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Error Analysis of the Bergman Kernel Method with Singular Basis Functions

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Abstract. Let G be a bounded Jordan domain in the complex plane with a piecewise analytic boundary. We present theoretical estimates and numerical evidence for certain phenomena, regarding the application of the Bergman kernel method with algebraic and pole singular basis functions, for approximating the conformal mapping of G onto the normalized disk. Thereby, we complete the task of providing full theoretical justification of this method.

Keywords. Bergman orthogonal polynomials, numerical conformal mapping, Bergman kernel method, singular basis function.

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1. Introduction

Let G be a bounded, simply-connected domain in the complex plane $\mathbb{C}$ whose boundary $\Gamma := \partial G$ is a Jordan curve and let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ denote the complement of $\overline{G}$ with respect to the extended complex plane. Fix $z_0 \in G$ and let f_0 denote the conformal map of G onto the disk $D(0, r_0) := \{z \colon |z| < r_0\}$, normalized by the conditions $f_0(z_0) = 0$ and $f_0'(z_0) = 1$. The quantity $r_0 := r_0(G, z_0)$ is called the conformal radius of G with respect to z_0.

For the inner product

$$\langle f, g \rangle := \int_G f(z)\, \overline{g(z)}\, dA(z), \tag{1}$$

where dA denotes the differential of the area measure on $\mathbb{C}$, we consider the Hilbert space

$$L_a^2(G) := \big\{ f \colon f \text{ analytic in } G,\ \langle f, f \rangle < \infty \big\}, \tag{2}$$

with corresponding norm $\|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}$.

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Let $K(\cdot, z_0)$ denote the *Bergman kernel function* of G with respect to z_0. This is the unique function of $L^2_a(G)$ satisfying the reproducing property

$$\langle g, K(\cdot, z_0)\rangle = g(z_0) \qquad \text{for all } g \in L^2_a(G). \tag{3}$$

It follows from (3) that the kernel $K(\cdot, z_0)$ is related to the mapping function f_0 by means of

$$f_0'(z) = \frac{K(z, z_0)}{K(z_0, z_0)} \qquad \text{and} \qquad f_0(z) = \frac{1}{K(z_0, z_0)} \int_{z_0}^{z} K(\zeta, z_0) d\zeta, \tag{4}$$

see e.g. [3, p. 33]. These yield the two relations,

$$K(z, z_0) = \frac{1}{\pi {r_0}^2} f_0'(z) \qquad \text{and} \qquad r_0 = \frac{1}{\sqrt{\pi K(z_0, z_0)}}. \tag{5}$$

Now, let $\{P_n\}_{n=0}^{\infty}$ denote the sequence of the *Bergman polynomials* of G. This is defined as the sequence of polynomials

$$P_n(z) = \lambda_n z^n + \cdots, \qquad \lambda_n > 0,\, n = 0, 1, 2, \ldots,$$

that are orthonormal with respect to the inner product (1), i.e.

$$\int_G P_m(z)\overline{P_n(z)}\, dA(z) = \delta_{m,n}.$$

The Bergman polynomials form a complete orthonormal system in $L^2_a(G)$. Therefore, in view of the reproducing property (3),

$$K(z, z_0) = \sum_{j=0}^{\infty} \overline{P_j(z_0)}\, P_j(z), \tag{6}$$

locally uniformly with respect to $z \in G$.

The *Bergman kernel method* (BKM) is an orthonormalization method for computing approximations to the conformal map $f_0(z)$. It is based on the fact that the kernel $K(z, z_0)$ is given explicitly in terms of the Bergman polynomials $\{P_n(z)\}_{n=0}^{\infty}$. Thus, the partial sums of the Fourier series expansion of $K(z, z_0)$ are given by

$$K_n(z, z_0) := \sum_{j=0}^{n} \overline{P_j(z_0)}\, P_j(z), \qquad n \in \mathbb{N}. \tag{7}$$

The polynomials $\{K_n(z, z_0)\}_{n=0}^{\infty}$ are the so-called *kernel polynomials* of G, with respect to z_0. They provide the best $L^2(G)$-approximation to $K(\cdot, z_0)$ in the space $\mathbb{P}_n$ of complex polynomials of degree at most n.

In accordance with (4), the n-th BKM approximation to f_0 is given by

$$\pi_n(z) := \frac{1}{K_{n-1}(z_0, z_0)} \int_{z_0}^{z} K_{n-1}(\zeta, z_0)\, d\zeta, \qquad n \in \mathbb{N}. \tag{8}$$

This defines the sequence $\{\pi_n\}_{n=1}^{\infty}$ of the *Bieberbach polynomials* of G, with respect to z_0. The polynomial π_n solves the following minimal problem: Let

$$\mathbb{P}_n^* := \{p\colon p \in \mathbb{P}_n, \text{ with } p(z_0) = 0 \text{ and } p'(z_0) = 1\}.$$

Then, for each $n \in \mathbb{N}$, *the polynomial* π_n *minimizes uniquely the two norms* $\|f_0' - p'\|_{L^2(G)}$ *and* $\|p'\|_{L^2(G)}$ *over all* $p \in \mathbb{P}_n^*$; see e.g. [2, Kap. III, Sec. 1].

Regarding the convergence of the method, we note that in cases when f_0 has an analytic continuation across Γ, this is a consequence of Walsh's theory of maximal convergence [22, Sec. 4.7, Sec. 5.3]. In order to be more specific, let Φ denote the conformal map of Ω onto $\Delta := \{w\colon |w| > 1\}$, normalized so that near infinity,

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots, \qquad \gamma > 0. \tag{9}$$

Note that $\gamma = 1/\operatorname{cap}(\Gamma)$, where $\operatorname{cap}(\Gamma)$ denotes the (logarithmic) capacity of Γ. Then,

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} = \mathcal{O}\left(\frac{1}{R^n}\right), \tag{10}$$

holds for any $1 < R < |\Phi(z_1)|$, but for no $R > |\Phi(z_1)|$, where z_1 denotes the nearest singularity of f_0 in Ω; see also [3, Ch. I]. (We use $\|\cdot\|_{L^\infty(\overline{G})}$ to denote the sup-norm on $\overline{G}$.)

In cases when Γ is piecewise analytic and f_0 has singularities on Γ, then Levin, Papamichael and Siderides were the first to observe in [11] that the error (10) depends on the boundary singularities of the mapping function f_0 on Γ, and also on the singularities of the extension of f_0 across the segments of Γ into Ω. Accordingly, in order to improve the numerical performance of the BKM, they extended the method by orthonormalizing a system of basis functions consisting of monomials, as in the BKM, and also of functions that reflect the dominant singularities of f_0 on Γ and in Ω. This extension is known as BKM/AB (AB stands for *augmented basis*). The BKM/AB was used subsequently in [15, 16].

The most precise results regarding the convergence of the BKM are due to D. Gaier [6]. In particular, under the assumption that Γ is piecewise analytic without cusps, Gaier derived the estimate

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} = \mathcal{O}(\log n)\frac{1}{n^s}, \tag{11}$$

where $s := \lambda/(2-\lambda)$ and $\lambda\pi$, $0 < \lambda < 2$, denotes the smallest exterior angle where two analytic arcs of Γ meet. Regarding sharpness of the estimate (11), it was shown in [5, Thm. 4] that there are cases where the exponent s cannot be replaced by a smaller number. However, the factor $\log n$ can be replaced by $\sqrt{\log n}$, see [1] and [12, Rem. 3.1]. A lower estimate of the form

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} \geq c\,\frac{1}{n^s}, \tag{12}$$

provided that $1/(2-\lambda)$ is not a positive integer, where c is a constant that does not depend on n, was established in [12, Thm. 3.2] by Maymeskul, Saff and the second author.

The theoretical justification of the BKM/AB with basis function that reflect the corner singularities of f_0 was given in [12], by means of sharp estimates for the associated BKM/AB errors. The purpose of the present paper is to derive theoretical results that justify the use of basis functions that reflect (a) pole singularities of f_0 and (b) both corner and pole singularities of f_0. More specifically, we derive upper and lower estimates for the BKM/AB errors in the case (a), and upper estimates for the BKM/AB errors in the case (b). In doing so, we complete the task that was put forward by Yu. E. Khokhlov, reviewer of the introductory paper [11] of the BKM/AB in the Mathematical Reviews, who concluded that: "*A proof of the convergence of the numerical method given and an investigation of its convergence rate are lacking, so the results obtained are of a heuristic nature*".

The paper is organized as follows: In Section 2 we set up the notation and recall the BKM/AB. Section 3 is devoted to the study of the various BKM and BKM/AB errors, in cases when f_0 has an analytic continuation across Γ, hence only basis functions reflecting poles are used in the BKM/AB. In Section 4, we consider the case when both corner and pole basis functions are included in BKM/AB. Finally, in Section 5, we present numerical computations that illustrate the theory of Sections 3 and 4.

2. The Bergman kernel method with singular basis functions

2.1. Corner singularities. Throughout this section we assume that the boundary Γ of G consists of N analytic arcs that meet at corner points τ_k, $k=1,2,\ldots,N$, where they form interior angles $\alpha_k\pi$, $0<\alpha_k<2$. Then, we have the following asymptotic expansions for f_0, valid near τ_k:

(i) If α_k is irrational, then

$$f_0(z)=f_0(\tau_k)+\sum_{p,q}B_{p,q}(z-\tau_k)^{p+q/\alpha_k}, \tag{13}$$

where p and q run over all integers $p\geq 0$, $q\geq 1$ and $B_{0,1}\neq 0$.

(ii) If $\alpha_k=a/b$, with a and b relative prime numbers, then

$$f_0(z)=f_0(\tau_k)+\sum_{p,q,m}B_{p,q,m}(z-\tau_k)^{p+q/\alpha_k}(\log(z-\tau_k))^m, \tag{14}$$

where p,q and m run over all integers $p\geq 0$, $1\leq q\leq a$, $1\leq m\leq p/b$ and $B_{0,1,0}\neq 0$.

(iii) If τ_k is formed by two straight-line segments, then

$$f_0(z) = f_0(\tau_k) + \sum_{l=1}^{\infty} B_l (z-\tau_k)^{l/\alpha_k}, \tag{15}$$

where $B_1 \neq 0$. Furthermore, (13) holds in the case when τ_k is formed by two circular arcs, or a straight-line and a circular arc.

In the above, (i) and (ii) are due to Lehman [9], while (iii) emerges easily from the reflection principle; see also [6, Sec. 2.1] and [14, pp. 6–7].

It follows from (iii) that if G is a half-disk or a rectangle, then f_0 has a Taylor series expansion valid around each corner, and thus an analytic continuation across Γ into Ω. In this case, the only singularities of f_0 are simple poles in Ω. This shows that the study of the BKM/AB, even with only pole basis functions is important in the applications.

For simplicity in the exposition, we shall assume throughout this paper that *no logarithmic terms occur* in the asymptotic expansion of f_0 near the corner τ_k, $k = 1, 2, \ldots, N$. This, for example, will be the case in the expansions (13) and (15) above. Nevertheless, our method of study can be adjusted to cover logarithmic singularities as well.

Let M denote the number of corners of Γ for which α_k is not of the special form $1/m$, $m \in \mathbb{N}$. When we present results for corner singularities we shall assume that $M \geq 1$. We index such corners by τ_k, $k = 1, \ldots, M$. That is, if $N > M$, then the mapping function f_0 has an analytic continuation in some neighborhood of the corner τ_N.

For $k = 1, \ldots, M$, we denote by $\{\gamma_j^{(k)}\}_{j=1}^{\infty}$ the increasing arrangement of the possible powers $p + q/\alpha_k$ of $(z-\tau_k)$ that appear in the asymptotic expansion of $f_0(z)$ near τ_k. In particular, if τ_k is formed by two straight-line segments, then $\gamma_j^{(k)} = j/\alpha_k$, $j = 1, 2, \ldots$. Also, if α_k is irrational, or the corner τ_k is formed by two circular arcs, then

$$\gamma_1^{(k)} = \frac{1}{\alpha_k};$$

$$\gamma_2^{(k)} = \frac{1}{\alpha_k} + \min\left(\frac{1}{\alpha_k}, 1\right);$$

$$\gamma_3^{(k)} = \begin{cases} \dfrac{1}{\alpha_k} + 2, & 0 < \alpha_k < \dfrac{1}{2}, \\[2mm] \dfrac{2}{\alpha_k}, & \dfrac{1}{2} < \alpha_k < 1, \\[2mm] \dfrac{1}{\alpha_k} + 1, & 1 < \alpha_k < 2; \end{cases}$$

$$\vdots$$

Remark 2.1. Under the assumption regarding the no-appearance of logarithmic terms, the asymptotic expansion near τ_k, $k=1,2,\dots,M$, can be written in the form

$$f_0(z)=\sum_{j=0}^{\infty} a_j^{(k)}(z-\tau_k)^{\gamma_j^{(k)}}, \tag{16}$$

where, $\gamma_0^{(k)}:=0$ and $a_1^{(k)}\neq 0$. Note that, we always have $\gamma_1^{(k)}>1/2$, and since τ_k is not a special corner, $\gamma_1^{(k)}\notin\mathbb{N}$. Therefore $(z-\tau_k)^{\gamma_1^{(k)}}$ has an algebraic singularity at τ_k. However, when α_k is rational, it is possible that $\gamma_j^{(k)}\in\mathbb{N}$, for indices $j\geq 2$, so that $(z-\tau_k)^{\gamma_j^{(k)}}$ is analytic at τ_k.

