Domain Decomposition for Conformal Maps

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Abstract. This paper is concerned with certain aspects of the theory and application of a domain decomposition method for computing the conformal modules of long quadrilaterals. Our main purpose is to make use of the theory of the method in order to investigate the quality of certain heuristic rules that we have come across in the VLSI literature, in connection with the measurement of resistance values of integrated circuit networks.

1 Introduction

The work of this paper is concerned with certain aspects of the theory and application of a domain decomposition method (DDM) for computing the conformal modules of long quadrilaterals. The DDM was introduced by us [16], [17] for the purpose of computing the conformal modules and associated conformal maps of a special class of quadrilaterals, *i.e.* quadrilaterals that are bounded by two parallel straight lines and two Jordan arcs. In this context, the method was also studied by Gaier and Hayman [3], [4], in connection with the computation of conformal modules, and more recently by Laugesen [13], in connection with the determination of the conformal maps. These three papers contain several important results that enhance considerably the associated DDM theory. In particular, the results of Gaier and Hayman provided us with the necessary tools for extending the area of application of the DDM to a much wider class of quadrilaterals [18], [19].

Our objectives in this paper are as follows: (a) to present a number of results that improve somewhat the available DDM error analysis, (b) to consider the method from a more practical point of view, by studying certain engineering applications. More specifically, our main objective is to make use of the DDM theory in order to investigate the quality of certain heuristic rules that we have come across in the VLSI[1] literature [6], [7], [11], [14], [22], in connection with the measurement of resistance values of integrated circuit networks. We do this in Section 3. In addition, in Section 2 we show how a result which is given as a "note added in proof" in [3], p. 467 can be used to improve some of the DDM error estimates that we derived in our two earlier papers [18], [19].

[1]VLSI stands for *Very Large Scale Integration*. It is a subject that covers a broad area of study including semiconductor devices and proccesing, integrated electronic circuits, digital logic, design disciplines and tools for creating complex systems.

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For the sake of completeness, we list below several facts regarding the properties of conformal modules and the application of the DDM that are needed for our work in Sections 2 and 3. Further details of all these can be found in [15], [18], [19] and the other references cited there.

Remark 1: A system consisting of a Jordan domain Ω and four points z_j, $j = 1,2,3,4$, in counterclockwise order on $\partial\Omega$ is said to be a quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$. The conformal module $m(Q)$ of Q is defined as follows: Let $R_H := \{(\xi,\eta) : 0 < \xi < 1,\ 0 < \eta < H\}$. Then, $m(Q)$ is the unique value of H for which Q is conformally equivalent to the rectangular quadrilateral $\{R_H; 0, 1, 1+\mathrm{i}H, \mathrm{i}H\}$, in the sense that for $H = m(Q)$ and for this value only there exists a unique conformal map $F : \Omega \to R_H$ that takes the four points z_1, z_2, z_3, z_4 respectively onto the four vertices $0, 1, 1+\mathrm{i}H, \mathrm{i}H$ of R_H.

Remark 2: Assume that $\partial\Omega$ is piecewise analytic and let Δ and $\partial/\partial n$ denote respectively the two-dimensional Laplace operator and differentiation in the direction of the outward normal to $\partial\Omega$. Then we call the mixed boundary value problem

$$\begin{array}{rcll} \Delta u &=& 0 & \text{in } \Omega, \\ u &=& 0 & \text{on } (z_1, z_2), \\ u &=& 1 & \text{on } (z_3, z_4), \\ \frac{\partial u}{\partial n} &=& 0 & \text{on } (z_2, z_3) \cup (z_4, z_1), \end{array} \tag{1.1}$$

the harmonic problem associated with the quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$. It is easy to see that the solution of this problem is

$$u = \frac{1}{m(Q)}\, \Im F,$$

where F is the conformal map $\Omega \to R_H$.

Remark 3: The above observation leads to the following physical interpretation of $m(Q)$. Let Ω represent a thin plate of homogeneous electrically conducting material of specific resistance 1, and suppose that constant voltages are applied on the two boundary segments (z_1, z_2) and (z_3, z_4) while the remainder of $\partial\Omega$ is insulated. Then, the conformal module $m(Q)$, of the quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$, gives the resistance of the conducting plate. This physical interpretation of $m(Q)$ provides the link between the problems of computing conformal modules and of measuring resistance values of electrical networks.

Remark 4: The domain decomposition method (DDM) is based on decomposing the original quadrilateral Q into two or more component quadrilaterals Q_j and approximating $m(Q)$ by the sum $\sum_j m(Q_j)$ of the modules of the component quadrilaterals. The objectives for doing this are: (a) to overcome the crowding difficulties associated with the problem of computing the modules of long quadrilaterals, *i.e.* the difficulties associated with the conventional approach of seeking to determine $m(Q)$ by going via the unit disc or the half plane (see *e.g.* [15], p. 67-68 and [19] §1), (b) to take advantage of the fact that in many applications a complicated original quadrilateral Q can be decomposed into very simple components Q_j.

Remark 5: In general, $m(Q) \geq \sum_j m(Q_j)$ and equality occurs only in the special case where the crosscuts of subdivision, *i.e.* the crosscuts that subdivide Q into its components Q_j, are equipotentials of the harmonic problem (1.1) associated with Q. In other words, $m(Q) = \sum_j m(Q_j)$ only in the case where the images of these crosscuts, under the conformal map $F : \Omega \rightarrow R_H$, are straight lines parallel to the real axis.

Remark 6: In view of the above remark, the DDM consists of the following three steps:

Step 1 Determining appropriate crosscuts of subdivision, *i.e.* crosscuts that are "near" equipotentials of the associated harmonic problem.

Step 2 Computing the modules $m(Q_j)$ of the component quadrilaterals Q_j.

Step 3 Estimating the error in the DDM approximation $\tilde{m}(Q) = \sum_j m(Q_j)$.

Remark 7: In connection with the above, Steps 1 and 3 are closely related and involve the use of the available DDM theory given in [3], [4] and [18], [19]. As for Step 2, in some cases the modules of the component quadrilaterals are known exactly and can be written down immediately (see *e.g.* [18], Remark 2.4 and Section 3 of this paper). In some other cases, when Q_j is bounded by two straight lines and two Jordan arcs, $m(Q_j)$ can be computed by mapping directly the domain Ω_j onto the rectangle $R_{m(Q_j)}$ using the simple and efficient Garrick iterative algorithm described in [5] (see *e.g.* [16], [17], [18], Ex. 5.1, Ex. 5.2 and [19], Ex. 3.1). More generally, since by construction the component quadrilaterals are not long (and consequently the effects of crowding are not serious), the modules $m(Q_j)$ can be computed by first mapping the domain Ω_j onto the unit disc using one of the available computer packages for numerical conformal mapping such as: (a) the Schwarz-Christoffel package SCPACK of Trefethen [20] (see *e.g.* [18], Ex. 5.2, Ex. 5.3), (b) the integral equation package CONFPACK of Hough [10] (see *e.g.* [19], Ex. 3.3–Ex. 3.5), and (c) the orthonormalization package BKMPACK of Warby [21].

