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Complex Analysis and Operator Theory

Finite-Term Relations for Planar Orthogonal Polynomials

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To Peter Duren on the occasion of his seventieth birthday

Abstract. We prove by elementary means that, if the Bergman orthogonal polynomials of a bounded simply-connected planar domain, with sufficiently regular boundary, satisfy a finite-term relation, then the domain is algebraic and characterized by the fact that Dirichlet's problem with boundary polynomial data has a polynomial solution. This, and an additional compactness assumption, is known to imply that the domain is an ellipse. In particular, we show that if the Bergman orthogonal polynomials satisfy a three-term relation then the domain is an ellipse. This completes an inquiry started forty years ago by Peter Duren.

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1. Introduction

Let Ω be a bounded simply-connected domain in the complex plane, and let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of *Bergman orthogonal polynomials* of Ω. This is defined as the sequence

$$p_n(z) = \gamma_n z^n + \gamma_{n-1} z^{n-1} + \cdots\,, \quad \gamma_n > 0\,, \quad n = 0, 1, 2, \dots\,,$$

of polynomials which are orthonormal with respect to the inner product

$$\langle f, g\rangle := \int_\Omega f(z)\overline{g(z)}dA(z)\,,$$

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where dA stands for the area measure. Orthogonal polynomials over planar domains, as well as orthogonal polynomials over planar curves (the so-called Szegö polynomials), are becoming increasingly popular. This is partly due to their relevance in polynomial approximation and reconstruction problems, see for instance [8, 15, 18–21]. It does not mean that complex orthogonal polynomials were not studied before, but the amount of work and publications devoted to their one-dimensional cousins (orthogonal polynomials with respect to measures supported on the real line or the circle) was until recently disproportionately large.

The problem we address in this note is rather old: *characterize all planar domains on which the Bergman orthogonal polynomials satisfy a three-term recurrence relation* (for the precise definition see Section 3 below). For the corresponding problem related to Szegö orthogonal polynomials, and for Jordan domains with a smooth real-analytic boundary, Duren [5] has proved that they must be ellipses. His technique used the full information about the associated Jacobi-type matrix via a classical theorem due to Szegö, regarding the asymptotics of the elements in the three-term relation and the conformal mapping of the exterior domain $\overline{\mathbb{C}} \setminus \overline{\Omega}$, see [26, §16.4]. As it can be readily seen, Duren's method can be adapted to the case of Bergman polynomials, where a theorem of Carleman for the ratio asymptotics can be used (see, e.g., [7, p. 12]), to yield the same result. That is, *amongst all planar domains bounded by an analytic Jordan curve, only ellipses admit a three-term recurrence relation for the Bergman polynomials.* Furthermore, if we use the asymptotic formulae of Suetin [25, p. 20] for the Bergman polynomials, the requirement that the boundary should be analytic can be weaken to C^2-smooth.

In the present note we show that actually less is needed to infer that a simply-connected domain carrying a finite-term relation for its Bergman orthogonal polynomials is an ellipse. Namely, a regular enough boundary for the completeness of the Bergman polynomials and an algebraic compactness assumption on the boundary. Our proof is structurally different than Duren's. It relies on the adjoint of the Bergman shift and a basic characterization of the orthogonal complement to the associated Bergman space (in the corresponding Lebesgue space). Essentially the same technique was used by H. Shapiro and the first author in their study of the Friedrichs operator of a planar domain [22]. In particular, we obtain below the auxiliary statement: *on a simply-connected domain with regular enough boundary, the Bergman orthogonal polynomials satisfy a finite-term recurrence relation if and only if the associated Dirichlet problem with polynomial data has a polynomial solution.* The latter is a well studied question, both for its function theory implications and in connection with the propagation of singularities of the Cauchy problem for the complexified Laplacian, see [3, 6, 12, 24]. A well-known conjecture of Khavinson and Shapiro [12] states that the Dirichlet problem with polynomial data has a polynomial solution if and only if the domain is an ellipse. For the most recent account of the state of this conjecture we refer to [2, 3].

The technique we suggest here carries over, almost word by word, to the Hardy space of a smooth Jordan curve, as stated in Theorem 2 below. It is also clear how to adapt our method to more general measures, such as a polynomial weight or an entire function against the area measure. We note a relevant result of Lempert [14] to the effect that, although the orthogonal polynomials associated with measures supported on the real axis satisfy a three-term recurrence relation, there is no finite-term (for example, four-term or five-term) relation which is satisfied by all the Bergman or all the Szegö orthogonal polynomials. We also note that other results regarding recurrence relations for orthogonal planar polynomials can be found in [4, 16, 17].