2.2. Pole singularities. Since $f_0(z_0)=0$, $z_0\in G$, it follows from the reflection principle for analytic arcs that the extension of f_0 across any segment constituting Γ would have a pole or a pole-type singularity at the reflected images of z_0. For example, if Γ consists explicitly from straight-line segments and/or circular arcs, then f_0 has a simple pole (due to the univalency of f_0) at every mirror image of z_0 (with respect to the straight-lines) and at every geometric inverse of z_0 (with respect to the circular arcs), that lies in Ω. More generally, f_0 may have at points $z_j\in\Omega$ a pole or a poly-type singularity of the form

$$(z-z_j)^{-k_j/m_j},\qquad k_j,m_j\in\mathbb{N}. \tag{17}$$

According to [17, Sec. 5.1], the following three special cases occur frequently in the applications:

(i) $k_j=m_j=1$. In this case, f_0 has a simple pole at z_j.
(ii) $k_j=2$, $m_j=1$. In this case, f_0 has a double pole at z_j.
(iii) $k_j=1$, $m_j=2$. In this case, f_0 has a rational pole singularity at z_j.

In order to describe the BKM/AB, we assume that the nearest singularities of f_0 in Ω are poles or rational poles, of the form (17) at points z_j, $j=1,2,\dots\kappa$, where $|\Phi(z_1)|\leq|\Phi(z_2)|\leq\dots\leq|\Phi(z_\kappa)|$ and that the other singularities of f_0 in Ω occur at points $z_{\kappa+1},z_{\kappa+2},\dots$, where $|\Phi(z_\kappa)|<|\Phi(z_{\kappa+1})|\leq|\Phi(z_{\kappa+2})|\leq\dots$.

2.3. BKM/AB. Using the above notation, the BKM/AB with n monomials, κ poles and p_k corner singularities at each (non-special) corner τ_k, $k=1,2\dots,M$, can be summarized as follows:

(i) Start with the augmented system $\{\eta_j\}$ consisting of:
(i.1) the nearest poles or rational poles, i.e. for $j=1,2\dots\kappa$,

$$\eta_j(z)=\left[\left(\frac{1}{z-z_j}\right)^{k_j/m_j}\right]'; \tag{18}$$

(i.2) the dominant $r_M := \sum_{k=1}^{M} p_k$ algebraic singular functions, i.e. for each non-special corner τ_k, $k = 1, 2, \dots, M$,

$$\eta_{\kappa+j}(z) = [(z-\tau_k)^{\gamma_j^{(k)}}]', \qquad j = 1, 2, \dots p_k; \tag{19}$$

(i.3) the monomials

$$\eta_{\kappa+r_M+j}(z) = (z^j)', \qquad j = 1, 2, \dots, n. \tag{20}$$

(As noted in Remark 2.1, it might be possible that $\gamma_j^{(k)} \in \mathbb{N}$. If this happens, we avoid redundancy in the basis by omitting such $\gamma_j^{(k)}$.)

(ii) Orthonormalize $\{\eta_j\}$, by means of the Gram-Schmidt process to produce the orthonormal set $\{\widetilde{P}_j\}$, where

$$\widetilde{P}_j(z) = \sum_{i=1}^{j} b_{j,i}\, \eta_i(z), \qquad b_{j,j} > 0. \tag{21}$$

(iii) Approximate $K(z, z_0)$ by its finite Fourier expansion with respect to $\{\widetilde{P}_j\}$:

$$\widetilde{K}_n(z, z_0) := \sum_{j=1}^{\kappa+r_M+n} \overline{\widetilde{P}_j(z_0)}\widetilde{P}_j(z) = \sum_{j=1}^{\kappa+r_M+n} d_{n,j}\eta_j(z). \tag{22}$$

(iv) Approximate $f_0(z)$ by

$$\widetilde{\pi}_{n+1}(z) := \frac{1}{\widetilde{K}_n(z_0, z_0)} \int_{z_0}^{z} \widetilde{K}_n(\zeta, z_0)\, d\zeta = \sum_{j=1}^{\kappa+r_M+n} c_{n,j}\mu_j(z), \tag{23}$$

where

$$\mu_j(z) := \int_{z_0}^{z} \eta_j(\zeta)\, d\zeta. \tag{24}$$

We call the functions $\{\widetilde{P}_j\}$ the *augmented Bergman polynomials* of G, with respect to $\{\eta_j\}$, and the functions $\{\widetilde{\pi}_n\}$ the *augmented Bieberbach polynomials* over the system $\{\mu_j\}$. Clearly, $\widetilde{\pi}_n(z_0) = 0$ and $\widetilde{\pi}_n'(z_0) = 1$, $n \in \mathbb{N}$. Note that $\{\widetilde{P}_j\}_{j=1}^{\infty}$ forms a complete orthonormal system in $L_a^2(G)$. Consequently,

$$K(z, z_0) = \sum_{j=1}^{\infty} \overline{\widetilde{P}_j(z_0)}\widetilde{P}_j(z), \tag{25}$$

locally uniformly with respect to $z \in G$, cf. (6).

We conclude this section by presenting a result which shows that the two errors $\|f_0' - \widetilde{\pi}_{n+1}'\|_{L^2(G)}$ and $\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}$ are of the same order. This fact will be used below in Sections 3 and 4.

In what follows we denote by c, c_1, $c_2, \dots,$ constants that are independent of n. For quantities $A > 0$, $B > 0$, we use the notation $A \preceq B$ (inequality with respect to the order) if $A \le cB$. The expression $A \asymp B$ means that $A \preceq B$ and $B \preceq A$ simultaneously.

Lemma 2.1. *We have*

$$\|f_0' - \widetilde{\pi}_{n+1}'\|_{L^2(G)} \asymp \|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}. \tag{26}$$

Proof. We set $m := \kappa + r_M + n$ and note that (5), (22)–(25), imply:

$$f_0'(z) - \widetilde{\pi}_{n+1}'(z) = \pi {r_0}^2 \sum_{j=1}^{\infty} \overline{\widetilde{P}_j(z_0)} \widetilde{P}_j(z) - \left(\sum_{j=1}^{m} |\widetilde{P}_j(z_0)|^2 \right)^{-1} \sum_{j=1}^{m} \overline{\widetilde{P}_j(z_0)} \widetilde{P}_j(z)$$
$$= \sum_{j=1}^{m} \left[\pi {r_0}^2 - \left(\sum_{j=1}^{m} |\widetilde{P}_j(z_0)|^2 \right)^{-1} \right] \overline{\widetilde{P}_j(z_0)} \widetilde{P}_j(z)$$
$$+ \pi {r_0}^2 \sum_{j=m+1}^{\infty} \overline{\widetilde{P}_j(z_0)} \widetilde{P}_j(z).$$

Therefore, by using the orthonormality of $\widetilde{P}_j$ we see that,

$$\|f_0' - \widetilde{\pi}_{n+1}'\|_{L^2(G)}^2 = \sum_{j=1}^{m} \left[\pi {r_0}^2 - \frac{1}{\widetilde{K}_n(z_0, z_0)} \right]^2 |\widetilde{P}_j(z_0)|^2$$
$$+ \left(\pi {r_0}^2\right)^2 \sum_{j=m+1}^{\infty} |\widetilde{P}_j(z_0)|^2.$$

Now, using (5) once more, we obtain, after some trivial calculation

$$\|f_0' - \widetilde{\pi}_{n+1}'\|_{L^2(G)}^2 = \left[K(z_0, z_0) - \widetilde{K}_n(z_0, z_0) \right] \left[K(z_0, z_0)\, \widetilde{K}_n(z_0, z_0) \right]^{-1}.$$

This and (3), with $g(\cdot) = K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)$, leads to

$$\|f_0' - \widetilde{\pi}_{n+1}'\|_{L^2(G)}^2 = \frac{\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}{}^2}{K(z_0, z_0)\, \widetilde{K}_n(z_0, z_0)} \tag{27}$$

and the result (26) follows from the set of the obvious inequalities,

$$|\widetilde{P}_1(z_0)| = \widetilde{K}_1(z_0, z_0) \le \widetilde{K}_n(z_0, z_0) \le K(z_0, z_0) = \frac{1}{\pi {r_0}^2},$$

with constants depending on r_0 and $|\widetilde{P}_1(z_0)|$ only. ∎

Remark 2.2. It is clear from the proof that the result of Lemma 2.1 holds true for *any* complete orthonormal system. We note that for the system $\{\widetilde{P}_j\}_{j=1}^{\infty}$ to be complete it suffices that Γ is a bounded Jordan curve. In particular, (26) holds with π_{n+1} and K_n in the place of $\widetilde{\pi}_{n+1}$ and $\widetilde{K}_n$.

3. BKM/AB with pole singularities

In this section we study the BKM and BKM/AB errors $\|f_0 - \pi_n\|_{L^\infty(\overline{G})}$ and $\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})}$, under the assumption that f_0 *has an analytic continuation across* Γ in Ω and its only singularities are poles, or rational poles, of the type (17). More precisely, we refine the classical result (10) for the BKM error, and at the same time we obtain a lower estimate for it. Furthermore, we establish upper and lower estimates for the BKM/AB error. The lower estimates and the refinement are obtained by exploiting the assumption regarding the singularities of f_0 and by using certain important results of E. B. Saff on polynomial interpolation of meromorphic functions [18]. Since the results of [18] were established for domains with smooth boundaries, we show in the next lemma that they hold true for domains with corners.

In order to do so, we use the *Faber polynomials* $\{F_n\}_{n=0}^{\infty}$ of $\overline{G}$. We recall that $F_n(z)$ is defined as the polynomial part of the Laurent series expansion of Φ^n at infinity, i.e.

$$F_n(z) - \Phi^n(z) = \mathcal{O}\left(\frac{1}{z}\right), \qquad \text{as } z \to \infty. \tag{28}$$

This, in view of (9), gives $F_n \in \mathbb{P}_n$ and

$$F_n(z) = \gamma^n z^n + \cdots .$$

Let L_R, $R \geq 1$, denote the level curve of index R of Φ, i.e.

$$L_R := \{z \colon |\Phi(z)| = R\}, \tag{29}$$

so that $L_1 \equiv \Gamma$. Note that L_R, for $R > 1$, is an analytic Jordan curve. We use G_R to denote its interior, i.e. $G_R := \operatorname{int}(L_R)$. The following result gives the exact rate of convergence of the minimum uniform error in approximating meromorphic functions by polynomials.

Lemma 3.1. *Assume that the boundary* Γ *of* G *is piecewise Dini-smooth and consider a function* f *which is analytic on* $\overline{G_\varrho}$*, for some* $\varrho > 1$*, apart from a finite number of poles on* L_ϱ*. Let* m *denote the highest order of the poles of* f *on* L_ϱ*. Then,*

$$\inf_{p \in \mathbb{P}_n} \|f - p\|_{L^\infty(\overline{G})} \asymp \frac{n^{m-1}}{\varrho^n}. \tag{30}$$

A curve Γ is piecewise Dini-smooth if it consists of a finite number of Dini-smooth arcs. An arc $z = z(s)$, where $s \in [a,b]$ stands for the arclength, is called *Dini-smooth* if $z'(s)$ is continuous on $[a,b]$, and if $z'(s)$ has a modulus of continuity ω which satisfies

$$\int_0^\alpha \frac{\omega(t)}{t}\, dt < \infty,$$

for some $\alpha > 0$. We note, in particular, that a piecewise Dini-smooth curve may have corners or cusps and that a piecewise analytic Jordan curve is also piecewise Dini-smooth.

Proof. We recall the following two facts regarding Faber polynomials:

(i) For any r, R, with $1 < r < R$, it is true that

$$F_n(z) = \Phi^n(z)\left(1 + \mathcal{O}\left(\frac{r^n}{R^n}\right)\right), \qquad z \in L_R, \tag{31}$$

see e.g. [21, p. 43].

(ii) Under the assumption on Γ, the Faber polynomials are uniformly bounded on $\overline{G}$ (see [7]), i.e.

$$\|F_n\|_{L^\infty(\overline{G})} \le c(\Gamma), \qquad n \in \mathbb{N}, \tag{32}$$

where $c(\Gamma)$ is a positive constant that depends on Γ only.

Observe that (31) implies that the sequence $\{F_n(z)\}_{n=1}^\infty$ has no limit point of zeros exterior to $\overline{G}$. Also, from (31) and (32) we have for $z \in \overline{G}$ and $t \in L_\varrho$ that,

$$\frac{|F_n(z)|}{|F_n(t)|} \le \frac{c_1(\Gamma)}{\varrho^n}, \qquad n \in \mathbb{N}.$$

Now, following the proof of [18, Thm. 2] and using the sequence of the Faber polynomials $\{F_n\}$ in the place of $\{\omega_n\}$, we conclude that there exist polynomials $\{p_n\}_{n=1}^\infty$, such that

$$\|f - p_n\|_{L^\infty(\overline{G})} \le c_2(\Gamma)\frac{n^{m-1}}{\varrho^n}, \quad n \in \mathbb{N},$$

see also [19, p. 399]. This yields the upper bound in (30). The lower bound follows at once from [18, Thm. 10], by observing that Ω is simply-connected and hence its Green function with pole at infinity has no critical points. ■

The following result is the so-called Andrievskiĭ's Lemma for polynomials and rational polynomials. Its proof, for bounded Jordan domains such that the inverse conformal map $g\colon \mathbb{D} \to G$ satisfies a Lipschitz condition on $\overline{\mathbb{D}}$, can be found in [4]. This condition is certainly satisfied by the type of domains considered below.