Remark 8: In VLSI applications the quadrilaterals under consideration are bounded typically by straight lines inclined at angles of 90^o and 45^o. For this reason, all the numerical examples of this paper involve polygonal quadrilaterals with corners of 90^o and 45^o. It should be noted, however, that the DDM can also be applied to quadrilaterals with curved boundaries (see *e.g.* [19], Ex. 3.3 – Ex. 3.5).

Remark 9: We shall adopt throughout the notations that we used in our earlier papers [18], [19]. That is:

- Ω and $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ will denote respectively the original domain and corresponding quadrilateral.
- $\Omega_1, \Omega_2, \cdots$, and $Q_1, Q_2, \cdots$, will denote respectively the "principal" subdomains and corresponding quadrilaterals of the decomposition under consideration.
- The additional subdomains and associated quadrilaterals that arise when the decomposition of Q involves more than one crosscuts will be denoted by using (in an obvious manner) a multisubscript notation.

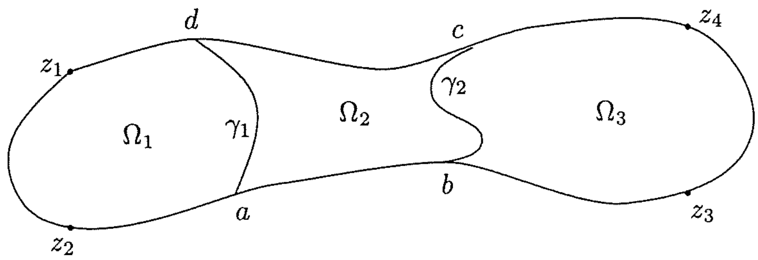


Figure 1.

For example, the five component quadrilaterals of the decomposition illustrated in Figure 1 are:

$$Q_1 := \{\Omega_1; z_1, z_2, a, d\}, \quad Q_2 := \{\Omega_2; d, a, b, c\}, \quad Q_3 := \{\Omega_3; c, b, z_3, z_4\}$$

and

$$Q_{1,2} := \{\Omega_{1,2}; z_1, z_2, b, c\}, \quad Q_{2,3} := \{\Omega_{2,3}; d, a, z_3, z_4\},$$

where

$$\bar{\Omega}_{1,2} := \bar{\Omega}_1 \cup \bar{\Omega}_2, \quad \bar{\Omega}_{2,3} := \bar{\Omega}_2 \cup \bar{\Omega}_3.$$

2 Improved error estimates

We first state three results due to Gaier and Hayman [3], [4].

Result 1 *Consider the decomposition illustrated in Figure 2, where:*

(i) *the domain Ω of the original quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ is bounded by two segments of the lines $x = 0$ and $x = 1$ and two Jordan arcs γ_1, γ_2,*

(ii) *the points z_1, z_2, z_3, z_4 are the four corners where the arcs γ_1, γ_2 meet the lines $x = 0$ and $x = 1$,*

(iii) *the crosscut of subdivision l is a straight line parallel to the real axis.*

Let $h := \min(h_1, h_2)$, where h_1, h_2 are respectively the distances of l from the arcs γ_1, γ_2. Then

$$0 \leq m(Q) - \{m(Q_1) + m(Q_2)\} \leq 0.761e^{-2\pi h}, \tag{2.1}$$

provided that $h \geq 1$.

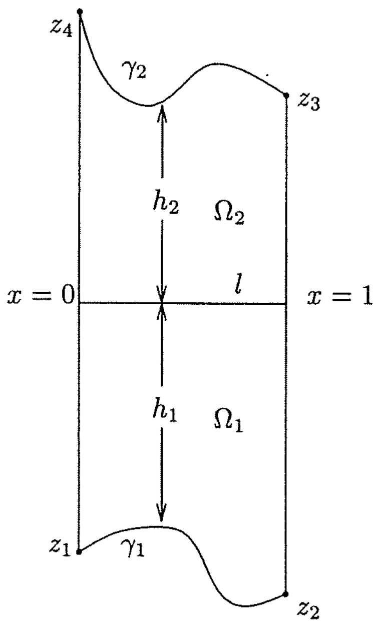


Figure 2.

Result 2 *If in Figure 2 the boundary arc γ_1 is a straight line parallel to the real axis (so that $m(Q_1) = h_1$), i.e. if the decomposition is of the form illustrated in Figure 3, then*

$$0 \le m(Q) - \{h_1 + m(Q_2)\} \le \frac{1}{2} \times 0.381 e^{-2\pi h_2}, \tag{2.2}$$

provided that $h_2 \ge 1$.

Result 3 *For the decomposition illustrated in Figure 3 we also have that*

$$0 \le m(Q) - \{h_1 + m(Q_2)\} \le \frac{4}{\pi} e^{-2\pi m(Q_2)}, \tag{2.3}$$

provided that $m(Q_2) \ge 1$. Here $4/\pi$ cannot be replaced by a smaller number.

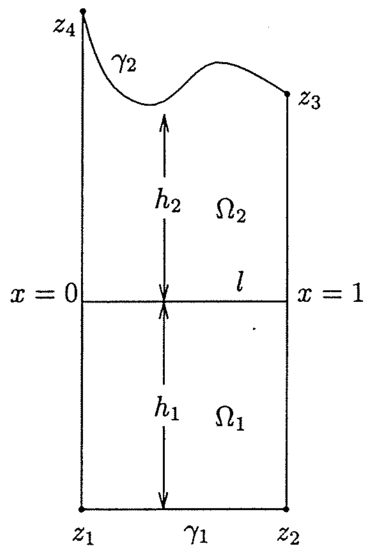


Figure 3.

As was previously remarked, all the above results are due to Gaier and Hayman [3], [4]: (2.1) and (2.2) follow easily from the precise estimates given in [4] (see [18], p. 218), whilst (2.3) is given as a "note added in proof" in [3], p. 467.

For comparison purposes, we note the following in connection with the two estimates given in Results 2 and 3:

Apart from the trivial case, where γ_2 is a straight line parallel to the real axis (so that $m(Q_2) = h_2$) we have that:

$$m(Q_2) = h_2 + c,$$

for some $c > 0$. Therefore, the condition $m(Q_2) \geq 1$ needed for estimate (2.3) is less restrictive than the corresponding condition $h_2 \geq 1$ needed for (2.2). On the other hand, (2.3) gives a sharper bound for the DDM error only if

$$c > \frac{1}{2\pi}\ln\{8/0.381\pi\} = 0.302\ 341\ 48\ \cdots.$$

Thus, it is not always advantageous to use (2.3), rather than (2.2), for estimating the DDM error of special decompositions of the form illustrated in Figure 3. However, by making use of (2.3) it is possible to improve some of the results derived in [18], [19] for the purpose of estimating the errors of more general decompositions. The details of these improvements are as follows:

By making use of (2.3) (rather than (2.2)) in the proof of Theorem 3.2 of [18], p. 222, we can state the corresponding result in improved form as follows:

Result 4 *Consider the decomposition of the quadrilateral* $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ *illustrated in Figure 1 and assume that the crosscut* γ_1 *is an equipotential of the harmonic problem associated with* Q *(see Remark 2). Then*

$$-\frac{4}{\pi}e^{-2\pi m(Q_2)} \leq m(Q) - \{m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2)\} \leq 0, \tag{2.4}$$

provided that $m(Q_2) \geq 1$.