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2. Preliminaries

Throughout this note Ω will be a bounded simply-connected planar domain. Hence its boundary $\partial\Omega$ will contain at least two points and, therefore, Ω will be regular with respect to the Dirichlet problem. This means that the Dirichlet problem for Ω with continuous data on $\partial\Omega$ has a unique solution, which coincides with the data on the boundary; see, e.g., [23, pp. 91–92]. We denote by $L^2_a(\Omega)$ the associated Bergman space, that is the Hilbert space of all complex analytic functions defined in Ω which are square summable with respect to the area measure dA. To derive certain results in Section 3, we need to assume in addition that the polynomials are dense in $L^2_a(\Omega)$. This will be, for example, the case when Ω is a Caratheodory domain, i.e., when $\Omega = \text{int}(\overline{\Omega})$.

In the case when the boundary of Ω is smooth one can speak, without ambiguity, about the associated Hardy space $H^2(\Omega)$, consisting of the closure of polynomials in $L^2(\partial\Omega, d\omega)$, where ω is the harmonic measure with respect to a fixed point z_0 of Ω. The Cauchy–Riemann operators will be simply denoted by $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial\overline{z}$. We also write $\Gamma = \partial\Omega$ and refer to an equality almost everywhere (a.e.) on Γ, as almost everywhere with respect to the arc length measure.

We need simple descriptions of the orthogonal complements $L^2(\Omega) \ominus L^2_a(\Omega)$, and respectively $L^2(\partial\Omega, d\omega) \ominus H^2(\Omega)$. These are classical results and, for completeness, we only recall the terminology and sketch their proofs. For details we refer to the monographs [1, 9]. For a bounded planar domain Ω we denote by $L^2(\Omega)$ the Lebesgue space with respect to the area measure supported on Ω; the corresponding norm will be denoted by

$$\|f\|_0 := \langle f, f\rangle^{\frac{1}{2}} = \left(\int_\Omega |f|^2 dA\right)^{\frac{1}{2}}.$$

The Sobolev space $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ (C^∞-smooth functions with compact support in Ω), in the norm defined by

$$\|\varphi\|_1^2 = \|\varphi\|_0^2 + \|\overline{\partial}\varphi\|_0^2 .$$

The space $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$ with respect to the non-degenerated sesquilinear form induced by $L^2(\Omega)$. According to Cauchy–Pompeiu's formula, for a function $\varphi \in C_0^\infty(\Omega)$ we have

$$\varphi(z) = \frac{-1}{\pi} \int_\Omega \frac{\overline{\partial}\varphi(w)}{w-z} dA(w)\,, \quad z \in \Omega\,,$$

whence, due to the integrability of the Cauchy kernel, there exists a constant C depending only on Ω, with the property

$$\|\varphi\|_1 \le C \|\overline{\partial}\varphi\|_0\,.$$

With the above abstract definition of the negative Sobolev space, it is easy to prove that the operator (defined in the sense of distributions) $\overline{\partial} : L^2(\Omega) \longrightarrow H^{-1}(\Omega)$ is onto. Indeed, if $f \in H^{-1}(\Omega)$, then there exists a positive constant K, such that, for every test function $\varphi \in C_0^\infty(\Omega)$ one has

$$|\langle f, \varphi \rangle| \le K\|\varphi\|_1 \le CK\|\overline{\partial}\varphi\|_0\,.$$

According to Hahn–Banach Theorem and Riesz Lemma, there exists an element $u \in L^2(\Omega)$ satisfying

$$\langle f, \varphi \rangle = -\langle u, \overline{\partial}\varphi \rangle\,, \quad \varphi \in C_0^\infty(\Omega)\,.$$

Thus $\partial u = f$, and similarly one proves, by complex conjugation, that there exists $v \in L^2(\Omega)$, such that $\overline{\partial} v = f$. The next result, known among function theorists as *Havin's Lemma* [10], describes the orthogonal complement to the Bergman space.

Lemma 1. *Let Ω be a bounded planar domain. Then,*

$$L^2(\Omega) \ominus L_a^2(\Omega) = \partial H_0^1(\Omega)\,.$$

Proof. We only need to remark that the dual of the operator $\overline{\partial}$ is ∂ and hence

$$L_a^2(\Omega)^\perp = \ker\left(\overline{\partial} : L^2(\Omega) \longrightarrow H^{-1}(\Omega)\right)^\perp = \operatorname{ran}\left(\partial : H_0^1(\Omega) \longrightarrow L^2(\Omega)\right). \qquad \square$$

Similarly, we can describe the orthogonal complement to the Hardy space.