Lemma 3.2. *Assume that Γ is piecewise analytic without cusps. Then:*

(i) *For any $P_n \in \mathbb{P}_n$, with $P_n(z_0) = 0$, it is true that*

$$\|P_n\|_{L^\infty(\overline{G})} \preceq \sqrt{\log n}\,\|P_n'\|_{L^2(G)}, \qquad n \ge 2. \tag{33}$$

(ii) *For any $P_n \in \mathbb{P}_n$, with $P_n(z_0) = 0$, and q a fixed polynomial with no zeros on $\overline{G}$, it is true that*

$$\left\|\frac{P_n}{q}\right\|_{L^\infty(\overline{G})} \preceq \sqrt{\log n}\,\left\|\left(\frac{P_n}{q}\right)'\right\|_{L^2(G)}, \qquad n \ge 2. \tag{34}$$

3.1. BKM. The next theorem complements the classical result (10) of Walsh, in the sense that it provides a lower estimate and, in addition, uses the precise $\varrho = |\Phi(z_1)|$ in the denominator, instead of any R, with $1 < R < \varrho$. This is done by utilizing extra information on the nature of the singularities of f_0 in Ω.

Theorem 3.1. *Assume that Γ is piecewise analytic without cusps. Assume further that the conformal map f_0 has an analytic continuation across Γ, such that f_0 is analytic on $\overline{G_\varrho}$, for some $\varrho > 1$, apart from a finite number of poles on L_ϱ. Let m denote the highest order of the poles of f_0 on L_ϱ. Then,*

$$\frac{n^{m-1}}{\varrho^n} \preceq \|f_0 - \pi_n\|_{L^\infty(\overline{G})} \preceq \frac{n^m\sqrt{\log n}}{\varrho^n}, \qquad n \geq 2. \tag{35}$$

Proof. We observe first that the kernel $K(z, z_0)$ shares the same analytic properties with f_0 on $\overline{G_\varrho}$, apart from an unit increase on the order of its poles on L_ϱ. Therefore, using Lemma 3.1 with $f \equiv K(\cdot, z_0)$, we conclude that

$$\|K(\cdot, z_0) - p_n\|_{L^\infty(\overline{G})} \preceq \frac{n^m}{\varrho^n}, \quad n \in \mathbb{N}, \tag{36}$$

for some sequence of polynomials $\{p_n\}_{n=1}^\infty$. Since the $L^2(G)$-norm is dominated by the $L^\infty(\overline{G})$-norm, (36) leads to the estimate

$$\|K(\cdot, z_0) - p_n\|_{L^2(G)} \preceq \frac{n^m}{\varrho^n}.$$

Then, the minimum property of the kernel polynomials implies that

$$\|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)} \preceq \frac{n^m}{\varrho^n}, \tag{37}$$

which, in conjunction with Remark 2.2, yields the estimate

$$\|f_0' - \pi_n'\|_{L^2(G)} \preceq \frac{n^m}{\varrho^n}.$$

Now, we use Andrievskiĭ's Lemma 3.2 (i) and employ the method of Andrievskiĭ and Simonenko, see e.g. [5, Sec. 2.1]. This method enables the transition from an upper bound of the error $\|f_0' - \pi_n'\|_{L^2(G)}$ to a similar bound for the error $\|f_0 - \pi_n\|_{L^\infty(\overline{G})}$, with the extra cost of a $\sqrt{\log n}$ factor, and leads to the upper estimate in (35). The lower estimate follows immediately from Lemma 3.1. ■

The following pointwise estimate is useful in the study of the distribution of the zeros of the Bergman polynomials; see e.g. [10, 13, 19, 8].

Corollary 3.1. *With the assumptions of Theorem 3.1, it is true that*

$$|P_n(z_0)| \preceq \frac{n^m}{\varrho^n}, \qquad n \in \mathbb{N},\ z_0 \in G. \tag{38}$$

Proof. The result emerges easily from (37), using the reproducing property of $K(\cdot, z_0)$, the fact that P_{n+1} is orthogonal to any polynomial in $\mathbb{P}_n$, and the Cauchy-Schwarz inequality:

$$\begin{aligned}|P_{n+1}(z_0)| &= |\langle P_{n+1}, K(\cdot, z_0)\rangle| = |\langle P_{n+1}, K(\cdot, z_0) - K_n(\cdot, z_0)\rangle| \\ &\le \|P_{n+1}\|_{L^2(G)} \|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)} \\ &= \|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)}.\end{aligned}$$

■

3.2. BKM/AB with pole singularities. We exploit now the specific assumptions on the singularities of the analytic extension of f_0 studied in Section 2.2. More precisely, the assumption that the nearest singularities of f_0 are κ poles, each one of order k_j at z_j, $j = 1, 2, \dots \kappa$, where

$$|\Phi(z_1)| \le |\Phi(z_2)| \le \cdots \le |\Phi(z_\kappa)|,$$

and that the other singularities of f_0 occur at points $z_{\kappa+1}, z_{\kappa+2}, \dots$, where

$$|\Phi(z_\kappa)| < |\Phi(z_{\kappa+1})| \le |\Phi(z_{\kappa+2})| \le \cdots .$$

Therefore, for the BKM/AB we consider the system $\{\eta_j\}$, defined by the singular functions in (18), with $m_j = 1$, $j = 1, 2, \dots, \kappa$, and the n monomials in (20). Accordingly, we let $\mathbb{P}_n^{A_1}$ denote the following space of augmented polynomials:

$$\mathbb{P}_n^{A_1} := \left\{ p \colon p(z) = \sum_{j=1}^{\kappa+n} t_j \eta_j(z),\ t_j \in \mathbb{C} \right\}.$$

We note that the associated augmented kernel polynomial $\widetilde{K}_n(z, z_0)$ is the best approximation to $K(z, z_0)$ in $L^2(G)$ out of the space $\mathbb{P}_n^{A_1}$, i.e.

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \le \|K(\cdot, z_0) - p\|_{L^2(G)}, \tag{39}$$

for any $p \in \mathbb{P}_n^{A_1}$.

The next theorem provides an estimate for the error in the resulting BKM/AB approximation $\widetilde{\pi}_n$ to f_0.

Theorem 3.2. *Assume that* Γ *is a piecewise analytic curve without cusps and set* $\varrho := |\Phi(z_{\kappa+1})|$. *Then,*

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \tag{40}$$

for any R, *with* $1 < R < \varrho$, *but for no* $R > \varrho$.

Proof. Observe that $K(\cdot, z_0)$ has poles of order $k_j + 1$ at each $z_j, j = 1, 2, \dots, \kappa$, and set $Q(z) := \prod_{j=1}^{\kappa} (z - z_j)^{k_j+1}$. Then, the function $K(z, z_0)Q(z)$ is analytic in the interior G_ϱ of the level curve of L_ϱ, and from Walsh's Maximal Convergence

Theorem [22, §4.7] it follows that, for any R, with $1 < R < \varrho$, there exists a sequence of polynomial $\{p_n\}_{n=1}^{\infty}$, such that

$$\|K(\cdot, z_0)Q - p_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \qquad n \in \mathbb{N}. \tag{41}$$

Let now $d := \min_{j=1,2,\ldots,\kappa}\{|z - z_j|\colon\ z \in \Gamma\}$ denote the distance of Γ from the poles $\{z_j\}_{j=1}^{\kappa}$, and set $\xi := \sum_{j=1}^{\kappa} k_j$. Then, $|Q(z)| \geq d^{\kappa+\xi}$, $z \in \Gamma$, and (41) gives

$$\|K(\cdot, z_0) - \frac{p_n}{Q}\|_{L^\infty(\overline{G})} \leq \frac{c}{d^{\kappa+\xi}}\frac{1}{R^n}.$$

Since the $L^2(G)$-norm is dominated by the $L^\infty(\overline{G})$-norm, we see that there exist a sequence of rational polynomials $\{Q_n\}_{n=1}^{\infty}$, with $Q_n \in \mathbb{P}_n^{A_1}$, such that,

$$\|K(\cdot, z_0) - Q_n\|_{L^2(G)} \preceq \frac{1}{R^n}, \qquad n \in \mathbb{N}.$$

Therefore, using the minimum property (39) of the augmented kernel polynomials, we have

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \preceq \frac{1}{R^n}, \qquad n \in \mathbb{N}, \tag{42}$$

and this, in conjunction with the Equivalence Lemma 2.1, yields the estimate

$$\|f_0' - \widetilde{\pi}_n'\|_{L^2(G)} \preceq \frac{1}{R^n}, \qquad n \in \mathbb{N}.$$

Next, we recall that

$$\widetilde{\pi}_n(z) = \sum_{j=1}^{\kappa} c_{n,j}\left[\frac{1}{(z - z_j)^{k_j}} - \frac{1}{(z_0 - z_j)^{k_j}}\right] + \sum_{j=1}^{n} c_{n,\kappa+j}\left[z^j - z_0^j\right], \tag{43}$$

i.e.,

$$\widetilde{\pi}_n(z) = \frac{P(z)}{q(z)}, \qquad \text{where } q(z) := \prod_{j=1}^{\kappa}(z - z_j)^{k_j}, \tag{44}$$

and $P(z)$ is a polynomial of degree $n + \xi$.

Then, the transition from the $L^2(G)$-norm to the $L^\infty(\overline{G})$-norm is done as in the proof of Theorem 3.1, where now, in view of (44), Lemma 3.2(ii) is applicable. This leads to,

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \preceq \frac{\sqrt{\log n}}{R^n}, \qquad n \geq 2, \tag{45}$$

and (40) follows with a different R, which is still less than ϱ.

Finally, the fact that (40) holds for no $R > \varrho$ is evident from [22, Thm. 6, Ch. IV], since the contrary assumption would lead to the contradictory conclusion that f_0 has no singularities on L_ϱ; see the next remark. ∎

Remark 3.1. From (43) it is clear that $\widetilde{\pi}_n(z) = \widetilde{q}_\kappa(z)+p_n(z)$, where $\widetilde{q}_\kappa$ is defined by the nearest κ poles of f_0 in Ω and $p_n \in \mathbb{P}_n$. Hence, (40) gives

$$\|(f_0 - \widetilde{q}_\kappa) - p_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n},$$

for any $1 < R < \varrho$, and [22, Ch. V, Thm. 6] implies that the function $f_0 - \widetilde{q}_\kappa$ is analytic in G_ϱ. This shows that the rational polynomial $\widetilde{q}_\kappa$, constructed by the BKM/AB considered above, *cancels out the specific poles of f_0 that contains.* In particular, this provides the theoretical justification for the heuristic observation made to that effect by Papamichael and Warby in [16, p. 652].

A finer estimate than (40) can be obtained if the singularities of f_0 on $L_{|\Phi(z_{\kappa+1})|}$ are a finite number of poles.

Theorem 3.3. *Assume that Γ is a piecewise analytic curve without cusps and set $\varrho := |\Phi(z_{\kappa+1})|$. Assume, in addition to Theorem 3.2, that f_0 has a finite number of poles and no other singularities on L_ϱ and let m denote their highest order. Then,*

$$\frac{n^{m-1}}{\varrho^n} \preceq \|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \preceq \frac{n^m\sqrt{\log n}}{\varrho^n}, \qquad n \geq 2.$$

Proof. The upper estimate follows by working in the same way as in the proof of Theorem 3.2, but using here the precise result of Lemma 3.1, in the place of Walsh's Theorem in (41).

To obtain the lower estimate, observe that $q\widetilde{\pi}_n$ is a polynomial of degree $n + \xi$ (see (44)) and that the function qf_0 is analytic on $\overline{G_\varrho}$, apart from a finite number of poles on L_ϱ. Hence, from Lemma 3.1 we have, for $n \in \mathbb{N}$,

$$\|qf_0 - q\widetilde{\pi}_n\|_{L^\infty(\overline{G})} \succeq \inf_{p\in\mathbb{P}_{n+\xi}} \|qf_0 - p\|_{L^\infty(\overline{G})} \succeq \frac{n^{m-1}}{\varrho^n},$$

which yields the estimate

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \geq \frac{c}{\|q\|_{L^\infty(\overline{G})}} \frac{n^{m-1}}{\varrho^n}$$

and hence the required result. ■

In the more general case, where the nearest κ singularities of f_0 in Ω are rational poles of type (17), we have the following result regarding the associated kernel polynomials $\widetilde{K}_n(\cdot, z_0)$.

Theorem 3.4. *Assume that Γ is a piecewise analytic curve without cusps and set $\varrho := |\Phi(z_{\kappa+1})|$. Then,*

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \preceq \frac{1}{R^n}, \tag{46}$$

for any R, $1 < R < \varrho$.

Proof. Set $Q(z) := \prod_{j=1}^{\kappa}(z - z_j)^{k_j/m_j+1}$ and follow the proof of Theorem 3.2 up to Equation (42). ■

4. BKM with pole and corner singularities

In this section we assume that f_0 *has a singularity on* Γ and study the BKM and BKM/AB errors, corresponding to a variety of different syntheses of the system $\{\eta_j\}$ of basis functions. In stating the results we use the notation and the assumptions set up in Sections 2.1 and 2.2.

