0.74	1.3e-02	1.24
100	7.7e-01	0.25	5.1e-02	0.75	9.7e-03	1.25

Table 4.3. L^2 polynomial approximations to $z^m \log z$: Case $\alpha = 0.5$

	$m = 1$ $\sigma = 3.0$		$m = 2$ $\sigma = 4.5$		$m = 3$ $\sigma = 6.0$	
n	$E_{n,2}(g_m, G_\alpha)$	σ_n	$E_{n,2}(g_m, G_\alpha)$	σ_n	$E_{n,2}(g_m, G_\alpha)$	σ_n
20	5.2e-04	-	3.9e-05	-	5.7e-06	-
40	6.7e-05	2.97	1.8e-06	4.46	9.2e-08	5.96
60	2.0e-05	2.98	3.0e-07	4.47	8.2e-09	5.96
80	8.5e-06	2.98	8.2e-08	4.48	1.5e-09	5.97
100	4.4e-06	2.99	3.0e-08	4.48	3.9e-10	5.97

BKM approximations to the conformal map.

The numerical results for the values $\alpha = 5/11$, $\alpha = 0.8$ and $\alpha = 4/3$, respectively for $n = 20(20)100$, $n = 20(20)120$ and $n = 20(20)120$, are given in Tables 4.5, 4.6 and 4.7. (When $\alpha = 5/11$ the Gram-Schmidt process breaks down before $n = 108$ is reached.) We consider these particular values of α because the corresponding power $1/\alpha - 1$ of the singular function $\eta_1(z) = z^{1/\alpha - 1}$ needed for the application of BKM/AB, can be presented exactly in finite precision.

In all three tables, the results associated with the errors $\mathcal{E}_{n,2}(f_0, G_\alpha)$ and $\widetilde{\mathcal{E}}_{n,2}(f_0, G_\alpha)$ indicate the convergence of σ_n to σ. Regarding the errors $\mathcal{E}_{n,\infty}(f_0, G_\alpha)$ and $\widetilde{\mathcal{E}}_{n,\infty}(f_0, G_\alpha)$, σ_n^* converges faster to σ than σ_n. This suggests, at least for the geometry under consideration, a behavior of the type (4.33) for the errors $\mathcal{E}_{n,\infty}(f_0, G_\alpha)$ and $\widetilde{\mathcal{E}}_{n,\infty}(f_0, G_\alpha)$.

In the application of BKM/AB with $\alpha = 5/11$, presented in Table 4.5, the slow convergence of σ_n and σ_n^* to σ can be explained by observing

Table 4.4. L^2 polynomial approximations to $z^m \log z$: Case $\alpha = 1.5$

	$m=1$ $\sigma=1.0$		$m=2$ $\sigma=1.5$		$m=3$ $\sigma=2.0$	
n	$E_{n,2}(g_m, G_\alpha)$	σ_n	$E_{n,2}(g_m, G_\alpha)$	σ_n	$E_{n,2}(g_m, G_\alpha)$	σ_n
20	3.2e-01	-	1.7e-01	-	1.2e-01	-
40	1.6e-01	0.99	6.1e-02	1.49	3.1e-02	1.99
60	1.1e-01	0.99	3.3e-02	1.49	1.4e-02	1.99
80	8.1e-02	0.99	2.2e-02	1.49	7.8e-03	1.99
100	6.5e-02	0.99	1.6e-02	1.49	5.0e-03	1.99

Table 4.5. BKM approximations to f_0: Case $\alpha = 5/11$

	BKM/MB: $\sigma = 3.4$					BKM/AB: $\sigma = 6.8$				
n	$\mathcal{E}_{n,2}(f_0, G_\alpha)$	σ_n	$\mathcal{E}_{n,\infty}(f_0, G_\alpha)$	σ_n	σ_n^*	$\widetilde{\mathcal{E}}_{n,2}(f_0, G_\alpha)$	σ_n	$\widetilde{\mathcal{E}}_{n,\infty}(f_0, G_\alpha)$	σ_n	σ_n^*
20	3.7e-04	-	2.0e-04	-	-	4.4e-04	-	1.8e-04	-	-
40	6.1e-06	5.90	1.0e-05	4.38	4.22	2.3e-07	10.89	7.6e-08	11.40	11.25
60	1.5e-06	3.37	2.7e-06	3.50	3.37	3.3e-10	16.16	4.2e-10	12.97	12.84
80	5.9e-07	3.38	1.0e-06	3.49	3.37	5.1e-11	6.50	6.2e-11	6.77	6.65
100	2.8e-07	3.38	4.7e-07	3.49	3.38	1.0e-11	6.84	1.3e-11	6.96	6.84