Further, by making use of Result 4 we can state Corollary 2.6 of [19] in the following improved form:

Result 5 *Consider a quadrilateral* $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ *of the form illustrated in Figure 4, and assume that the defining domain* Ω *can be decomposed by a straight line crosscut* l *into* Ω_1 *and* Ω_2*, so that* Ω_2 *is the reflection in* l *of some subdomain of* Ω_1*. Then, for the decomposition defined by* l*,*

$$0 \leq m(Q) - \{m(Q_1) + m(Q_2)\} \leq \frac{4}{\pi}e^{-2\pi m(Q_2)}, \tag{2.5}$$

provided that $m(Q_2) \geq 1$.

The improvements in the above two results are the constant $4/\pi$ (in the left hand side of (2.4) and the right hand side of (2.5)) and the condition $m(Q_2) \geq 1$. These replace respectively the larger constant 4.41 and the more restrictive condition $m(Q_2) \geq 1.5$ involved in the original versions of the results (see [18], Theorem 3.2 and [19], Theorem 2.2, Corollary 2.6).

Finally, by making use of (2.5), we can state Corollary 2.7 of [19] in the following improved form:

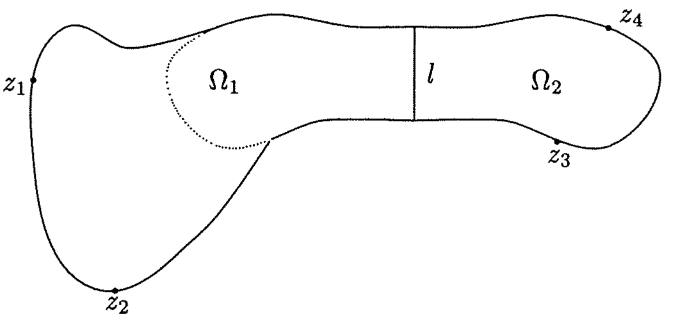


Figure 4.

Result 6 *Consider a quadrilateral* $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ *of the form illustrated in Figure 5, and assume that the defining domain* Ω *can be decomposed by means of a straight line crosscut* l *and two other crosscuts* γ_1 *and* γ_2 *into four subdomains* Ω_1*,* Ω_2*,* Ω_3 *and* Ω_4*, so that* Ω_3 *is the reflection in* l *of* Ω_2*. Then, for the decomposition of* Q *defined by* l*,*

$$0 \leq m(Q) - \{m(Q_{1,2}) + m(Q_{3,4})\} \leq 11.37e^{-2\pi m(Q_2)},$$

provided that $m(Q_2) \geq 1.5$*.*

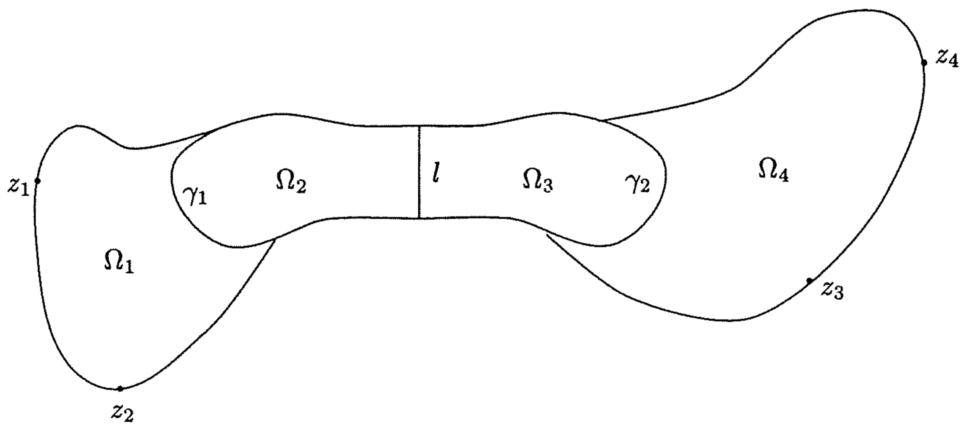


Figure 5.

The improvement here is the constant 11.37, which replaces the constant 17.64 involved in the original version of the result.

We consider next the following useful inequality:

Result 7 *Consider the decomposition of the quadrilateral* $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ *illustrated in Figure 1, and assume that the crosscut* γ_2 *is an equipotential of the harmonic problem associated with the quadrilateral* $Q_{2,3}$. *Then,*

$$0 \le m(Q_{1,2}) - \{m(Q_1) + m(Q_2)\} \le m(Q) - \{m(Q_1) + m(Q_{2,3})\}. \qquad (2.6)$$

Proof: The well-known composition law for conformal modules implies that

$$m(Q_{1,2}) + m(Q_3) \le m(Q).$$

Hence,

$$m(Q_{1,2}) + m(Q_3) - \{m(Q_1) + m(Q_{2,3})\} \le m(Q) - \{m(Q_1) + m(Q_{2,3})\}$$

and the result follows because the hypothesis about γ_2 implies that

$$m(Q_{2,3}) = m(Q_2) + m(Q_3). \qquad \blacksquare$$

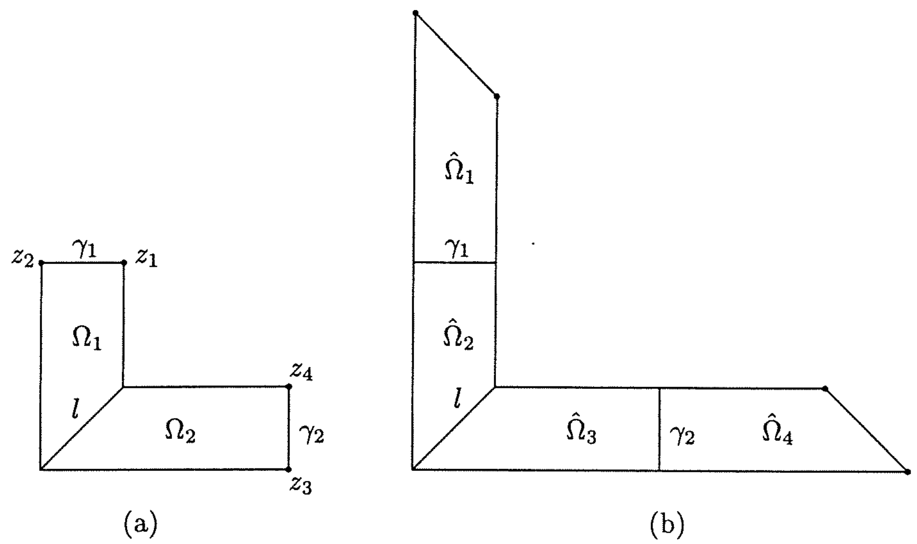


Figure 6.