4.1. BKM. Our first result is a straightforward consequence of [19, Thm. 3.1] and Lemma 3.2 above.

Theorem 4.1. *Assume that* Γ *is a piecewise analytic curve without cusps and set* $\varrho := |\Phi(z_1)|$ *and* $s := \min\{(2-\alpha_k)/\alpha_k\colon 1 \le k \le M\}$. *Then,*

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} \le c_1\sqrt{\log n}\,\frac{1}{n^s} + c_2\frac{1}{R^n}, \qquad n \ge 2, \tag{47}$$

for any R, $1 < R < \varrho$.

Proof. Observe that ζ_1 in [19, Thm. 3.1] can be chosen arbitrarily close to z_1. Thus, from the minimum property of the kernel polynomials $K_n(\cdot, z_0)$ we have

$$\|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)} \le c_1\frac{1}{n^s} + c_2\frac{1}{R^n}, \qquad n \in \mathbb{N}, \tag{48}$$

for any R, $1 < R < \varrho$, and the transition from the $L^2(G)$-error in (48) to the $L^\infty(\overline{G})$-error in (47), follows along the same lines as in the proof of Theorem 3.2. ■

Remark 4.1. Clearly, as $n \to \infty$, (47) yields the result (11). However, Theorem 4.1 does more: it captures, in a very precise form, the dependence of the BKM error $\|f_0 - \pi_n\|_{L^\infty(\overline{G})}$ for "small" values of n, on both the corner and pole singularities of f_0. This dependence has been tested numerically in [11] and has given rise to the introduction of the BKM/AB.

The following result is a simple consequence of (48). Its proof is similar to that of Corollary 3.1.

Corollary 4.1. *With the assumptions of Theorem 4.1, it is true that*

$$|P_n(z_0)| \le c_1\frac{1}{n^s} + c_2\frac{1}{R^n}, \qquad n \in \mathbb{N},\ z_0 \in G. \tag{49}$$

Remark 4.2. Since $|P_n(z_0)| \le \|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)}$, it follows from Corollary 4.1 that, if for small values of n, $P_n(z_0)$ decays geometrically to zero, then the most "serious" singularity of $K(\cdot, z_0)$, and hence of f_0, is the nearest pole in Ω and not an algebraic singularity on the boundary, as the asymptotic estimate (11) would suggest. On the other hand, given that f_0 has a singularity on Γ, [10, Thm. 2.1] implies that any point of Γ is a point of accumulation of the zeros of the sequence $\{P_n\}_{n=1}^{\infty}$. Therefore, an easy way to check whether a pole singularity is more serious than an algebraic singularity, for a range of values of n, is by plotting the zeros of P_n for the same range: *if the zeros stay away from a specific part of the boundary, this indicates that $P_n(z_0)$ decays geometrically and therefore the presence of a pole singularity near that part.* We refer to [19, Ex. 2, 3], where (49) was used as the tool for explaining the misleading nature of such plots.

4.2. BKM/AB with corner singularities. From our assumptions on Γ, it follows that the conformal map f_0 can be extended analytically, by means of the reflection principle, beyond Γ to a larger Jordan domain $\widetilde{G}$, such that the boundary $\partial\widetilde{G}$ of $\widetilde{G}$ consists of analytic arcs to be defined below. For this, we recall our assumptions on the position of the nearest poles z_j, $j=1,\dots,\kappa$, of f_0 in Ω and pick up a point ζ_1 near z_1, but interior to the level curve L_ϱ, with $\varrho := |\Phi(z_1)|$. Next, we draw the level curve $L_{\widetilde{\varrho}}$, with $\widetilde{\varrho} := |\Phi(\zeta_1)|$ and fix on it points ζ_k, $k=2,\dots,N$, "between" τ_k and τ_{k+1}, where we set $\tau_{N+1}=\tau_1$. We connect each non-special corner τ_k, $k=1,\dots,M$, with the two adjacent ζ_k's, by using two analytic arcs. Next, we denote by l_k the two arcs emanating from τ_k and call l_N the part (or parts) of the level line $L_{\widetilde{\varrho}}$ that joins together those consecutive points ζ_k that have only one connection with τ_k. See Figure 1, for a possible arrangement of corners τ_k, points ζ_k, and arcs l_k and l_N. Finally, we define $\widetilde{G}$ by taking $\partial\widetilde{G} := \{\cup_{k=1}^{M} l_k\} \cup l_N$.

The above construction is such that:

(i) $\partial\widetilde{G}$ is a piecewise analytic Jordan curve that meets Γ at the non-special corner τ_k, $k=1,\dots,M$.
(ii) f_0 is continuous on $\widetilde{G}\cup\partial\widetilde{G}$ and analytic in $\widetilde{G}$ and on $\partial\widetilde{G}$, except for the endpoints τ_k.
(iii) The asymptotic expansion (16) holds for $z\in l_k$, $k=1,\dots,M$, in the sense that, for any $p_k\in\mathbb{N}_0$,

$$\begin{aligned} f_0(z) &= \sum_{j=0}^{p_k} a_j^{(k)}(z-\tau_k)^{\gamma_j^{(k)}} + \widetilde{f}_{\gamma_{p_k+1}^{(k)},\tau_k}(z), \\ \widetilde{f}_{\gamma_{p_k+1}^{(k)},\tau_k}(z) &= \mathcal{O}\Big((z-\tau_k)^{\gamma_{p_k+1}^{(k)}}\Big). \end{aligned} \tag{50}$$

Let us consider now the application of BKM/AB with only corner singularities, where we use $p_k\in\mathbb{N}_0$ singular functions for each non-special corner τ_k, $k=1,2,\dots,M$. In order to measure the BKM/AB error we set

$$\nu_k := \min\Big\{j>p_k\colon\ \gamma_j^{(k)}\notin\mathbb{N},\ a_j^{(k)}\neq 0\Big\}, \tag{51}$$

and assume that at least one of ν_k's is finite, otherwise the results become trivial. The associated BKM/AB system $\{\eta_j\}$ is thus defined by $r_M=\sum_{k=1}^{M}p_k$ singular functions of the form (19) and n monomials (20). Accordingly, we let $\mathbb{P}_n^{A_2}$ denote the space of augmented polynomials:

$$\mathbb{P}_n^{A_2} := \left\{ p\colon p(z) = \sum_{j=1}^{r_M+n} t_j \eta_j(z),\, t_j \in \mathbb{C} \right\}.$$

Clearly, the associated augmented polynomial $\widetilde{K}_n(z, z_0)$ is the best approximation to $K(z, z_0)$ in $L^2(G)$ out of the space $\mathbb{P}_n^{A_2}$.

Let $\widetilde{\pi}_n$ denote the BKM/AB approximation resulting from $\mathbb{P}_n^{A_2}$. Then we have the following result.

Theorem 4.2. *Assume that Γ is a piecewise analytic curve without cusps and set $\varrho := |\Phi(z_1)|$ and $s^\star := \min\{(2-\alpha_k)\gamma_{\nu_k}^{(k)}\colon\ 1 \le k \le M\}$. Then,*

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \le c_1 \sqrt{\log n}\,\frac{1}{n^{s^\star}} + c_2 \frac{1}{R^n}, \qquad n \ge 2, \tag{52}$$

for any R, $1 < R < \varrho$.

Proof. Using Cauchy's integral formula for the derivative of the extension of f_0 we have, for $z \in G$,

$$\begin{aligned} f_0'(z) &= \frac{1}{2\pi i} \int_{\partial \widetilde{G}} \frac{f_0(t)}{(t-z)^2}\, dt \\ &= \frac{1}{2\pi i} \sum_{k=1}^{M} \int_{l_k} \frac{f_0(t)}{(t-z)^2}\, dt + \frac{1}{2\pi i} \int_{l_N} \frac{f_0(t)}{(t-z)^2}\, dt. \end{aligned} \tag{53}$$

For each τ_k, $k = 1, \ldots, M$, we consider the first terms up to p_k, of the Lehman expansion (50) for f_0:

$$F_k(z) := \sum_{j=0}^{p_k} a_j^{(k)} (z - \tau_k)^{\gamma_j^{(k)}}. \tag{54}$$

Since the function $F_k(z)$, is analytic in $\widetilde{G}$ and continuous on $\partial\widetilde{G}$ we have, as in (53), for $z \in G$,

$$F_k'(z) = \frac{1}{2\pi i} \sum_{r=1}^{M} \int_{l_r} \frac{F_k(t)}{(t-z)^2}\, dt + \frac{1}{2\pi i} \int_{l_N} \frac{F_k(t)}{(t-z)^2}\, dt.$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{M} F_k'(z) &= \frac{1}{2\pi i}\sum_{k=1}^{M}\sum_{r=1}^{M}\int_{l_r}\frac{F_k(t)}{(t-z)^2}\,dt + \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_N}\frac{F_k(t)}{(t-z)^2}\,dt \\ &= \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{F_k(t)}{(t-z)^2}\,dt + \frac{1}{2\pi i}\sum_{k=1}^{M}\sum_{\substack{r=1\\ r\neq k}}^{M}\int_{l_r}\frac{F_k(t)}{(t-z)^2}dt \\ &\quad + \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_N}\frac{F_k(t)}{(t-z)^2}\,dt.\end{aligned}$$

Hence, for $z\in G$,

$$f_0'(z) - \sum_{k=1}^{M} F_k'(z) = g(z) + h(z), \tag{55}$$

where,

$$g(z) := \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{f_0(t)-F_k(t)}{(t-z)^2}dt, \tag{56}$$

and

$$\begin{aligned}h(z) := \frac{1}{2\pi i}\int_{l_N}\frac{f_0(t)}{(t-z)^2}\,dt &- \frac{1}{2\pi i}\sum_{k=1}^{M}\sum_{\substack{r=1\\ r\neq k}}^{M}\int_{l_r}\frac{F_k(t)}{(t-z)^2}\,dt \\ &- \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_N}\frac{F_k(t)}{(t-z)^2}\,dt.\end{aligned} \tag{57}$$

Now, we denote by l_r', $r=1,\dots,M$, the part of the level line $L_{\widetilde{\varrho}}$ that shares the same endpoints with l_r, so that $L_{\widetilde{\varrho}} = \{\bigcup_{r=1}^{M} l_r'\}\cup l_N$ and $l_r\cup l_r'$ is the boundary of a Jordan domain in Ω; see Figure 1. Since, for $k\neq r$, the function $F_k(z)$, $z\in G$, is analytic in the interior of $l_r\cup l_r'$ and continuous on $l_r\cup l_r'$, we can replace in (57) the path of integration l_r by l_r', with suitable orientation, i.e. for $z\in G$,

$$\begin{aligned}h(z) = \frac{1}{2\pi i}\int_{l_N}\frac{f_0(t)}{(t-z)^2}\,dt &- \frac{1}{2\pi i}\sum_{k=1}^{M}\sum_{\substack{r=1\\ r\neq k}}^{M}\int_{l_r'}\frac{F_k(t)}{(t-z)^2}\,dt \\ &- \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_N}\frac{F_k(t)}{(t-z)^2}\,dt.\end{aligned} \tag{58}$$

Observe that, by construction, f_0 is continuous on l_N and F_k is continuous on $l_N\cup l_r'$, for $k\neq r$. Thus, the function h in (58) is analytic in $G_{\widetilde{\varrho}}$ and by Walsh's

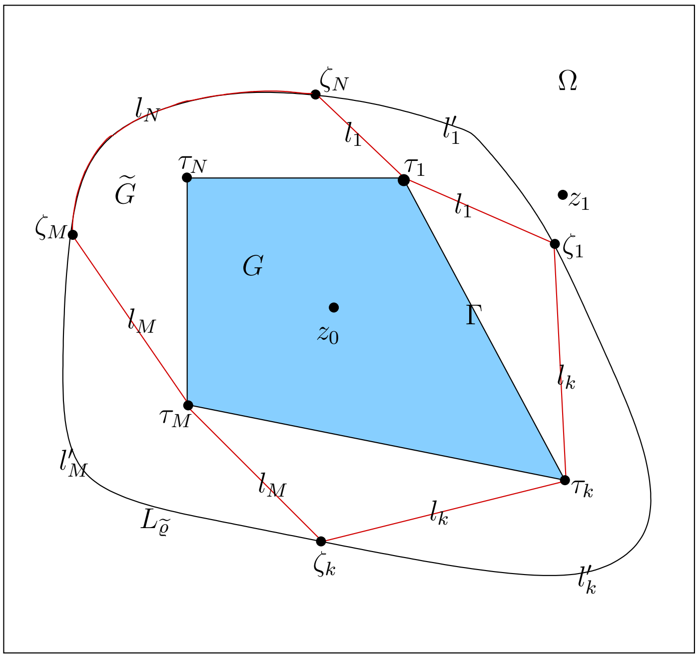

FIGURE 1. The domain $\widetilde{G}$ in the proof of Theorem 4.2.

Maximal Convergence Theorem there exist a sequence of polynomials $\{t_n\}_{n=1}^{\infty}$ such that,

$$\|h - t_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \qquad n \in \mathbb{N}, \tag{59}$$

where $1 < R < \widetilde{\varrho}$. Since we can choose ζ_1 arbitrarily close to z_1, (59) is valid for any $1 < R < \varrho$.