Lemma 2. *Assume that Γ is C^2-smooth. A function $f \in L^2(\Gamma, d\omega) \ominus H^2(\Omega)$ extends anti-analytically inside Ω and vanishes at z_0.*

Stated differently, $L^2(\Gamma, d\omega) = H^2(\Omega) \oplus \overline{H_0^2(\Omega)}$, where the latter symbol stands for the complex conjugate of the Hardy space, with the additional assumption that its elements vanish at z_0. A direct proof of Lemma 2 can be obtained by pushing forward such a decomposition from the disk, via a conformal mapping.

3. Main results

Let Ω be a bounded simply-connected planar domain. Then, there exists an orthonormal sequence $\{p_n\}_{n=0}^{\infty}$ of $L_a^2(\Omega)$, consisting of polynomials arranged by their degree $\deg p_n = n, n \geq 0$. These are the Bergman orthogonal polynomials of (or associated to) Ω. Under fairly weak regularity assumptions on the boundary of the domain Ω, the polynomials are dense in $L_a^2(\Omega)$ (or equivalently the Bergman polynomials form a complete orthonormal system of $L_a^2(\Omega)$); see the precise statement in [1, 11]. We tacitly adopt this assumptions.

The multiplication operator T_z by the complex variable (sometimes called the *Bergman shift*), $(T_z f)(z) = zf(z)$, is linear and bounded on the Hilbert space $L_a^2(\Omega)$. It can be represented, with respect to the basis $\{p_n\}_{n=0}^{\infty}$ by an infinite matrix $(a_{m,n})$, $\;a_{m,n} = \langle zp_n, p_m\rangle$:

$$zp_n(z) = \sum_{k=0}^{\infty} a_{k,n} p_k(z)\,,$$

satisfying

$$\sum_{k=0}^{\infty} |a_{k,n}|^2 < \infty\,, \quad n \geq 0\,.$$

In the approximation theory literature this is called the associated *Hessenberg matrix.* Orthogonal polynomials on the real line produce in this way (three-diagonal) Jacobi matrices.

Definition. The Bergman orthogonal polynomials of a bounded planar domain Ω satisfy a *finite-term recurrence relation* if, for every fixed $k \geq 0$, there exists an $N(k) \geq 0$, such that

$$a_{k,n} = \langle zp_n, p_k\rangle = 0\,, \quad n \geq N(k)\,.$$

The above definition represents a relaxation of the following standard terminology. For a fixed positive integer d, the orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a $(d+1)$-*term recurrence relation* if

$$zp_n(z) = a_{n+1,n}p_{n+1}(z) + a_{n,n}p_n(z) + \cdots + a_{n-d+1,n}p_{n-d+1}(z)\,, \quad n \geq d-1\,.$$

In this case the adjoint T_z^* of the Bergman shift, and its multiples, increase the degree of a polynomial p by a controlled value:

$$\deg\left[(T_z^*)^m p\right] \leq m(d-1) + \deg p\,. \tag{1}$$

The next technical result isolates a minimal necessary condition from the finite-term recurrence assumption.

Proposition 1. *Let Ω be a bounded simply-connected planar domain, such that the polynomials are dense in $L_a^2(\Omega)$. Assume that the matrix of the Bergman shift T_z has only finitely many non-zero entries on its first row* (*i.e.*, $a_{0,n} = 0,\; n \geq d \geq 2$). *Then Ω is bounded by an algebraic curve of equation $x^2 + y^2 = H(x, y)$, where H is a harmonic polynomial of degree at most d.*