Table 4.6. BKM approximations to f_0: Case $\alpha = 0.8$

	BKM/MB: $\sigma = 1.5$					BKM/AB: $\sigma = 3.0$				
n	$\mathcal{E}_{n,2}(f_0, G_\alpha)$	σ_n	$\mathcal{E}_{n,\infty}(f_0, G_\alpha)$	σ_n	σ_n^*	$\widetilde{\mathcal{E}}_{n,2}(f_0, G_\alpha)$	σ_n	$\widetilde{\mathcal{E}}_{n,\infty}(f_0, G_\alpha)$	σ_n	σ_n^*
20	4.4e-03	-	1.4e-02	-	-	2.2e-04	-	4.1e-04	-	-
40	1.5e-03	1.57	4.6e-03	1.70	1.55	2.1e-05	3.38	4.6e-05	3.31	3.16
60	8.0e-04	1.53	2.5e-03	1.65	1.52	5.9e-06	3.12	1.3e-05	3.23	3.11
80	5.2e-04	1.51	1.6e-03	1.63	1.51	2.4e-06	3.08	5.4e-06	3.19	3.07
100	3.7e-04	1.51	1.2e-03	1.62	1.51	1.2e-06	3.05	2.7e-06	3.16	3.05
120	2.8e-04	1.50	8.8e-04	1.61	1.50	7.1e-07	3.04	1.6e-06	3.14	3.04

Table 4.7. BKM approximations to f_0: Case $\alpha = 4/3$

	BKM/MB: $\sigma = 0.5$					BKM/AB: $\sigma = 1.0$				
n	$\mathcal{E}_{n,2}(f_0, G_\alpha)$	σ_n	$\mathcal{E}_{n,\infty}(f_0, G_\alpha)$	σ_n	σ_n^*	$\widetilde{\mathcal{E}}_{n,2}(f_0, G_\alpha)$	σ_n	$\widetilde{\mathcal{E}}_{n,\infty}(f_0, G_\alpha)$	σ_n	σ_n^*
20	8.2e-02	-	2.5e-01	-	-	7.2e-03	-	2.4e-02	-	-
40	5.5e-02	0.59	1.8e-01	0.62	0.47	4.7e-03	0.62	1.4e-02	0.92	0.77
60	4.3e-02	0.60	1.5e-01	0.63	0.50	3.3e-03	0.90	9.4e-03	1.10	0.97
80	3.6e-02	0.60	1.3e-01	0.63	0.51	2.5e-03	0.99	7.0e-03	1.14	1.03
100	3.2e-02	0.59	1.1e-01	0.63	0.52	2.0e-03	1.03	5.5e-03	1.16	1.05
120	2.8e-02	0.58	1.0e-01	0.62	0.52	1.6e-03	1.04	4.5e-03	1.17	1.06

that in this case the mapping function f_0 has a pole singularity close to the boundary L. This affects the quality of the obtained approximation. (See [15] and the references cited there for ways of determination and treatment of pole-type singularities in numerical conformal mapping.)

We note in passing that Maple V may not be regarded as a well-suited environment for the construction of BKM approximations to conformal mappings of complicated geometries. The main restriction is the enormous

Table 4.8. Rate of decrease of $|P_n(\zeta)|$: Case $\alpha = 5/11$, $\sigma = 3.90$

	$\zeta = 0.5$		$\zeta = 1.0$		$\zeta = 1.5$	
n	$\|P_n(\zeta)\|$	σ_n	$\|P_n(\zeta)\|$	σ_n	$\|P_n(\zeta)\|$	σ_n
20	1.2e-02	-	2.5e-04	-	3.2e-04	-
40	1.5e-04	6.31	2.4e-06	6.69	8.2e-07	8.53
60	3.5e-06	9.42	5.4e-07	3.69	1.8e-07	3.72
80	1.6e-06	2.81	1.8e-07	3.90	5.9e-08	3.90
100	6.6e-07	3.93	7.4e-08	3.90	2.5e-08	3.90

Table 4.9. Rate of decrease of $|P_n(\zeta)|$: Case $\alpha = 0.8$, $\sigma = 2.0$

	$\zeta = 0.5$		$\zeta = 1.0$		$\zeta = 1.5$	
n	$\|P_n(\zeta)\|$	σ_n	$\|P_n(\zeta)\|$	σ_n	$\|P_n(\zeta)\|$	σ_n
20	2.0e-02	-	1.9e-03	-	2.5e-03	-
40	1.9e-03	3.40	4.2e-04	2.81	2.1e-04	3.56
60	8.0e-04	2.08	1.8e-04	2.04	9.1e-05	2.05
80	4.4e-04	2.12	1.0e-04	2.03	5.1e-05	2.01
100	2.8e-04	2.06	6.5e-05	2.02	3.3e-05	2.01
120	1.9e-04	2.04	4.5e-05	2.02	2.3e-05	2.00

Table 4.10. Rate of decrease of $|P_n(\zeta)|$: Case $\alpha = 4/3$, $\sigma = 1.0$

	$\zeta = 0.5$		$\zeta = 1.0$		$\zeta = 1.5$	
n	$\|P_n(\zeta)\|$	σ_n	$\|P_n(\zeta)\|$	σ_n	$\|P_n(\zeta)\|$	σ_n
20	5.3e-02	-	2.0e-02	-	6.3e-03	-
40	3.1e-02	0.79	9.6e-03	0.99	4.8e-03	0.38
60	2.1e-02	0.96	6.1e-03	1.10	3.1e-03	1.08
80	1.5e-02	1.05	4.4e-03	1.12	2.3e-03	1.07
100	1.2e-02	1.10	3.4e-03	1.12	1.8e-03	1.06
120	9.8e-03	1.12	2.8e-03	1.12	1.5e-03	1.06

amount of C.P.U. time required, in general, for the accurate computation of the inner products needed by the Gram-Schmidt process.