The function g in (56) consists of sums of integrals of the type,

$$G(z) = \int_{l_k} \frac{g_k(t)}{(t-z)^2}\, dt,$$

where in view of (50) and (54) we have, for $t \in l_k$,

$$|g_k(t)| \preceq |t - \tau_k|^{\gamma^{(k)}_{p_k+1}}.$$

Hence, by using the result of Lemma 6 in [1], in conjunction with the remark following Theorem 1 of the same paper and the triangle inequality, we conclude that there exists a sequence of polynomials $\{q_n\}_{n=1}^{\infty}$ satisfying

$$\|g - q_n\|_{L^2(G)} \preceq \frac{1}{n^{\widetilde{s}}}, \qquad n \in \mathbb{N}, \tag{60}$$

where

$$\widetilde{s} := \min\left\{(2-\alpha_k)\gamma^{(k)}_{p_k+1} :\ k = 1, 2, \ldots, M\right\}.$$

This, combined with (55), (59) and the triangle inequality, yields

$$\left\| f_0' - \sum_{k=1}^{M} F_k' - (t_n + q_n) \right\|_{L^2(G)} \le c_1 \frac{1}{n^{\widetilde{s}}} + c_2 \frac{1}{R^n}, \qquad n \in \mathbb{N}. \tag{61}$$

Note that $\widetilde{s} = s^\star$, if $\gamma^{(k)}_{p_k+1} \notin \mathbb{N}$ for the index k for which the minimum is attained in the definition of $\widetilde{s}$. In the opposite case, where for the same index k, it is true that $\gamma^{(k)}_{p_k+1} \in \mathbb{N}$, we get $\widetilde{s} = s^\star$ in (61) by simply subtracting from $g(z)$ and adding to $h(z)$, in the right hand side of (55), the derivative of the Cauchy integral on l_N, with density function $a^{(k)}_{p_k+1}(z-\tau_k)^{\gamma^{(k)}_{p_k+1}}$. This observation and (5) imply that there exists a sequence of augmented polynomials $\{\widetilde{p}_n\}$, with $\widetilde{p}_n \in \mathbb{P}_n^{A_2}$, such that,

$$\|K(\cdot, z_0) - \widetilde{p}_n\|_{L^2(G)} \le c_3 \frac{1}{n^{s^\star}} + c_4 \frac{1}{R^n}, \qquad n \in \mathbb{N},$$

and the rest is similar to the proof of Theorem 3.1, except here we use the version of Andrievskiĭ's Lemma for functions with anti-derivatives in the space $\mathbb{P}_n^{A_2}$, given in [12, Cor. 2.5]. These yield,

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \le c_5 \sqrt{\log n} \frac{1}{n^{s^\star}} + c_6 \sqrt{\log n} \frac{1}{R^n}, \qquad n \ge 2,$$

and (52) follows with a different R, but one which is still less than ϱ. ■

Remark 4.3. Note that $n^{s^\star} \le R^n$, as $n \to \infty$. Therefore from (52) we recover the result of [12, Thm. 3.1]. However, Theorem 4.2 above gives, in addition, the precise dependence of the BKM/AB error on the pole singularities of f_0 for small values of n. We also note the lower estimate

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \ge c \frac{1}{n^{s^\star}}, \qquad n \in \mathbb{N},$$

established in [12, Thm. 3.2].

4.3. BKM/AB with pole and corner singularities. We consider now the application of the BKM/AB with both pole and corner singular basis function of the form studied in Sections 3.2 and 4.2. Regarding poles we recall, in particular, our assumptions in Section 3.2. That is, the nearest singularities of f_0 in Ω are κ poles, each one of order k_j at z_j, $j = 1, 2, \dots \kappa$, where

$$|\Phi(z_1)| \le |\Phi(z_2)| \le \cdots \le |\Phi(z_\kappa)|,$$

while the other singularities of f_0 occur at points $z_{\kappa+1}, z_{\kappa+2}, \dots$, where

$$|\Phi(z_\kappa)| < |\Phi(z_{\kappa+1})| \le |\Phi(z_{\kappa+2})| \le \cdots .$$

Therefore, for the BKM/AB we consider the system $\{\eta_j\}$, defined by:

(i) the κ pole singular functions (18), with $m_j = 1$, $j = 1, 2, \dots, \kappa$;
(ii) the $r_M = \sum_{k=1}^{M} p_k$ corner singular functions of the form (19);

(iii) and the n monomials (20).

Accordingly, we let $\mathbb{P}_n^{A_3}$ denote the space,

$$\mathbb{P}_n^{A_3} := \left\{ p \colon p(z) = \sum_{j=1}^{\kappa+r_M+n} t_j \eta_j(z),\, t_j \in \mathbb{C} \right\},$$

and note that the associated augmented polynomial $\widetilde{K}_n(z,z_0)$ is the best approximation to $K(z,z_0)$ in $L^2(G)$ out of the space $\mathbb{P}_n^{A_3}$.

The following result is a version of Andrievskiĭ's Lemma for functions with anti-derivatives in $\mathbb{P}_n^{A_3}$. It will be used below, in the proof of the concluding theorem of this section (in the transition from the $L^2(G)$-norm to the $L^\infty(\overline{G})$-norm), where we establish the BKM/AB error in approximating f_0 by the augmented polynomials $\widetilde{\pi}_n$ derived from $\mathbb{P}_n^{A_3}$.

Lemma 4.1. *Assume that Γ is piecewise analytic without cusps and let $t_k \in \Gamma$, $k = 1, 2, \ldots, m$. Also, let $P \in \mathbb{P}_n$ and q be a fixed polynomial with no zeros on $\overline{G}$. Assume further that for some constants $a_{n,k,j}$, $k = 1, 2, \ldots, m$, $j = 1, 2, \ldots, r_k$, the function*

$$L(z) := \frac{P(z)}{q(z)} + \sum_{k=1}^{m} \sum_{j=1}^{r_k} a_{n,k,j} f_{\beta_j^{(k)},t_k}(z),$$

where $f_{\beta_j^{(k)},t_k}(z) = (z - t_k)^{\beta_j^{(k)}}$, with $\beta_j^{(k)} > 0$ non-integer, satisfies: $L(z_0) = 0$ and $\|L'\|_{L^2(G)} \le M$. Then,

$$\|L\|_{L^\infty(\overline{G})} \le CM\sqrt{\log n}, \tag{62}$$

where C is a constant independent of n and of $\{\{a_{n,k,j}\}_{j=1}^{r_k}\}_{k=1}^{m}$.

Proof. The proof is based on Andrievskiĭ's Lemma for singular algebraic functions given in [12, Cor. 2.5] and relies on the results contained in [12, Sec. 2]. The details of the derivation are as follows:

First, we note that our assumption implies that Γ is a quasiconformal curve. Then, it is straightforward to verify that the results of Theorems 2.1 and 2.2 (and hence the result of Corollary 2.2) in [12] hold true for functions of the form $q^2(z) f_{\beta,\tau}(z)$, where $f_{\beta,\tau}(z) := (z - \tau)^\beta$, with $\tau \in \Gamma$ and $\beta > 0$ non-integer. That is,

$$\inf_{p \in \mathbb{P}_n} \left\| q^2 f_{\beta,\tau} - p \right\|_{L^2(G)} \asymp \frac{1}{n^{(2-a)(\beta+1)}}, \tag{63}$$

where $\alpha\pi$, $0 < \alpha < 2$, denotes the interior angle of Γ at τ.

With (63) at hand it is, again, straightforward to verify consequentially that the results of Theorem 2.3, Corollaries 2.3 and 2.4, Lemma 2.3 and Corollary 2.5, of [12], hold true if we replace $f'_{\beta,\tau}$ by $q^2 f'_{\beta,\tau}$. In particular, Corollary 2.5 of [12] applied to the function

$$S(z) := \int_{z_0}^{z} q^2(z) L'(z)\, dz,$$

where the path of integration $[z_0, z]$ is any rectifiable arc in G, gives that

$$\|S\|_{L^\infty(\overline{G})} \le c_1 \sqrt{\log n}\, \|S'\|_{L^2(G)}.$$

(Note that $S(z_0) = 0$ and $S'(z) = q^2(z)L'(z)$.) Therefore, our hypothesis on $\|L'\|_{L^2(G)}$ yields the inequality

$$\|S\|_{L^\infty(\overline{G})} \le c_2 \sqrt{\log n}\, M. \tag{64}$$

On the other hand we have,

$$L(z) = \int_{z_0}^{z} q^{-2}(z) S'(z)\, dz = q^{-2}(z) S(z) + 2 \int_{z_0}^{z} q^{-3}(z) q'(z) S(z)\, dz,$$

which implies

$$\|L\|_{L^\infty(\overline{G})} \le c_3 \, \|S\|_{L^\infty(\overline{G})}$$

and (62) follows from (64); cf. [4, p. 122]. ∎

The concluding result of this section provides the theoretical justification for the use of the BKM/AB, with both corner and pole singularities.

Let $\widetilde{\pi}_n$ denote the BKM/AB approximation to f_0 resulting from the space $\mathbb{P}_n^{A_3}$. Then we have the following result.

Theorem 4.3. *Assume that Γ is a piecewise analytic curve without cusps and set $\varrho := |\Phi(z_{\kappa+1})|$ and*

$$s^\star := \min\left\{(2-\alpha_k)\gamma_{\nu_k}^{(k)} \colon \ 1 \le k \le M\right\}.$$

Then,

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \le c_1 \sqrt{\log n}\, \frac{1}{n^{s^\star}} + c_2 \frac{1}{R^n}, \qquad n \ge 2, \tag{65}$$

for any R, $1 < R < \varrho$.

Proof. As in the proof of Theorem 3.2, we set $Q(z) := \prod_{j=1}^{\kappa}(z - z_j)^{k_j+1}$. The result (65) will emerge by working as in the proof of Theorem 4.2. The basic idea is to consider, in a bigger domain $\widetilde{G}$, the anti-derivatives F and G_k of the functions Qf'_0 and QF'_k, respectively, in place of the functions f_0 and F_k. The details of the derivation are as follows:

We note that the function Qf'_0 shares the same analytic properties with f'_0, apart from the fact that it has the singularities at the points z_j, $j = 1, \ldots, \kappa$, all removed. Therefore, the function

$$F(z) := \int_{z_0}^{z} Q(\zeta) f'_0(\zeta)\, d\zeta, \tag{66}$$

can be extended analytically to a larger domain $\widetilde{G}$ than the one considered in Section 4.2. This larger domain $\widetilde{G}$ is obtained by choosing the point ζ_1 close to the nearest pole $z_{\kappa+1}$ of Qf_0' in Ω, but inside the level curve L_ϱ, where now $\varrho := |\Phi(z_{\kappa+1})|$. The remaining part of the construction of $\widetilde{G}$ is exactly the same as in Section 4.2.

It follows therefore that (66) is valid for $z \in \widetilde{G}$, provided the arc of integration $[z_0, z]$ lies on $\widetilde{G} \cup \partial\widetilde{G} \setminus \{\bigcup_{k=1}^{M} \tau_k\}$ and is rectifiable. (This is always possible because $\partial\widetilde{G}$ is piecewise analytic.) Since the derivative of f_0 near τ_k can be obtained by termwise differentiation of the expansion (50), (cf. [9, p. 1448]) and since any power in the resulting expansion is bigger than $-1/2$, we see that f_0' is integrable along any rectifiable arc in $\widetilde{G}$ with one endpoint at τ_k. Therefore, integration by parts gives, for $z \in \widetilde{G} \cup \partial\widetilde{G}$,

$$F(z) = Q(z)f_0(z) - \int_{z_0}^{\tau_k} Q'(\zeta)f_0(\zeta)\,d\zeta - \int_{\tau_k}^{z} Q'(\zeta)f_0(\zeta)\,d\zeta, \tag{67}$$

where we made use of the normalization of f_0 at z_0. This shows that F is continuous on $\partial\widetilde{G}$ and analytic and on $\partial\widetilde{G}$, except for the endpoints τ_k. By arguing as in (53) we have, for $z \in G$,

$$Q(z)f_0'(z) = F'(z) = \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{F(t)}{(t-z)^2}\,dt + \frac{1}{2\pi i}\int_{l_N}\frac{F(t)}{(t-z)^2}\,dt. \tag{68}$$

Similar properties to those of F apply to the anti-derivative

$$G_k(z) := \int_{z_0}^{z} Q(\zeta)F_k'(\zeta)\,d\zeta, \tag{69}$$

of QF_k', $k = 1, \ldots, M$. That is, for $z \in \widetilde{G} \cup \partial\widetilde{G}$,

$$\begin{aligned} G_k(z) &= Q(z)F_k(z) - Q(z_0)F_k(z_0) \\ &\quad - \int_{z_0}^{\tau_k} Q'(\zeta)F_k(\zeta)\,d\zeta - \int_{\tau_k}^{z} Q'(\zeta)F_k(\zeta)\,d\zeta, \end{aligned} \tag{70}$$

and, for $z \in G$,

$$Q(z)F_k'(z) = G_k'(z) = \frac{1}{2\pi i}\sum_{r=1}^{M}\int_{l_r}\frac{G_k(t)}{(t-z)^2}\,dt + \frac{1}{2\pi i}\int_{l_N}\frac{G_k(t)}{(t-z)^2}\,dt. \tag{71}$$