As an example of the usefulness of the above result, consider the decomposition of the L-shaped quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ illustrated in Figure 6(a), and let

$$h := \min\{m(Q_1), m(Q_2)\}.$$

Then the direct application of Result 5 gives that

$$0 \le m(Q) - \{m(Q_1) + m(Q_2)\} \le \frac{4}{\pi} e^{-2\pi h},$$

provided $h \geq 1$. It is, however, possible to obtain a much sharper estimate by making use of the inequality (2.6) as follows:

Reflect Ω_1 in γ_1 and Ω_2 in γ_2 and denote the resulting quadrilateral, illustrated in Figure 6(b), by $\hat{Q}$. Then, by construction,

$$m(\hat{Q}_1) = m(\hat{Q}_2) = m(Q_1), \; m(\hat{Q}_3) = m(\hat{Q}_4) = m(Q_2), \; m(\hat{Q}_{2,3}) = m(Q),$$

and the lines γ_1 and γ_2 are equipotentials of the harmonic problems associated with the quadrilaterals $\hat{Q}_{1,2}$ and $\hat{Q}_{3,4}$ respectively. Therefore, by applying the inequality of Result 7 twice, we get

$$\begin{aligned} 0 \leq m(Q) - \{m(Q_1) + m(Q_2)\} &= m(\hat{Q}_{2,3}) - \{m(\hat{Q}_2) + m(\hat{Q}_3)\} \\ &\leq m(\hat{Q}_{1,2,3}) - \{m(\hat{Q}_{1,2}) + m(\hat{Q}_3)\} \leq m(\hat{Q}) - \{m(\hat{Q}_{1,2}) + m(\hat{Q}_{3,4})\}, \end{aligned}$$

where, by making use of Result 5,

$$m(\hat{Q}) - \{m(\hat{Q}_{1,2}) + m(\hat{Q}_{3,4})\} \leq \frac{4}{\pi} e^{-2\pi H},$$

provided that

$$H := \min\{m(\hat{Q}_{1,2}), m(\hat{Q}_{3,4})\} \geq 1.$$

Therefore, since $m(\hat{Q}_{1,2}) = 2m(Q_1)$ and $m(\hat{Q}_{3,4}) = 2m(Q_2)$, we have the following substantially improved estimate for the decomposition of the L-shaped quadrilateral of Figure 6(a):

$$0 \leq m(Q) - \{m(Q_1) + m(Q_2)\} \leq \frac{4}{\pi} e^{-4\pi h}, \tag{2.7}$$

provided that $h := \min\{m(Q_1), m(Q_2)\} \geq 0.5$.

As might be expected, by using the results of this section we can improve somewhat the DDM error estimates that we derived in some of the examples of our earlier papers [18], [19]. This can be achieved by following, in each example, the same error estimation technique and replacing, where appropriate, the basic error estimates by their improved counterparts. For example, by considering the decomposition of the spiral quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ illustrated in Figure 7 and applying the error estimation technique described in [19], Ex. 3.2, but with the estimate (2.5) of Result 5 (rather than its original version given in [19], Corollary 2.6), we find that

$$132.704\,539\,3 < m(Q) < 132.704\,540\,3. \tag{2.8}$$

This improves somewhat our previous estimate

$$132.704\,539 < m(Q) < 132.704\,543,$$

and provides further evidence that the approximation

$$m(Q) \approx 132.704\,54,$$

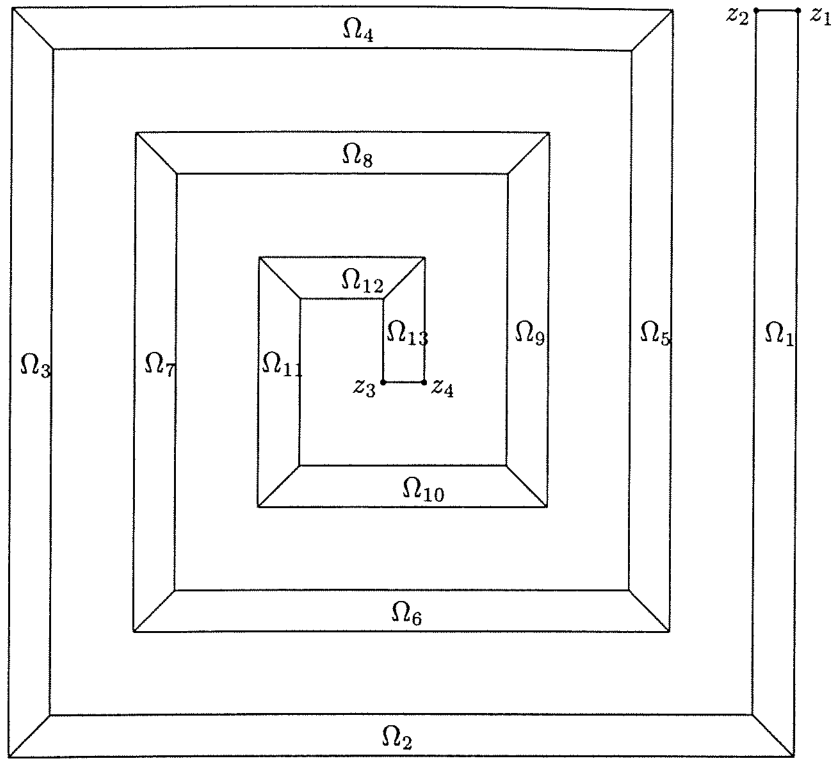


Figure 7.

which was obtained by Howell and Trefethen in [12], p. 943, using a modified Schwarz-Christoffel technique is, in fact, correct to all the figures given.

We consider next another example taken from the paper of Howell and Trefethen [12]. This is the quadrilateral $Q := \{\Omega, z_1, z_2, z_3, z_4\}$ of Figure 8, for which the modified Schwarz-Christoffel technique gives the approximation

$$m(Q) \approx 49.436\ 547,$$

(see [12], p. 944).