Proof. By assumption, the first column of the adjoint operator T_z^* has non-zero entries only up to the index $d-1$. Thus,

$$T_z^* 1 = q(z)\,,$$

where q is a polynomial of degree less than d. Therefore,

$$\overline{z} = q(z) + h(z)\,,$$

where $h \in L^2(\Omega) \ominus L^2_a(\Omega)$. Let $Q(z)$ be a polynomial satisfying $Q' = q$. According to Lemma 1, $h = \partial g$ with $g \in H^1_0(\Omega)$. By integrating against ∂ we find

$$|z|^2 = Q(z) + g(z) + \overline{f(z)}\,,$$

where $f \in L^2_a(\Omega)$. Thus, the weak form of the Dirichlet problem

$$(\Delta u)|_\Omega = 4\,, \quad u \in H^1_0(\Omega)\,,$$

is solved by the function $u(z) = |z|^2 - Q(z) - \overline{f(z)}$. On the other hand, the conjugate function $\overline{u}$ solves the same boundary value problem. By the uniqueness of the weak solution to Dirichlet's problem we infer that

$$Q(z) + \overline{f(z)} = f(z) + \overline{Q(z)}\,, \quad z \in \Omega\,.$$

But the simply-connectedness of Ω implies $f = Q + C$, where C is a real constant. Thus $u(z) - |z|^2 = -2\,\mathrm{Re}\,Q(z) - C$ is a harmonic polynomial of degree at most d. By assumption, the boundary Γ of Ω is regular for the inner Dirichlet problem, and $u \in H^1_0(\Omega)$. We conclude that $v(z) = u(z) - |z|^2$ is a solution to the classical problem

$$(\Delta v)|_\Omega = 0, \quad v|_\Gamma = -|z|^2\,.$$

Therefore,

$$\Gamma \subset \{z : \ u(z) = 0\}\,,$$

and the proof is complete. For terminology and details about weak versus classical solutions to Dirichlet's problem we refer to [9]. □

In the case when $d = 2$, the above proposition, in conjunction with the fact that Γ is bounded, yields:

Corollary 1. *If the Bergman orthogonal polynomials associated to Ω satisfy a three-term recurrence relation, then Ω is an ellipse.*

It is well-known that the ellipse carries a three-term recurrence relation and that the associated polynomials are (suitably normalized) Chebyshev polynomials of the second kind. Next, we adapt the above proof to a more general context, which naturally links our investigation to the study of Dirichlet's problem with algebraic data.

Theorem 1. *Let Ω be a bounded simply-connected planar domain, such that the polynomials are dense in $L^2_a(\Omega)$. Then the corresponding Bergman orthogonal polynomials satisfy a finite-term recurrence relation if and only if Dirichlet's problem for Ω with polynomial data on Γ has a polynomial solution.*

Proof. Fix two positive integers m, n and assume that the matrix attached to T_z^* has only finitely many non-zero entries on each column. Then both $(T_z^*)^m z^{n-1}$ and $(T_z^*)^n z^{m-1}$ are polynomials and arguing as in the proof of Proposition 1, we deduce that there exist polynomials $Q(z)$ and $R(z)$ such that

$$\overline{\zeta}^m \zeta^n = Q(\zeta) + \overline{R(\zeta)}\,, \quad \zeta \in \Gamma\,.$$

The inverse implication is almost identical. Assume that Dirichlet's problem for Ω with data $\overline{z}\frac{z^{n+1}}{n+1}$ has a polynomial solution $u(z) = Q(z) + \overline{R(z)}$, with Q, R complex analytic polynomials. Then Γ is piecewise smooth, as a part of a real algebraic curve. Let $h(z)$ be an analytic function in Ω with finite boundary values on Γ. Then,

$$\int_\Omega \overline{z} z^n \overline{h(z)} dz \wedge d\overline{z} = \int_\Gamma \overline{\zeta} \frac{\zeta^{n+1}}{n+1} \overline{h(\zeta)} d\overline{\zeta} = \int_\Omega Q'(z) \overline{h(z)} dz \wedge d\overline{z}\,,$$

whence

$$T_z^* z^n = Q'(z)\,.$$

□

For the relevance of the polynomial Dirichlet question and related works see [2, 3, 6, 12]. We confine to derive, from the latest known results about the polynomial Dirichlet problem, a single significant corollary, cf. [3, p. 216].

Corollary 2. *Let Ω be a simply-connected bounded planar domain, such that the polynomials are dense in $L_a^2(\Omega)$. Assume that the corresponding Bergman orthogonal polynomials satisfy a finite-term recurrence relation, with*

$$\Gamma \subset \left\{(x, y) \in \mathbb{R}^2 : \psi(x, y) = 0\right\},$$

where ψ is a polynomial with bounded zero set. Then Ω is an ellipse.