Next, by combining (67) and (70) we get,

$$F(z) - d_k - G_k(z) = Q(z)[f_0(z) - F_k(z)] - \int_{\tau_k}^{z} Q'(\zeta)[f_0(z) - F_k(\zeta)]\,d\zeta, \tag{72}$$

where

$$d_k := Q(z_0)F_k(z_0) - \int_{z_0}^{\tau_k} Q'(\zeta)[f_0(\zeta) - F_k(\zeta)]\,d\zeta.$$

This and (50) lead to

$$|F(z) - d_k - G_k(z)| \preceq |z - \tau_k|^{\gamma^{(k)}_{p_k+1}}, \qquad z \in l_k. \tag{73}$$

By reasoning as in the proof of Theorem 4.2 we conclude, by using (68) and (71) that, for $z \in G$,

$$Q(z) f_0'(z) - Q(z) \sum_{k=1}^{M} F_k'(z) = g(z) + h(z), \tag{74}$$

where the singular part

$$g(z) := \frac{1}{2\pi i} \sum_{k=1}^{M} \int_{l_k} \frac{F(t) - d_k - G_k(t)}{(t-z)^2}\, dt, \tag{75}$$

of the splitting (74) can be approximated, eventually, by a sequence of polynomials $\{q_n\}_{n=1}^{\infty}$ at a polynomial rate, viz.,

$$\|g - q_n\|_{L^2(G)} \preceq \frac{1}{n^{s^\star}}, \qquad n \in \mathbb{N}, \tag{76}$$

with $s^\star := \min\{(2-\alpha_k)\gamma^{(k)}_{\nu_k} \colon\ 1 \le k \le M\}$ and the analytic part

$$\begin{aligned} h(z) := \frac{1}{2\pi i} \sum_{k=1}^{M} \int_{l_k} \frac{d_k}{(t-z)^2}\, dt + \frac{1}{2\pi i} \int_{l_N} \frac{F(t)}{(t-z)^2}\, dt \\ -\frac{1}{2\pi i} \sum_{k=1}^{M} \sum_{\substack{r=1 \\ r\neq k}}^{M} \int_{l_r} \frac{G_k(t)}{(t-z)^2}\, dt - \frac{1}{2\pi i} \sum_{k=1}^{M} \int_{l_N} \frac{G_k(t)}{(t-z)^2}\, dt \end{aligned} \tag{77}$$

can be approximated by a sequence of polynomials $\{t_n\}_{n=1}^{\infty}$ at a geometric rate, viz.,

$$\|h - t_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \qquad n \in \mathbb{N},$$

where $1 < R < \varrho$. Hence using the triangle inequality we get

$$\|Q(f_0' - \sum_{k=1}^{M} F_k') - (t_n + q_n)\|_{L^2(G)} \le c_1 \frac{1}{n^{s^\star}} + c_2 \frac{1}{R^n}.$$

This implies

$$\left\| f_0' - \sum_{k=1}^{M} F_k' - \frac{t_n + q_n}{Q} \right\|_{L^2(G)} \le \frac{c}{d^{\kappa+\xi}} \left(c_1 \frac{1}{n^{s^\star}} + c_2 \frac{1}{R^n} \right),$$

where $d := \min_{j=1,2,\dots,\kappa}\{|z - z_j| \colon z \in \Gamma\}$ and $\xi := \sum_{j=1}^{\kappa} k_j$. Thus, from (5) we conclude there exists a sequence of augmented polynomials $\{\widetilde{p}_n\}$, where $\widetilde{p}_n \in \mathbb{P}_n^{A_3}$, such that,

$$\|K(\cdot, z_0) - \widetilde{p}_n\|_{L^2(G)} \le c_3 \frac{1}{n^{s^\star}} + c_4 \frac{1}{R^n}, \qquad n \in \mathbb{N}.$$

Therefore, using the minimum property of the augmented kernel polynomials, we have

$$\|K(\cdot,z_0)-\widetilde{K}_n(\cdot,z_0)\|_{L^2(G)}\le c_3\frac{1}{n^{s^\star}}+c_4\frac{1}{R^n},\qquad n\in\mathbb{N},$$

and this, in conjunction with the Equivalence Lemma 2.1, yields that

$$\|f_0'-\widetilde{\pi}_n'\|_{L^2(G)}\le c_5\frac{1}{n^{s^\star}}+c_6\frac{1}{R^n},\qquad n\in\mathbb{N}.$$

Since,

$$\widetilde{\pi}_n(z)=\frac{P(z)}{q(z)}+\sum_{k=1}^{M}\sum_{j=0}^{p_k}a_{n,k,j}(z-\tau_k)^{\gamma_j^{(k)}},$$

where $q(z):=\prod_{j=1}^{\kappa}(z-z_j)^{k_j}$ and $P(z)$ is a polynomial of degree $n+\xi$, the rest follows as in the concluding part of the proof of Theorem 4.2, except here we use the result of Lemma 4.1 in the place of [12, Cor. 2.5]. ∎

5. Numerical results

In this section we present numerical examples, that illustrate the convergence results predicted by the theory of Sections 3 and 4, regarding the following four errors:

$$E_{n,2}(K,G):=\|K(\cdot,z_0)-K_n(\cdot,z_0)\|_{L^2(G)},\tag{78}$$

$$E_{n,\infty}(f_0,G):=\|f_0-\pi_n\|_{L^\infty(\overline{G})},\tag{79}$$

$$\widetilde{E}_{n,2}(K,G):=\|K(\cdot,z_0)-\widetilde{K}_n(\cdot,z_0)\|_{L^2(G)},\tag{80}$$

$$\widetilde{E}_{n,\infty}(f_0,G)\ :\ =\|f_0-\widetilde{\pi}_n\|_{L^\infty(\overline{G})}.\tag{81}$$

We do this by considering two different geometries: (a) lens-shaped domains; and (b) circular sectors. In both cases the normalized conformal map f_0, and hence the kernel function $K(\cdot,z_0)$, are known explicitly in terms of elementary functions. In addition, we present results illustrating the decay of the two sequences of points $\{P_n(z_0)\}_{n=1}^\infty$ and $\{\widetilde{P}_n(z_0)\}_{n=1}^\infty$ of the Bergman polynomials.

5.1. Computational details. Let $\{\eta_j\}$ denote the set of linearly independent functions defined in (18)–(20). For the application of the BKM/AB (or BKM), we compute the associated orthonormal set $\{\widetilde{P}_j\}$ by using the Arnoldi variant of the Gram-Schmidt (GS) process studied in [20], rather than the conventional GS, which is based on the orthonormalization of the monomials $\{z^j\}$, as suggested in [11, 16]. In the Arnoldi GS we construct first the polynomial part of the set $\{\widetilde{P}_j\}$ by orthonormalizing consequently the functions $1,z\widetilde{P}_0,z\widetilde{P}_1,\dots,z\widetilde{P}_{n-1}$. Then, we orthonormalize the singular basis functions (18) and (19). As it is shown in [20], in this way we avoid the instability difficulties associated with the application of the conventional GS method. For a comprehensive report of experiments illustrating the instability of the conventional GS in BKM and BKM/AB we refer to [16, Sec. 5].

The GS process, requires the computation of inner products of the form

$$\langle \eta_k, \eta_l \rangle = \int_G \eta_k(z)\, \overline{\eta_l(z)}\, dA(z). \tag{82}$$

For our purposes here, we compute these inner products by using Green's formula in order to transform the area integral into a line integral. For instance, when $\eta_k = z^k$, $\eta_l = z^l$, we have

$$\langle z^k, z^l \rangle = \frac{1}{2(l+1)i} \int_\Gamma z^k\, \overline{z}^{\,l+1}\, dz.$$

In all cases considered below this leads to explicit formulas for the inner products (82).

Regarding the computation of the errors (78)–(81) we note the following:

(i) The two errors $\|K(\cdot,z_0) - K_n(\cdot,z_0)\|_{L^2(G)}$ and $\|K(\cdot,z_0) - \widetilde{K}_n(\cdot,z_0)\|_{L^2(G)}$ are computed by using Parseval's identity, i.e.

$$\|K(\cdot,z_0) - K_n(\cdot,z_0)\|^2_{L^2(G)} = K(z_0,z_0) - K_n(z_0,z_0),$$

and

$$\|K(\cdot,z_0) - \widetilde{K}_n(\cdot,z_0)\|^2_{L^2(G)} = K(z_0,z_0) - \widetilde{K}_n(z_0,z_0).$$

(ii) Estimates for the two errors $\|f_0 - \pi_n\|_{L^\infty(\overline{G})}$ and $\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})}$ are obtained by using the exact formula for f_0 and then sampling the differences $f_0 - \pi_n$ and $f_0 - \widetilde{\pi}_n$ on 100 uniformly distributed points on each analytic arc forming the boundary Γ.

All results were obtained with Maple 11, using the systems facility for 64-digit floating point arithmetic, on a Pentium PC.

5.2. BKM and BKM/AB approximation.

5.2.1. Lens-shaped domains. Let $G_{a,b}$ denote the lens-shaped domain, whose boundary Γ consists of two circular arcs Γ_a and Γ_b that join together the points i and $-i$ (Γ_a being to the left of Γ_b) and form angles a and b, respectively, with the linear segment $[-i,i]$. (Thus, with the notation of Section 2.1 we have $\alpha_1 = \alpha_2 = \alpha$, where $\alpha := (a+b)/\pi$.) Let f_0 denote the normalized conformal map from $G_{a,b}$ onto $D(0,r_0)$, with $f_0(0)=0$ and $f_0'(0)=1$. By working as in [13, Sec. 4], it is easy to check that, if $a+b = k\pi/m$, where $k,m \in \mathbb{N}$, then f_0 is given by

$$f_0(z) = r_0 \frac{\left(\frac{z-i}{z+i}\right)^{m/k} - (-1)^{m/k}}{\left(\frac{z-i}{z+i}\right)^{m/k} - (-1)^{m/k} e^{-2iam/k}}, \qquad z \in \overline{G}_{a,b}, \tag{83}$$

where $r_0 = (k/m)\sin(ma/k)$. Also,

$$K(z,0) = -\frac{4m^2}{\pi k^2}\frac{[(z-i)(z+i)]^{m/k-1}}{\left[e^{iam/k}(-i)^{m/k}(z-i)^{m/k} - e^{-iam/k}(i)^{m/k}(z+i)^{m/k}\right]^2}, \tag{84}$$

and thus

$$K(0,0) = \frac{m^2}{\pi k^2}\frac{1}{\sin^2(ma/k)}.$$

It is also easy to verify that the formulas (71)–(73) of [13] work as well for the exterior conformal map $\Phi\colon \overline{\mathbb{C}} \setminus \overline{G}_{a,b} \to \Delta$ considered here. That is, $w = \Phi(z)$ is given by the composition of the following three transformations:

$$\xi(z) := e^{i((m-k)\pi/m+a)}\frac{z-i}{z+i}, \tag{85}$$

$$t(\xi) := \xi^{m/(2m-k)}, \qquad \arg\xi \in \left(\frac{-k\pi}{m}, \frac{(2m-k)\pi}{m}\right], \tag{86}$$

$$w(t) := \frac{1-\lambda_a t}{t-\lambda_a}, \qquad \lambda_a := e^{i((m-k)\pi+ma)/(2m-k)}. \tag{87}$$

We consider separately the following three cases:

(i) $\alpha = 1/2$, with $a = \pi/6$ and $b = \pi/3$;
(ii) $\alpha = 1/2$, with $a = \pi/4$ and $b = \pi/4$;
(iii) $\alpha = 2/13$, with $a = \pi/13$ and $b = \pi/13$.

Cases (i) *and* (ii): In the first two cases the conformal map f_0 is a rational function, and hence it has an analytic continuation across Γ into Ω. When $a = \pi/6$, then the two nearest singularities of f_0 in Ω are the two simple poles at $z_1 = -\sqrt{3}/3$ and $z_2 = \sqrt{3}$, where $|\Phi(z_1)| \approx 1.347$ and $|\Phi(z_2)| \approx 2.532$. Accordingly, in our experiments, we use the singular function

$$\left[\frac{1}{z-z_1}\right]'.$$

This cancels out the nearest singularity at z_1. In the symmetric case, where $a = b = \pi/4$, we have

$$f_0(z) = \frac{-2iz}{z^2-1},$$

and the only singularities of f_0 are the two simple poles at $z_1 = -1$ and $z_2 = 1$, where $|\Phi(z_1)| = |\Phi(z_2)| = \sqrt{3}$. In this case, we use the singular function

$$\left[\frac{z}{z^2-{z_1}^2}\right]',$$

which takes care of both poles at z_1 and z_2. It follows from Remark 3.1 that this cancels out all the singularities of f_0.