For the application of the DDM we consider the decomposition illustrated in Figure 8 and note that the module of the component quadrilateral Q_1 is known exactly in terms of elliptic functions and integrals (see (3.11)). Thus, to eight decimal places,

$$m(Q_1) = 3.469\ 394\ 16.$$

Also,

$$m(Q_2) = 6, \quad m(Q_4) = 30.5,$$

and by using the subroutine RESIST of the Schwarz-Christoffel package SC-PACK of Trefethen [20] we find that $m(Q_3)$ and $m(Q_5)$ are given correct to eight decimal places by

$$m(Q_3) = 4.970\ 221\ 11 \quad \text{and} \quad m(Q_5) = 4.496\ 930\ 26.$$

Therefore, the DDM approximation to $m(Q)$ is

$$\tilde{m}(Q) = \sum_{j=1}^{5} m(Q_j) = 49.436\ 545\ 53. \tag{2.9}$$

For the error in (2.9), the repeated use of Result 6 gives:

$$\begin{aligned}
0 &\le m(Q) - \{m(Q_1) + m(Q_{2,\ldots,5})\} \le 11.37e^{-6\pi},\\
0 &\le m(Q_{2,\ldots,5}) - \{m(Q_2) + m(Q_{3,4,5})\} \le 11.37e^{-4\pi},\\
0 &\le m(Q_{3,4,5}) - \{m(Q_3) + m(Q_{4,5})\} \le 11.37e^{-4\pi},\\
0 &\le m(Q_{4,5}) - \{m(Q_4) + m(Q_5)\} \le 11.37e^{-5\pi}.
\end{aligned}$$

Thus,

$$0 \le m(Q) - \sum_{j=1}^{5} m(Q_j) \le E, \tag{2.10}$$

where

$$E := 11.37\{2e^{-4\pi} + e^{-5\pi} + e^{-6\pi}\} < 8.11 \times 10^{-5}. \tag{2.11}$$

Therefore, from (2.9),

$$49.436\ 545 < m(Q) < 49.436\ 627.$$

We end this section by considering two quadrilaterals that we shall need, for comparison purposes, in the next section.

With reference to Figure 8, let $Q := \{\Omega; z_1, z_2, z_3, \hat{z}_4\}$, *i.e.* Q is the same quadrilateral as the one considered above except that we now take 10i, rather than 8i, as the fourth specified point. Then, the modules of all the component quadrilaterals of the decomposition remain as before, except for $m(Q_5)$ for which the subroutine RESIST of [20] gives

$$m(Q_5) = 4.719\ 790\ 96.$$

Hence, the DDM approximation to $m(Q)$ is

$$\tilde{m}(Q) = \sum_{j=1}^{5} m(Q_j) = 49.659\ 406\ 23$$

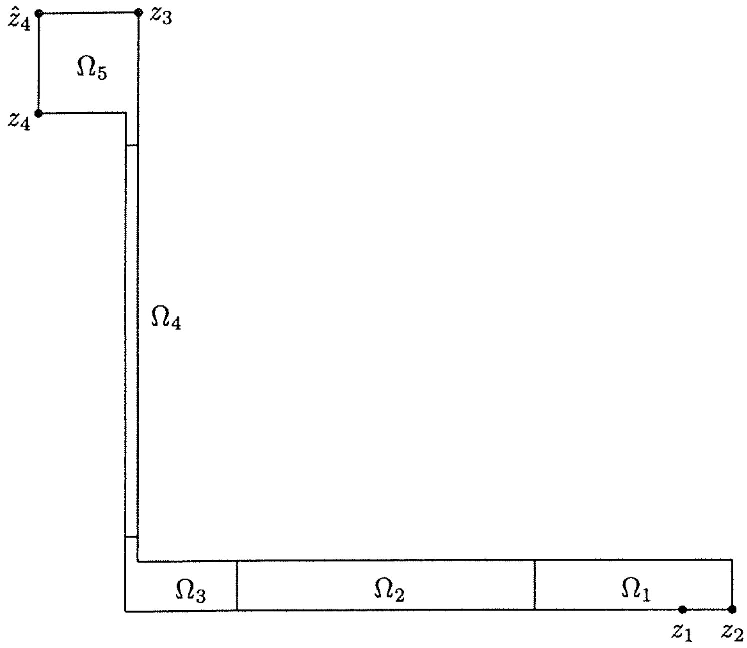


Figure 8. The coordinates of the corners, starting from z_2 and moving in counter-clockwise order, are (14.,0.), (14.,1.), (2.,1.), (2.,10.), (0.,10.), (0.,8.), (1.8,8.), (1.8,0.).

and, as before, the corresponding DDM error is given by (2.10) and (2.11). Therefore, with the notations of Figure 8, if $Q := \{\Omega; z_1, z_2, z_3, \hat{z}_4\}$, then

$$49.659\,406 < m(Q) < 49.659\,488. \tag{2.12}$$

Finally, we consider the quadrilateral $Q := \{\Omega, z_1, z_2, z_3, z_4\}$ illustrated in Figure 9, and note that the modules of the trapezoidal component quadrilaterals Q_5, Q_6, Q_7 and Q_8 are known exactly in terms of elliptic integrals (see *e.g.* [18], Remark 2.4, and Section 3 of this paper). Thus, to eight decimal places,

$$\begin{aligned} m(Q_5) &= 2.279\,364\,21, \\ m(Q_6) = m(Q_7) &= 5.279\,364\,40, \\ m(Q_8) &= 3.279\,364\,40. \end{aligned}$$

Also,

$$m(Q_2) = 2,$$

and by using again the subroutine RESIST of [20] we find that $m(Q_1)$, $m(Q_3)$, $m(Q_4)$ and $m(Q_9)$ are given correct to eight decimal places by

$$m(Q_1) = 3.810\,473\,74,$$

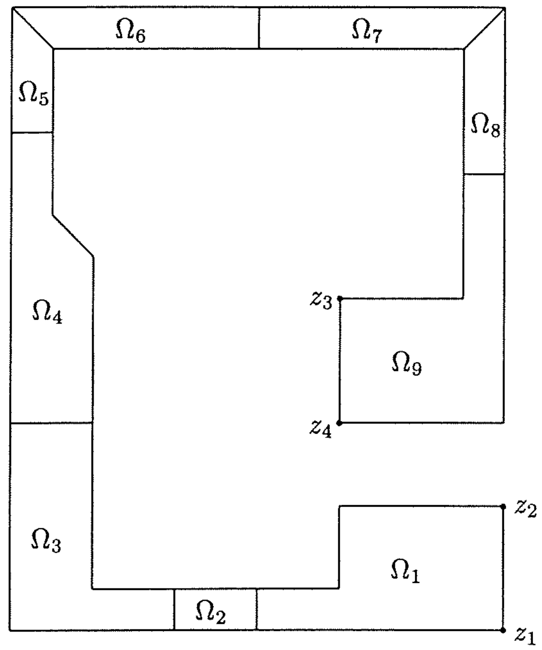


Figure 9. The coordinates of the corners, starting from z_1 and moving in counter-clockwise order, are (12,0), (12,3), (8,3), (8,1), (2,1), (2,9), (1,10), (1,14), (11,14), (11,8) (8,8), (8,5), (12,5), (12,15), (0,15), (0,0).

$$m(Q_3) = 4.643\ 536\ 66,$$
$$m(Q_4) = 4.834\ 640\ 84,$$
$$m(Q_9) = 4.763\ 034\ 81.$$