Next we focus on a class of planar domains which, apart from the trivial case, do not possess finite-term relations for their Bergman orthogonal polynomials. We call Ω a *quadrature domain*, if it admits a finite-point quadrature identity for all integrable analytic functions in Ω. The number of points in this quadrature rule is called the order of Ω. It turns out that the boundary curve Γ is algebraic and its defining equation $\Gamma = \{z : \ Q(z, \overline{z}) = 0\}$ is given by an irreducible polynomial of degree d in each variable. Moreover, a quadrature domain carries a meromorphic function $S(z)$ (called the Schwarz function) satisfying $S(\zeta) = \overline{\zeta},\ \zeta \in \Gamma$. The poles of $S(z)$ in Ω coincide with the quadrature nodes. For all these facts the reader can consult Shapiro's monograph [24].

Proposition 2. *Assume that the Bergman orthogonal polynomials associated to a quadrature domain Ω satisfy a finite-term recurrence relation. Then Ω is a disk.*

Proof. Let $p(z)$ be the monic polynomial vanishing at the quadrature nodes, so that $p(z)S(z)$ is an analytic function in Ω. We repeat the proof of Proposition 1, remarking that $T_z^* p' = q$ must be by our assumption a polynomial. As before,

$$\overline{z} p'(z) = q(z) + h(z)\,,$$

where $h \in L^2(\Omega) \ominus L^2_a(\Omega)$. Let, as before, Q be an antiderivative of q, i.e., let $Q' = q$. By integrating once with respect to the complex variable we find that

$$S(\zeta)p(\zeta) = Q(\zeta) + \overline{f(\zeta)}\,, \quad \zeta \in \Gamma\,,$$

where $f(z)$ is an analytic function. Thus, there exists a constant C such that

$$S(z)p(z) = Q(z) + C\,, \quad z \in \Omega\,,$$

which implies that $S(z)$ is a rational function. But the disk is the only quadrature domain having a rational Schwarz function; see [24]. (This can also be seen directly from the defining equation $\overline{z}p(z) = Q(z) + C$ of Γ. By the involute nature of the minimal defining polynomial, the degree in z must be equal to that in $\overline{z}$, that is one). □

The proof of Proposition 1 can easily be adapted to the case of the Hardy space $H^2(\Omega)$, see also the proof of Theorem 1 in [4]. This leads to the following result.

Proposition 3. *Let Γ be a C^2-smooth Jordan curve and assume that the matrix associated to the multiplication operator T_z on the Hardy space $H^2(\Omega)$ has only finitely non-zero entries on the first and second rows (i.e., $|a_{0,n}| + |a_{1,n}| = 0,\ n > d \geq 1$). Then Γ is contained in an algebraic curve of equation $x^2 + y^2 = H(x, y)$, where H is a harmonic polynomial of degree at most d.*

Proof. By our assumptions, $T_z^* z = q$, where q is a polynomial of degree at most d. This means, via Lemma 2,

$$\overline{\zeta}\zeta = q(\zeta) + \overline{f(\zeta)}\,, \quad \zeta \in \Gamma\,(a.\,e.\,on\ \Gamma)\,,$$

where $f \in H^2_0(\Omega)$. Arguing as before, we conclude that the real polynomial

$$|z|^2 = 2\,\mathrm{Re}\,q(z) + C$$

vanishes on Γ. □

In particular, if the associated Szegö orthogonal polynomials on Γ satisfy a three-term recurrence relation, then Γ is an ellipse.

An adaptation of the proof of Theorem 1 yields:

Theorem 2. *The Szegö orthogonal polynomials w.r.t. $L^2(\Gamma, d\omega)$ on a C^2-Jordan curve Γ satisfy a finite-term recurrence relation if and only if Dirichlet's problem with polynomial data on Γ has a polynomial solution.*

Again, as in the case of Bergman polynomials, in a good number of cases we can infer that Γ is an ellipse; see Corollary 2.

We conclude by translating the row vanishing conditions in the statements of Propositions 1 and 3 into equivalent terms. An element $f \in L^2_a(\Omega)$ is orthogonal to p_0 if and only if,

$$\int_\Omega f dA = 0\,.$$

Thus, the assumption in Proposition 1 becomes: *the vector space*

$$[z]' = \left\{ f \in L^2_a(\Omega) : \int_\Omega z f dA = 0 \right\} ,$$

contains all p_n *for* n *large enough.* Similarly, the condition needed into the proof of Proposition 3 is equivalent to: *the vector space*

$$[|z|^2]' = \left\{ g \in H^2(\Omega) : \int_\Gamma g|z|^2 d\omega = \int_\Gamma z g d\omega = 0 \right\} ,$$

contains all p_n *for* n *large enough.*

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