We recall from Theorems 3.1 and 3.3 (and their proofs) the four estimates,

$$E_{n,2}(K,G) \preceq \frac{n}{|\Phi(z_1)|^n}, \tag{88}$$

$$\frac{1}{|\Phi(z_1)|^n} \preceq E_{n,\infty}(f_0,G) \preceq \frac{n\sqrt{\log n}}{|\Phi(z_1)|^n}, \tag{89}$$

and

$$\widetilde{E}_{n,2}(K,G) \preceq \frac{n}{|\Phi(z_2)|^n}, \tag{90}$$

$$\frac{1}{|\Phi(z_2)|^n} \preceq \widetilde{E}_{n,\infty}(f_0,G) \preceq \frac{n\sqrt{\log n}}{|\Phi(z_2)|^n}. \tag{91}$$

Below, we present numerical results that illustrate the laws of the above errors and rates. In presenting the numerical results we use the following notation:

- ϱ: This denotes the order of approximation (the base of n) in the errors (88)–(91).
- ϱ_n: This denotes the estimate of ϱ, corresponding to n, and is determined as follows: with E_n denoting any of the two errors $E_{n,2}(K,G)$ or $\widetilde{E}_{n,2}(K,G)$, we assume that
$$E_n \approx c\frac{n}{\varrho^n} \tag{92}$$
and seek to estimate ϱ by means of the formula,
$$\varrho_n = \left(\frac{n}{n-m}\frac{E_{n-m}}{E_n}\right)^{\frac{1}{m}}. \tag{93}$$
(Here we take $m=4$, or $m=5$.) If E_n denotes either of the two errors $E_{n,\infty}(f_0,G)$ or $\widetilde{E}_{n,\infty}(f_0,G)$, then we assume that
$$E_n \approx c\frac{n\sqrt{\log n}}{\varrho^n},$$
and seek to estimate ϱ by means of the formula,
$$\varrho_n = \left(\frac{n}{n-m}\frac{\sqrt{\log n}}{\sqrt{\log(n-m)}}\frac{E_{n-m}}{E_n}\right)^{1/m}, \tag{94}$$
with $m=4$, or $m=5$.
- $\varrho_n^\star$: With E_n denoting either of the errors $E_{n,\infty}(f_0,G)$ or $\widetilde{E}_{n,\infty}(f_0,G)$, we also test the law
$$E_n \approx c\frac{1}{\varrho^n}, \tag{95}$$

thereby estimating ϱ by means of

$$\varrho_n^\star = \left(\frac{E_{n-m}}{E_n}\right)^{1/m}. \tag{96}$$

The results presented show clearly the advantage of the BKM/AB over the BKM. In addition, they indicate a close agreement between the theoretical and the computed order of approximation. In Tables 1 and 3, the results associated with the errors $E_{n,2}(K,G)$ and $\widetilde{E}_{n,2}(K,G)$ indicate the convergence of ϱ_n to ϱ. Regarding the errors $E_{n,\infty}(f_0,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$, the results of the Tables 2 and 4 show that $\varrho_n^\star$ converges faster to ϱ than ϱ_n. This suggest, at least for the geometry under consideration, a behavior of the type (95) for the errors $E_{n,\infty}(f_0,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$. As it is predicted by Remark 3.1, in Case (ii) the two errors $\widetilde{E}_{n,2}(K,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$ vanish. This was demonstrated in our experiments, in the sense that the computed errors $\widetilde{E}_{n,2}(K,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$ were zero within machine precision, thus they are not quoted in Tables 3 and 4.

	BKM: $\varrho \approx 1.347$		BKM/AB: $\varrho \approx 2.532$	
n	$E_{n,2}(K,G)$	ϱ_n	$\widetilde{E}_{n,2}(K,G)$	ϱ_n
5	4.4e-01	-	2.7e-02	-
10	1.3e-01	1.47	3.6e-04	2.72
15	3.5e-02	1.41	4.1e-06	2.65
20	8.9e-03	1.39	4.6e-08	2.60
25	2.2e-03	1.38	4.9e-10	2.59
30	5.4e-04	1.37	5.2e-12	2.57
35	1.3e-04	1.36	5.4e-14	2.57

Table 1. BKM approximations to K: Lens-shaped, Case (i).

Case (iii): In this case the conformal map f_0 has a branch point singularity at each of the two corners $\tau_1 = i$ and $\tau_2 = -i$, and therefore Lehman's expansions (16) are valid with $\gamma_1^{(1)} = \gamma_1^{(2)} = 13/2$ and $\gamma_2^{(1)} = \gamma_2^{(2)} = 1 + 1/\alpha = 15/2$. This gives $(2-\alpha)/\alpha = 12$ and $(2-\alpha)(1+1/\alpha) = 180/13 = 13.84\ldots$. Furthermore, it follows from (83) that the nearest singularities of f_0 in the domain Ω, are the two simple poles at $z_1 = \tan(\pi/13)$ and $z_2 = -\tan(\pi/13)$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.119$, and the next singularity occurs at a point z_3, where $|\Phi(z_3)| \approx 2.055$.

Therefore, from Theorem 4.1 we have that,

$$E_{n,2}(K,G) \le c_1\frac{1}{n^{12}} + c_2\frac{1}{R^n}, \tag{97}$$

	BKM: $\varrho \approx 1.347$			BKM/AB: $\varrho \approx 2.532$		
n	$E_{n,\infty}(f_0, G)$	$\varrho_n^\star$	ϱ_n	$\widetilde{E}_{n,\infty}(f_0, G)$	$\varrho_n^\star$	ϱ_n
5	2.5e-01	-	-	1.4e-02	-	-
10	6.8e-02	1.299	1.54	1.3e-04	2.541	3.03
15	1.6e-02	1.331	1.47	1.3e-06	2.528	2.79
20	3.8e-03	1.342	1.43	1.2e-08	2.537	2.71
25	8.5e-04	1.346	1.42	1.1e-10	2.532	2.67
30	1.9e-04	1.347	1.41	1.1e-12	2.532	2.64
35	4.3e-05	1.347	1.40	1.1e-14	2.532	2.62

TABLE 2. BKM approximations to f_0: Lens-shaped, Case (i).

	BKM: $\varrho \approx 1.732$	
n	$E_{n,2}(K, G)$	ϱ_n
4	2.7e-01	-
8	4.0e-02	1.92
12	5.3e-03	1.83
16	6.7e-04	1.80
20	8.3e-05	1.78
24	1.0e-05	1.78
28	1.2e-06	1.77
32	1.4e-07	1.76
36	1.7e-08	1.75

TABLE 3. BKM approximations to K: Lens-shaped, Case (ii).

and

$$E_{n,\infty}(f_0, G) \leq c_3 \sqrt{\log n} \frac{1}{n^{12}} + c_4 \frac{1}{R^n}, \tag{98}$$

where $1 < R < |\Phi(z_1)|$. In order to decide which singular functions to include in the BKM/AB the following estimates, valid for $n = 32$, are relevant; see also Theorem 4.3:

$$\frac{1}{n^{(2-\alpha)/\alpha}} \approx 8.7 \times 10^{-19}, \qquad \frac{1}{|\Phi(z_1)|^n} \approx 2.7 \times 10^{-2},$$
$$\frac{1}{n^{(2-\alpha)(1+1/\alpha)}} \approx 1.4 \times 10^{-21}, \qquad \frac{1}{|\Phi(z_3)|^n} \approx 1.0 \times 10^{-10}.$$

	BKM: $\varrho \approx 1.732$		
n	$E_{n,\infty}(f_0, G)$	$\varrho_n^\star$	ϱ_n
4	1.3e-01	-	-
8	1.6e-02	1.685	2.11
12	1.8e-03	1.718	1.95
16	2.0e-04	1.729	1.89
20	2.3e-05	1.732	1.83
24	2.5e-06	1.732	1.83
28	2.8e-07	1.732	1.81
32	3.1e-08	1.732	1.80
36	3.4e-09	1.732	1.79

Table 4. BKM approximations to f_0: Lens-shaped, Case (ii).

The estimates in the first line indicate that for $n = 32$ (even for bigger values of n), the dominant term in the errors (97) and (98) is $1/R^n$. As it is suggested by the estimate in the second line, we use in our BKM/AB approximations only the singular function $[z/(z^2 - z_1^2)]'$, which takes care of the two symmetric poles at z_1 and z_2, and we include no basis functions reflecting the corner singularities of f_0 on Γ. Then, from Theorem 4.3 we have for the resulting approximations that

$$\widetilde{E}_{n,2}(K, G) \le c_1 \frac{1}{n^{12}} + c_2 \frac{1}{R^n}, \tag{99}$$

$$\widetilde{E}_{n,\infty}(f_0, G) \le c_3 \sqrt{\log n} \frac{1}{n^{12}} + c_4 \frac{1}{R^n}, \tag{100}$$

where $1 < R < |\Phi(z_3)|$.

Below, we present numerical results that illustrate the rates in (97)–(100). In presenting the numerical results we use the following notation:

- ϱ: This denotes the order of approximation (the base of n) in the errors (97)–(100).
- ϱ_n: This denotes the estimate of ϱ, corresponding to n, and is determined as follows: with E_n denoting any of the four errors $E_{n,2}(K, G)$, $\widetilde{E}_{n,2}(K, G)$, $E_{n,\infty}(f_0, G)$ or $\widetilde{E}_{n,\infty}(f_0, G)$ we assume that

$$E_n \approx c \frac{1}{\varrho^n}, \tag{101}$$

and seek to estimate ϱ by means of the formula,

$$\varrho_n = \left(\frac{E_{n-4}}{E_n} \right)^{1/4}. \tag{102}$$

The results quoted in Tables 5 and 6, show the remarkable approximation achieved by the BKM/AB by using as little as 32 monomials. Moreover, they highlight the significance of Theorem 4.3, as it is compared to the estimate (11), in the sense that they confirm fully the theoretical prediction that the two poles at z_1 and z_2 are the most serious singularities of f_0 for small values of n; see also Remark 4.3.

	BKM: $\varrho \approx 1.119$		BKM/AB: $\varrho \approx 2.055$	
n	$E_{n,2}(K,G)$	ϱ_n	$\widetilde{E}_{n,2}(K,G)$	ϱ_n
4	2.8819	-	7.3e-02	-
8	2.3812	1.049	5.6e-03	1.898
12	1.3864	1.145	3.9e-04	1.934
16	0.9188	1.108	2.6e-05	1.965
20	0.5961	1.114	1.7e-06	1.974
24	0.3812	1.118	1.1e-07	1.982
28	0.2413	1.121	7.2e-09	1.981
32	0.1538	1.119	4.6e-10	1.992

Table 5. BKM approximations to K: Lens-shaped, Case (iii).

	BKM: $\varrho \approx 1.119$		BKM/AB: $\varrho \approx 2.055$	
n	$E_{n,\infty}(f_0,G)$	ϱ_n	$\widetilde{E}_{n,\infty}(f_0,G)$	ϱ_n
4	0.8820	-	1.2e-02	-
8	0.3817	1.232	6.1e-04	2.101
12	0.2044	1.170	3.4e-05	2.053
16	0.1180	1.147	2.0e-06	2.029
20	0.0702	1.139	1.2e-07	2.028
24	0.0424	1.135	7.0e-09	2.028
28	0.0259	1.131	4.1e-10	2.028
32	0.0160	1.128	2.5e-11	2.028

Table 6. BKM approximations to f_0: Lens-shaped, Case (iii).

5.2.2. Circular sectors. Let G_α denote the symmetric circular sector of radius 2 and opening angle $\alpha\pi$, $0 < \alpha < 2$, at the origin, i.e.

$$G_\alpha := \left\{ z\colon |z| < 2,\ -\alpha\frac{\pi}{2} < \arg z < \alpha\frac{\pi}{2} \right\}.$$

Let f_0 denote the normalized conformal mapping from G_α onto $D(0, r_0)$, with $f_0(1) = 0$ and $f_0'(1) = 1$. For each value of the parameter α the conformal map $f_0(z)$ can be computed by means of the transformations (see [12, p. 532]):

$$f_0(z) = \left[\frac{2\alpha(4^{1/\alpha} - 1)}{4^{1/\alpha} + 1}\right] \frac{t - d}{td - 1},$$

where

$$t = \left(\frac{iz^{1/\alpha} + 2^{1/\alpha}}{iz^{1/\alpha} - 2^{1/\alpha}}\right)^2 \qquad \text{and} \qquad d = \left(\frac{i + 2^{1/\alpha}}{i - 2^{1/\alpha}}\right)^2.$$

This gives

$$r_0 = \frac{2\alpha(4^{1/\alpha} - 1)}{4^{1/\alpha} + 1} \qquad \text{and} \qquad K(1,1) = \frac{1}{\pi}\left(\frac{4^{1/\alpha} + 1}{2\alpha(4^{1/\alpha} - 1)}\right)^2.$$

The normalized exterior map $\Phi\colon \overline{\mathbb{C}} \setminus \overline{G}_\alpha \to \Delta$ is given, as can be easily verified, by the composition of the following three transformations:

$$\xi(z) := \frac{i(2^{1-1/\alpha}z^{1/\alpha} - 2i)}{2^{1-1/\alpha}z^{1/\alpha} + 2i}, \qquad \arg z \in (-\pi, \pi],$$

$$t(\xi) := \xi^{2/3}, \qquad \arg \xi \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right],$$

$$w(t) := \frac{1 - e^{i\pi/3}t}{t - e^{i\pi/3}}.$$

We consider separately the following two cases:

(i) $\alpha = 1$ (half-disk);
(ii) $\alpha = 3/2$ (three-quarter disk).