Therefore, the DDM approximation to $m(Q)$ is

$$\tilde{m}(Q) = \sum_{j=1}^{9} m(Q_j) = 36.169\ 143\ 46. \tag{2.13}$$

For the error in (2.13), the repeated use of Results 5 and 6, gives that

$$0 \le m(Q) - \{m(Q_1) + m(Q_{2,\dots,9})\} \le 11.37e^{-4\pi},$$
$$0 \le m(Q_{2,\dots,9}) - \{m(Q_2) + m(Q_{3,\dots,9})\} \le (4/\pi)e^{-4\pi},$$
$$0 \le m(Q_{3,\dots,9}) - \{m(Q_3) + m(Q_{4,\dots,9})\} \le 11.37e^{-4\pi},$$
$$0 \le m(Q_{4,\dots,9}) - \{m(Q_4) + m(Q_{5,\dots 9})\} \le 11.37e^{-2\pi m(Q_5)},$$
$$0 \le m(Q_{5,\dots,9}) - \{m(Q_5) + m(Q_{6,\dots,9})\} \le (4/\pi)e^{-2\pi m(Q_5)},$$
$$0 \le m(Q_{6,\dots,9}) - \{m(Q_6) + m(Q_{7,8,9})\} \le (4/\pi)e^{-2\pi m(Q_6)},$$

$$0 \le m(Q_{7,8,9}) - \{m(Q_7) + m(Q_{8,9})\} \le (4/\pi)e^{-2\pi m(Q_7)},$$
$$0 \le m(Q_{8,9}) - \{m(Q_8) + m(Q_9)\} \le (4/\pi)e^{-2\pi m(Q_8)}.$$

Thus, by combining the above,

$$0 \le m(Q) - \sum_{j=1}^{9} m(Q_j) < 5.18 \times 10^{-5}$$

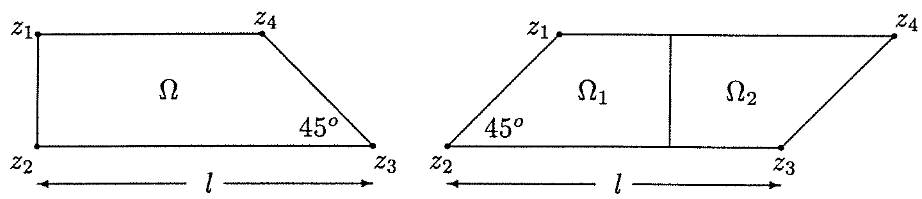
and therefore, from (2.13),

$$36.169\,143 < m(Q) < 36.169\,196. \tag{2.14}$$

3 On certain heuristic rules for computing conformal modules

In this section we make use of the DDM theory to investigate the validity of certain heuristic rules that we have come across in the VLSI literature, in connection with the measurement of resistance values of integrated circuit networks. In terms of our conformal mapping terminology these rules are, in fact, rules of thumb for computing by domain decomposition the conformal modules of long quadrilaterals.

We consider first two approximations which (in our terminology) are used for estimating the modules of component quadrilaterals in a heuristic domain decomposition algorithm due to Horowitz and Dutton [11] (see also [22], p. 122).



(a) $T_l := \{\Omega; z_1, z_2, z_3, z_4\}$ (b) $P_l := \{\Omega; z_1, z_2, z_3, z_4\}$

Figure 10.

Let T_l and P_l denote respectively the trapezium and parallelogram quadrilaterals illustrated in Figure 10(a) and Figure 10(b). Then, Horowitz and Dutton [11] propose the following estimates for the conformal modules $m(T_l)$ and $m(P_l)$:

$$\tilde{m}(T_l) = \{1 + 4(l-1)\}/4 = l - 0.75, \tag{3.1}$$

and

$$\tilde{m}(P_l) = l. \tag{3.2}$$

The actual errors in the above two heuristic approximations can be determined as follows:

For the trapezium of Figure 10(a), the error estimate (2.2) implies that for any $c > 0$,

$$0 \leq m(T_{l+c}) - \{c + m(T_l)\} \leq \frac{1}{2} \times 0.381 e^{-2\pi(l-1)},$$

provided that $l \geq 2$. Therefore, by using the known exact values of $m(T_2), m(T_3)$ and $m(T_4)$ listed in [18], p. 219 and [19], Remark 2.8, we can conclude that

$$m(T_l) = l - 0.720\,635\,600\,5 + E_l, \tag{3.3}$$

where

$$\begin{aligned} -1.03 \times 10^{-4} < E_l < 2.53 \times 10^{-4}, &\quad \text{if } 2 \leq l < 3, \\ -1.92 \times 10^{-7} < E_l < 4.73 \times 10^{-7}, &\quad \text{if } 3 \leq l < 4, \\ 0 < E_l < 1.25 \times 10^{-9}, &\quad \text{if } l \geq 4. \end{aligned} \tag{3.4}$$

Thus, for the approximation (3.1),

$$m(T_l) = \tilde{m}(T_l) + \epsilon,$$

where, for all $l \geq 2$,

$$2.9 \times 10^{-2} < \epsilon < 3.0 \times 10^{-2}.$$

To determine the error in (3.2) we decompose the parallelogram into two equal trapezia, as illustrated in Figure 10(b), and note that each of the resulting component quadrilaterals has modulus $m(T_{(l+1)/2})$. Hence, by using the error estimate of Result 1,

$$m(P_l) = 2m(T_{(l+1)/2}) + \epsilon_l,$$

where, for all $l \geq 3$,

$$0 \leq \epsilon_l \leq 0.761 e^{-\pi(l-1)}. \tag{3.5}$$

Therefore, from (3.3),

$$m(P_l) = 2\{\frac{l+1}{2} - 0.720\,635\,600\,5\} + \mathcal{E}_l = l - 0.441\,271\,201 + \mathcal{E}_l,$$

where, for all $l \geq 3$,

$$\mathcal{E}_l = 2E_{(l+1)/2} + \epsilon_l,$$

and E_l, ϵ_l are given respectively by (3.4) and (3.5), *i.e.*

$$\begin{aligned} -2.06 \times 10^{-4} < \mathcal{E}_l < 1.93 \times 10^{-3}, &\quad \text{if } 3 \leq l < 5, \\ -3.84 \times 10^{-7} < \mathcal{E}_l < 3.60 \times 10^{-6}, &\quad \text{if } 5 \leq l < 7, \\ 0 < \mathcal{E}_l < 7.46 \times 10^{-9}, &\quad \text{if } l \geq 7. \end{aligned}$$

Thus, for the approximation (3.2),

$$m(P_l) = \tilde{m}(P_l) - \epsilon,$$

where, for all $l \geq 3$,

$$4.3 \times 10^{-1} < \epsilon < 4.4 \times 10^{-1}.$$

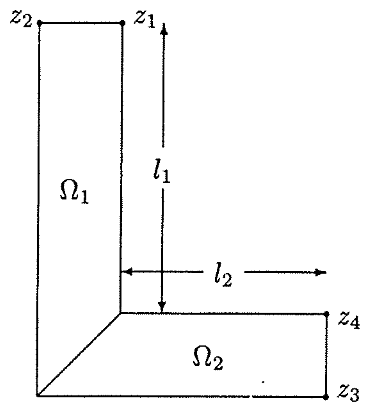


Figure 11.