Rates of decrease of the Bergman polynomials.

The numerical results for the rates of decrease of the values $|P_n(\zeta)|$ of the Bergman polynomials $\{P_n\}_{n=0}^{\infty}$ of G_α, for $\zeta = 0.5$, $\zeta = 1$ and $\zeta = 1.5$, corresponding to $\alpha = 5/11$, $\alpha = 0.8$, $\alpha = 4/3$, and for $n = 20(20)100$, $n = 20(20)120$, $n = 20(20)120$, respectively, are given in Tables 4.8, 4.9 and 4.10.

The numerical results contained in the three tables indicate a behavior of the type

$$|P_n(\zeta)| \approx C\frac{1}{n^\sigma} = C\frac{1}{n^{(2-\alpha)/\alpha+1/2}},$$

hence provide experimental support to the remark made at the end of Sect. 3, regarding the rate of decrease of the Bergman polynomials.

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List of notations

G	Quasidisk, i.e., a Jordan domain in the complex plane $\mathbb{C}$ with quasiconformal boundary L.
Ω	The complement of $\overline{G}$.
$D(z, r)$	Open disk centered at z and of the radius r.
$\mathbb{D}$	The unit disk.
$f_0(z)$	The Riemann mapping function, i.e., conformal and univalent map of the domain G onto $D(0, r_0)$ normalized by $f_0(z_0) = 0$, $f_0'(z_0) = 1$.
$\Phi(z)$	Conformal and univalent map of Ω onto the complement of $\mathbb{D}$ normalized by $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$, and extended to a quasiconformal map of the complex plane onto itself.
$\Psi(w)$	The inverse of $\Phi(z)$.
$L_r,\ r > 0$	The r-th level line of $\Phi(z)$, that is, the set $\{\zeta \in \mathbb{C} : \lvert\Phi(\zeta)\rvert = r\}$.
$G_r,\ \Omega_r$	The bounded and unbounded components of $\overline{\mathbb{C}} \setminus L_r$, respectively.
$d_u(z)$	The quantity $\max_{0 \le \varphi < 2\pi} \left\lvert z - \Psi\left[\Phi(z) + u e^{i\varphi}\right]\right\rvert$, $z \in \mathbb{C}$, $u > 0$.
$L^2(G)$	The Hilbert space of functions that are analytic and square-summable over the domain G.
$\mathbb{P}_n$	The class of all polynomials of degree at most n.
$E_{n,2}(f, G)$	The error of the best L^2-approximation to a function $f \in L^2(G)$ out of $\mathbb{P}_n$.
$E_{n,\infty}(f, G)$	The error of the best uniform approximation of a continuous function f on $\overline{G}$ out of $\mathbb{P}_n$.
$\pi_n(z)$	The n-th Bieberbach polynomial for G.
$f_{\beta,\tau}(z)$	A single-valued, analytic branch of the function $(z - \tau)^\beta$ in G, with $\beta > -1$ noninteger.
$g_{m,l,\tau}(z)$	A single-valued, analytic branch of the function $(z - \tau)^m \log^l(z - \tau)$ in G, with $m > -1$, $l \neq 0$.
$\{\eta_k(z)\}_1^\infty$	The extended system of power functions, that is, the set $\left\{\left\{f'_{\beta_k,\tau_k}\right\}_{k=1}^r, \left\{z^{k-r-1}\right\}_{k=r+1}^\infty\right\}$ with β_k, τ_k, and r depending on G.
$\{\eta_k^*(z)\}_1^\infty$	The orthonormal system obtained from $\{\eta_k(z)\}_1^\infty$.
$\mathbb{P}_{n-1}^A$	The class of "augmented polynomials", i.e., polynomials over the system $\{\eta_k\}_{k=1}^{n+r+1}$.
$\widetilde{\pi}_n(z)$	The n-th Bieberbach "augmented polynomial" for G.
$K(z, \zeta)$	The Bergman kernel (reproducing kernel for $L^2(G)$).
$\widetilde{K}_n(z, \zeta)$	The n-th partial Fourier sum for $K(z, \zeta)$, i.e., the n-th partial sum of the expansion of $K(z, \zeta)$ into Fourier series over the system $\{\eta_k^*\}_1^\infty$.