Case (i): When $\alpha = 1$, then the domain G_α is the half-disk

$$G_1 = \{z\colon |z| < 2, \operatorname{Re} z > 0\}.$$

In this case the conformal map f_0 has an analytic continuation across Γ into Ω. The nearest singularities of f_0 in Ω, are the two simple poles at $z_1 = -1$ and $z_2 = 4$, where $|\Phi(z_1)| \approx 1.452$ and $|\Phi(z_2)| \approx 2.212$. Accordingly, in our experiments we use the singular function

$$\left[\frac{1}{z - z_1}\right]',$$

which cancels out the nearest pole at z_1. This case is similar to the lens-shaped domain with $\alpha = 1/2$. Hence, the errors $E_{n,2}(K, G)$, $E_{n,\infty}(f_0, G)$, $\widetilde{E}_{n,2}(K, G)$ and $\widetilde{E}_{n,\infty}(f_0, G)$ satisfy respectively (88), (89), (90) and (91). Our purpose here, is to illustrate that the error bounds in (88)–(91) reflect the actual errors. We do so by computing estimates to ϱ_n and $\varrho_n^\star$ of ϱ by using (92)–(96).

In Table 7, the results associated with the errors $E_{n,2}(K, G)$ and $\widetilde{E}_{n,2}(K, G)$ indicate clearly the convergence of ϱ_n to ϱ. Regarding the errors $E_{n,\infty}(f_0, G)$ and

$\widetilde{E}_{n,\infty}(f_0,G)$, the results of Table 8 show that $\varrho_n^\star$ converges faster to ϱ than ϱ_n. This suggest a behavior of the type (95) for $E_{n,\infty}(f_0,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$. In both tables the numbers confirm the remarkable advantage of the BKM/AB over the BKM.

	BKM: $\varrho \approx 1.452$		BKM/AB: $\varrho \approx 2.212$	
n	$E_{n,2}(K,G)$	ϱ_n	$\widetilde{E}_{n,2}(K,G)$	ϱ_n
5	7.1e-02	-	2.8e-02	-
10	1.6e-02	1.55	7.2e-04	2.39
15	2.8e-03	1.54	1.7e-05	2.29
20	5.1e-04	1.49	3.6e-07	2.29
25	8.7e-05	1.49	7.6e-09	2.26
30	1.5e-05	1.48	1.6e-10	2.24
35	2.4e-06	1.47	3.2e-12	2.24
40	4.0e-07	1.47	6.4e-14	2.24
45	6.6e-08	1.47	1.3e-15	2.23
50	1.1e-08	1.46	2.6e-17	2.23

TABLE 7. BKM approximations to K: Half-disk.

	BKM: $\varrho \approx 1.452$			BKM/AB: $\varrho \approx 2.212$		
n	$E_{n,\infty}(f_0,G)$	$\varrho_n^\star$	ϱ_n	$\widetilde{E}_{n,\infty}(f_0,G)$	$\varrho_n^\star$	ϱ_n
5	1.1e-01	-	-	4.0e-02	-	-
10	2.2e-02	1.401	1.64	7.7e-04	2.20	2.62
15	3.4e-03	1.446	1.60	1.5e-05	2.20	2.42
20	5.3e-04	1.450	1.55	2.8e-07	2.22	2.37
25	8.3e-05	1.452	1.53	5.2e-09	2.22	2.34
30	1.3e-05	1.452	1.51	9.8e-11	2.21	2.31
35	2.0e-06	1.452	1.51	1.8e-12	2.22	2.30
40	3.1e-07	1.452	1.50	3.5e-14	2.20	2.27
45	4.8e-08	1.452	1.49	6.6e-16	2.21	2.27
50	7.4e-09	1.452	1.49	1.2e-17	2.22	2.27

TABLE 8. BKM approximations to f_0: Half-disk.

Case (ii): In this case f_0 has a branch point singularity at the point $\tau_1 = 0$ with

$$f_0(z) = f_0(0) + \sum_{j=1}^{\infty} a_j z^{j/\alpha}, \qquad a_1 \neq 0,$$

valid for z close to 0. The nearest singularity of f_0 in Ω is a simple pole at $z_1 = 4$, where $|\Phi(z_1)| \approx 2.04$. For the application of BKM, Theorem 4.1 gives that

$$E_{n,2}(K,G) \leq c_1 \frac{1}{n^{1/3}} + c_2 \frac{1}{R^n}, \tag{103}$$

and

$$E_{n,\infty}(f_0,G) \leq c_3 \sqrt{\log n} \frac{1}{n^{1/3}} + c_4 \frac{1}{R^n}, \tag{104}$$

where $1 < R < |\Phi(z_1)|$.

Since $1/|\Phi(z_1)|^{50} \approx 3.3 \times 10^{-16}$, and in view of Theorem 4.3, we include in our basis only singular functions that reflect the branch point singularity of f_0 at τ_1. More precisely, in order to keep the contribution of both sources of error balanced, we choose to use the first 15 singular function of the form

$$z^{j/\alpha - 1},$$

where $j/\alpha \notin \mathbb{N}$. This gives $s^\star = 23/3$ in Theorem 4.3, and hence the following estimates for the errors in the resulting BKM/AB approximations,

$$\widetilde{E}_{n,2}(K,G) \leq c_1 \frac{1}{n^{23/3}} + c_2 \frac{1}{R^n}, \tag{105}$$

and

$$\widetilde{E}_{n,\infty}(f_0,G) \leq c_3 \sqrt{\log n} \frac{1}{n^{23/3}} + c_4 \frac{1}{R^n}, \tag{106}$$

where $1 < R < |\Phi(z_1)|$.

Below, we present numerical results that illustrate the rates in (105)–(106), where we use the following notation:

- σ: This denotes the exponent of $1/n$ in the errors (105)–(106).
- σ_n: This denotes the estimate of σ corresponding to n, and is determined as follows: with E_n denoting any of the two errors $E_{n,2}(K,G)$, $\widetilde{E}_{n,2}(K,G)$, we assume that

$$E_n \approx c \frac{1}{n^\sigma}$$

  and seek to estimate σ by means of the formula

$$\sigma_n = \frac{\log\left(\frac{E_{n-5}}{E_n}\right)}{\log\left(\frac{n}{n-5}\right)}.$$

If E_n denotes either of the two errors $E_{n,\infty}(f_0, G)$ or $\widetilde{E}_{n,\infty}(f_0, G)$, then we assume that

$$E_n \approx c\sqrt{\log n}\frac{1}{n^\sigma},$$

and seek to estimate σ by means of the formula

$$\sigma_n = \frac{\log\left(\frac{E_{n-5}}{E_n}\right) - \frac{1}{2}\log\left(\frac{\log(n-5)}{\log n}\right)}{\log\left(\frac{n}{n-5}\right)}.$$

In addition, we check a behavior of the form (101) for the errors $\widetilde{E}_{n,2}(K, G)$ and $\widetilde{E}_{n,\infty}(f_0, G)$, by computing ϱ_n as in (102), with 5 in the place of 4.

Our purpose here is to show that the change of the dominant term in both (105) and (106) can actually be detected in the computed errors. This is indeed the case in the results quoted in Table 9. More precisely, the results associated with the errors $\widetilde{E}_{n,2}(K, G)$ and $\widetilde{E}_{n,\infty}(f_0, G)$ indicate the convergence of ϱ_n to ϱ for values of n up to 50 and the convergence of σ_n to σ for values larger than 50. Furthermore, the results show that the two constants c_1 and c_2 in (105) and c_3 and c_4 in (106) are, respectively, of the same magnitude.

	BKM/AB: $\sigma \approx 7.67$ $\varrho \approx 2.04$					
n	$\widetilde{E}_{n,2}(K, G)$	σ_n	ϱ_n	$\widetilde{E}_{n,\infty}(f_0, G)$	σ_n	ϱ_n
20	7.2e-05	-	-	8.2e-05	-	-
25	1.6e-05	6.74	1.35	1.5e-05	7.62	1.40
30	2.9e-06	9.31	1.41	2.6e-06	9.70	1.42
35	2.2e-07	16.84	1.67	1.8e-07	17.37	1.71
40	1.0e-08	22.81	1.84	7.7e-09	23.44	1.86
45	4.1e-10	27.41	1.90	2.8e-10	28.17	1.94
50	1.3e-11	32.83	1.99	1.0e-11	31.25	1.95
55	7.5e-12	5.84	1.12	5.3e-12	7.05	1.14
60	2.6e-12	11.84	1.23	2.0e-12	11.20	1.22
65	1.3e-12	8.68	1.15	9.9e-13	8.73	1.15
70	7.4e-13	7.50	1.12	5.9e-13	7.04	1.11
75	4.4e-13	7.58	1.11	3.5e-13	7.57	1.11
80	2.7e-13	7.66	1.10	2.1e-13	7.62	1.10

TABLE 9. BKM approximations to f_0 and K: 3/4-disk.

5.3. Rates of decrease of the Bergman polynomials. First, we present results illustrating the rate of decrease of the sequence $\{P_n(1)\}$ for the circular sector considered in Section 5.2.2, with $\alpha = 2/5$. In this case, the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{2i\pi/5}$, $z_2 = e^{-2i\pi/5}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.145$. From the proof of Corollary 3.1 and (103) we have that

$$|P_n(z_0)| \le \|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)} \le c_1 \frac{1}{n^{(2-\alpha)/\alpha}} + c_2 \frac{1}{R^n}, \tag{107}$$

where $1 < R < |\Phi(z_1)|$. Accordingly, we check to detect the decay in the following two forms:

$$|P_n(1)| \approx c\frac{1}{\varrho^n} \qquad \text{and} \qquad |P_n(1)| \approx c\frac{1}{n^\sigma}, \tag{108}$$

with $\varrho = |\Phi(z_1)|$ and $\sigma = (2-\alpha)/\alpha + 1/2$ (cf. the remark in [12, pp. 530–531]). We do so, by estimating ϱ and σ, respectively, by means of the formulas

$$\varrho_n = \left(\frac{|P_{n-10}(1)|}{|P_n(1)|}\right)^{1/10},$$

and

$$\sigma_n = \frac{\log\left(\frac{|P_{n-10}(1)|}{|P_n(1)|}\right)}{\log\left(\frac{n}{n-10}\right)}.$$

The results listed in Table 10 show clearly the transition from one dominant term to the other in (107) for values of n around 50.

	$\sigma = 4.5$		$\varrho \approx 1.145$
n	$\lvert P_n(1)\rvert$	σ_n	ϱ_n
10	2.6e-02	-	-
20	1.2e-03	4.51	1.37
30	7.6e-06	12.38	1.65
40	1.7e-06	5.14	1.16
50	4.0e-07	6.57	1.16
60	1.8e-07	4.35	1.08
70	9.1e-08	4.49	1.07
80	5.0e-08	4.50	1.06
90	2.9e-08	4.50	1.05
100	1.8e-08	4.50	1.05

TABLE 10. Rate of decrease of $|P_n(1)|$: Circular sector, $\alpha = 2/5$.

We end, by presenting results that illustrate the rate of decrease of the augmented sequence $\{\widetilde{P}_n(1)\}$, for the circular sector considered in Section 5.2.2, where now we consider the two cases $\alpha = 3/4$ and $\alpha = 4/5$. When $\alpha = 3/4$, then the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{3i\pi/4}$ and $z_2 = e^{-3i\pi/4}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.349$. When $\alpha = 4/5$, then the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{4i\pi/5}$ and $z_2 = e^{-4i\pi/5}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.372$. In both cases, we construct the sequence $\{\widetilde{P}_n(z)\}$ by augmenting the monomial basis functions with the singular function

$$z^{1/\alpha-1},$$

which reflects the branch point singularity of f_0 at $\tau_1 = 0$, and we seek to detect the decay of the sequence $\{\widetilde{P}_n(1)\}$ in the form

$$|\widetilde{P}_n(1)| \approx c\frac{1}{n^\sigma},$$

where, in view of Theorem 4.3 and [12, pp. 530–531], $\sigma = 2(2-\alpha)/\alpha + 1/2$. As above, we estimate σ by means of the formula

$$\sigma_n = \frac{\log\left(\frac{|\widetilde{P}_{n-10}(1)|}{|\widetilde{P}_n(1)|}\right)}{\log\left(\frac{n}{n-10}\right)}.$$

The results listed in Tables 11 and 12 indicate clearly the convergence of σ_n to the predicted value of σ, indicating that the argument in [12, pp. 530–531] applies also to the case of the augmented Bergman polynomials.

	$\sigma \approx 3.833$	
n	$\lvert\widetilde{P}_n(1)\rvert$	σ_n
10	2.8e-03	-
20	7.2e-05	5.30
30	1.1e-05	4.60
40	3.2e-06	4.36
50	1.3e-06	3.86
60	6.6e-07	3.89
70	3.6e-07	3.89
80	2.2e-07	3.88
90	1.4e-07	3.88
100	9.1e-08	3.87

TABLE 11. Rate of decrease of $|\widetilde{P}_n(1)|$: Circular sector, $\alpha = 3/4$.

	$\sigma = 3.5$	
n	$\lvert\widetilde{P}_n(1)\rvert$	σ_n
10	2.3e-03	-
20	2.7e-04	3.10
30	2.2e-05	6.17
40	8.7e-06	3.20
50	3.8e-06	3.72
60	2.0e-06	3.66
70	1.1e-06	3.63
80	6.9e-07	3.61
90	4.5e-07	3.59
100	3.1e-07	3.58

TABLE 12. Rate of decrease of $|\widetilde{P}_n(1)|$: Circular sector, $\alpha = 4/5$.

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