We consider next the L-shaped quadrilateral $L_{(l_1,l_2)}$ and the two rectangular quadrilaterals $R_{(l,x)}$ and $S_{(l,x)}$ illustrated in Figure 11, Figure 12(a) and Figure 13(a). For these three quadrilaterals the heuristic methodology of Horowitz and Dutton [11] leads easily to the following approximations to $m(L_{(l_1,l_2)})$, $m(R_{(l,x)})$ and $m(S_{(l,x)})$:

$$\tilde{m}(L_{(l_1,l_2)}) = l_1 + l_2 + 0.5, \tag{3.6}$$

$$\tilde{m}(R_{(l,x)}) = \begin{cases} l + \frac{x}{2} - 1 + \frac{4(1-x)}{1+3x}, & \text{if } 0 < x \leq 1, \\ l - x + 0.5, & \text{if } 1 < x < l, \end{cases} \tag{3.7}$$

and, for $l \geq 1$ and $0 < x \leq 1$,

$$\tilde{m}(S_{(l,x)}) = l + x - 1 + \frac{4(1-x)}{1+3x}. \tag{3.8}$$

(It is of interest to note that the approximation (3.6) is equivalent to taking

$$\tilde{m}(L_{l_1,l_2})) = \tilde{m}(T_{l_1+1}) + \tilde{m}(T_{l_2+1}).\ )$$

To determine the error in (3.6) we decompose the L-shape into the two trapezia T_{l_1+1} and T_{l_2+1}, as illustrated in Figure 11, and let

$$h := \min\{m(T_{l_1+1}), m(T_{l_2+1})\} \quad \text{and} \quad l := \min\{l_1, l_2\}.$$

Then, for $h \geq 0.5$, (2.7) gives that

$$0 \leq m(L_{(l_1,l_2)}) - \{m(T_{l_1+1}) + m(T_{l_2+1})\} \leq \frac{4}{\pi} e^{-4\pi h}.$$

Hence, by using (3.3), (3.4) and the known exact values for $m(T_2)$, $m(T_3)$ and $m(T_4)$, we get that

$$m(L_{(l_1,l_2)}) = l_1 + l_2 + 0.558\ 728\ 799 + \mathcal{E}_l, \tag{3.9}$$

where

$$\begin{aligned} -2.06 \times 10^{-4} < \mathcal{E}_l < 5.07 \times 10^{-4}, &\quad \text{if } 1 \leq l < 2, \\ -3.84 \times 10^{-7} < \mathcal{E}_l < 9.47 \times 10^{-7}, &\quad \text{if } 2 \leq l < 3, \\ 0 < \mathcal{E}_l < 2.51 \times 10^{-9}, &\quad \text{if } l \geq 3. \end{aligned} \tag{3.10}$$

Thus, for the approximation (3.6),

$$m(L_{(l_1,l_2)}) = \tilde{m}(L_{(l_1;l_2)}) + \epsilon,$$

where, for all $l := \min\{l_1, l_2\} \geq 1$,

$$5.8 \times 10^{-2} < \epsilon < 6.0 \times 10^{-2}.$$

If l is "small" (so that there are no crowding difficulties), then the modules $m(R_{(l,x)})$ and $m(S_{(l,x)})$ can be determined exactly for specific values of x in terms of elliptic functions and integrals (see *e.g.* [2]). For example, the exact values of $m(R_{(2.5,1)})$, $m(R_{(3.0,1)})$, $m(R_{(4.0,1)})$ and $m(S_{(1.5,0.5)})$, $m(S_{(2.0,0.5)})$, $m(S_{(3.0,0.5)})$ are given to twelve decimal places respectively by

$$\begin{aligned} m(R_{(2.5,1.0)}) &= 1.969\ 386\ 554\ 304, \\ m(R_{(3.0,1.0)}) &= 2.469\ 393\ 831\ 816, \\ m(R_{(4.0,1.0)}) &= 3.469\ 394\ 159\ 886, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} m(S_{(1.5,0.5)}) &= 1.720\ 609\ 913\ 476, \\ m(S_{(2.0,0.5)}) &= 2.220\ 634\ 490\ 099, \\ m(S_{(3.0,0.5)}) &= 3.220\ 635\ 598\ 080. \end{aligned} \tag{3.12}$$

Regarding the errors in (3.7) and (3.8), these can be determined for specific values of x by using exact values of $m(R_{(l,x)})$ and $m(S_{(l,x)})$, such as those given in (3.11) and (3.12), and applying the estimate of Result 6, in conjunction with inequality (2.6), to the decompositions illustrated in Figure 12(b) and Figure 13(b). We illustrate this error estimation technique by applying it to the cases $x = 1.0$ of (3.7) and $x = 0.5$ of (3.8), *i.e.* to the approximations

$$\tilde{m}(R_{(l,1.0)}) = l - 0.5, \tag{3.13}$$

and

$$\tilde{m}(S_{(l,0.5)}) = l + 0.3. \tag{3.14}$$

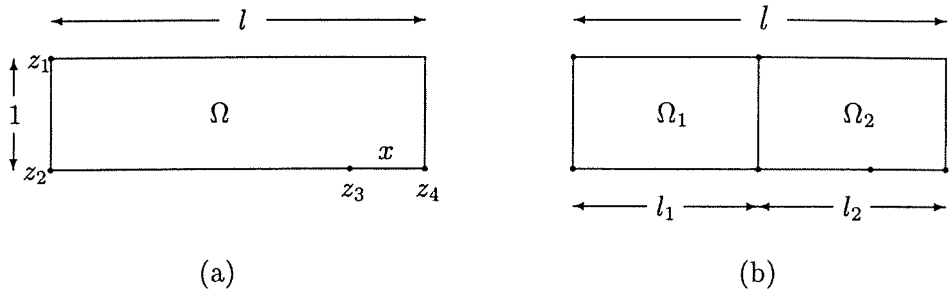


Figure 12.

Let $x = 1$ and consider the decomposition illustrated in Figure 12(b). Then, the error estimate of Result 6 (used, if necessary, in conjunction with inequality (2.6) of Result 7) implies that

$$0 \le m(R_{(l,1.0)}) - \{l_1 + m(R_{(l_2,1.0)})\} \le 11.37e^{-2\pi(l_2-1)},$$

provided $l_2 \ge 2.5$. Hence, by using the exact values given in (3.11), we can conclude that

$$m(R_{(l,1.0)}) = l - 0.530\,605\,840\,1 + \mathcal{E}_l, \tag{3.15}$$

where

$$\begin{aligned} -7.61 \times 10^{-6} < \mathcal{E}_l < 9.10 \times 10^{-4}, &\quad \text{if } 2.5 \le l < 3, \\ -3.29 \times 10^{-7} < \mathcal{E}_l < 3.94 \times 10^{-5}, &\quad \text{if } 3 \le l < 4, \\ 0 \le \mathcal{E}_l < 7.41 \times 10^{-8}, &\quad \text{if } l \ge 4. \end{aligned} \tag{3.16}$$

Thus, for the approximation (3.13),

$$m(R_{(l,1.0)}) = \tilde{m}(R_{(l,1.0)}) - \epsilon,$$

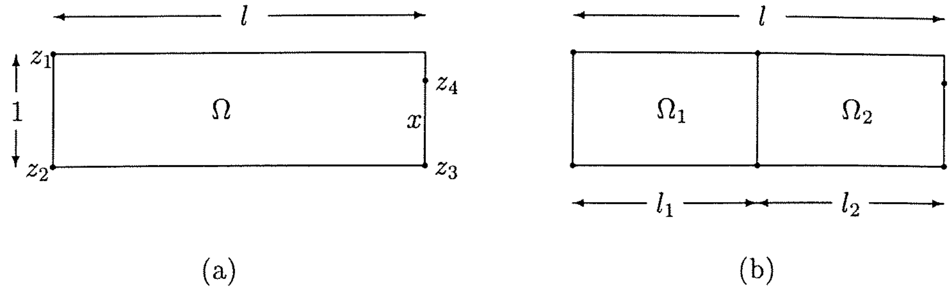


Figure 13.

where, for all $l \geq 2.5$,

$$2.9 \times 10^{-2} < \epsilon < 3.1 \times 10^{-2}.$$

Similarly, by applying the error estimate of Result 6, in conjunction with inequality (2.6), to the case $x = 0.5$ of the decomposition illustrated in Figure 13(b), and using the exact values given in (3.12) we find that

$$m(S_{(l,0.5)}) = l + 0.220\ 635\ 598\ 1 + \mathcal{E}_l, \tag{3.17}$$

where

$$\begin{aligned} -2.57 \times 10^{-5} < \mathcal{E}_l < 8.92 \times 10^{-4}, \quad & \text{if } 1.5 < l \leq 2, \\ -1.11 \times 10^{-6} < \mathcal{E}_l < 3.86 \times 10^{-5}, \quad & \text{if } 2 < l \leq 3, \\ 0 \leq \mathcal{E}_l < 7.41 \times 10^{-8}, \quad & \text{if } l > 3. \end{aligned} \tag{3.18}$$

That is, for the approximation (3.14),

$$m(S_{(l,0.5)}) = \tilde{m}(S_{(l,0.5)}) - \epsilon,$$

where

$$7.8 \times 10^{-2} < \epsilon < 8.0 \times 10^{-2}.$$

Apart from [11], the other references that we have come across in the VLSI literature are concerned mainly with the problem of estimating the modules of meander shaped quadrilaterals involving only right-angle bends ([6], [7], [14]). The associated heuristic methods of these references may be regarded as domain decomposition techniques where all the components are L-shaped quadrilaterals $L_{(l_1,l_2)}$ of the form illustrated in Figure 11 and where, in each case, the modules $m(L_{(l_1,l_2)})$ are approximated by

$$\tilde{m}(L_{(l_1,l_2)}) = l_1 + l_2 + c, \tag{3.19}$$

with

$$(i)\ c = 2/3 \ \text{ in [14]}, \quad (ii)\ c = 0.55 \ \text{ in [6]} \ \text{ and } \ (iii)\ c = 0.559 \ \text{ in [7]}. \tag{3.20}$$

For example, for the spiral quadrilateral $Q := \{\Omega, z_1, z_2, z_3, z_4\}$, illustrated in Figure 7, the methodologies given in each of [14], [6] and [7] are equivalent to approximating $m(Q)$ by

$$\begin{aligned}\tilde{m}(Q) &= \tilde{m}(L_{(17,8.5)}) + \tilde{m}(L_{(8.5,8)}) + \tilde{m}(L_{(8,7)}) + \tilde{m}(L_{(7,6.5)}) \\ &\quad + \tilde{m}(L_{(6.5,5.5)}) + \tilde{m}(L_{(5.5,5)}) + \tilde{m}(L_{(5,4)}) + \tilde{m}(L_{(4,3.5)}) \\ &\quad + \tilde{m}(L_{(3.5,2.5)}) + \tilde{m}(L_{(2.5,2)}) + \tilde{m}(L_{(2,1)}) + \tilde{m}(L_{(1,2)}),\end{aligned}$$

i.e. , from (3.19), by

$$\tilde{m}(Q) = 126 + 12c, \tag{3.21}$$

where c is as shown in (3.20). Similarly, the methodology of [11] leads to an approximation of the form (3.21) with $c = 0.5$; see (3.6). Therefore, for the spiral quadrilateral Q of Figure 7 the heuristic methods of [14], [6], [7] and [11] lead respectively to the following approximations

$$\begin{array}{llll} (i) & \tilde{m}(Q) = 134, & (iii) & \tilde{m}(Q) = 132.708, \\ (ii) & \tilde{m}(Q) = 132.6, & (\text{iv}) & \tilde{m}(Q) = 132. \end{array} \tag{3.22}$$

This should be compared with the true value which, according to (2.8), is given to five decimal places by

$$m(Q) = 132.704\,54.$$

We note the following in connection with the approximations (3.19), (3.20) and (3.22):

- As can be seen from (3.9) and (3.10), the most accurate of (3.19) and (3.20) is the approximation

  $$\tilde{m}(L_{(l_1,l_2)}) \approx l_1 + l_2 + 0.559, \tag{3.23}$$

  which is used in [7]. Thus, the best of (3.22) is the approximation

  $$\tilde{m}(Q) = 132.708,$$

  which was obtained by the method of [7].

- The approximation (3.23) is not based on heuristic arguments. It was obtained, together with several other similar and interesting results by Hall [9], using conformal transformation techniques (see also [1] and [8]). In fact, [9] contains several asymptotic formulae that lead to remarkably accurate approximations. For example, four such formulae give the following approximations to the modules of the quadrilaterals T_l, $L_{(l_1,l_2)}$,

$R_{(l,1.0)}$ and $S_{(l,0.5)}$, *i.e.* to the modules associated with the results (3.3), (3.4), (3.9), (3.10), (3.15), (3.16), (3.17) and (3.18):

$$\begin{array}{rcll} m(T_l) & \approx & l - 0.720\,635\,600, & \text{[9], Eq. (23)}, \\ m(L_{(l_1,l_2)}) & \approx & l_1 + l_2 + 0.558\,728\,800, & \text{[9], Fig. 39}, \\ m(R_{(l,1.0)}) & \approx & l - 0.530\,605\,840, & \text{[9], Eq. (12)}, \\ m(S_{(l,0.5)}) & \approx & l + 0.220\,635\,600, & \text{[9], Eq. (39)}. \end{array}$$

We end this section by applying the heuristic algorithm of Horowitz and Dutton [11] to the estimation of the modules of the last two quadrilaterals of Section 2.

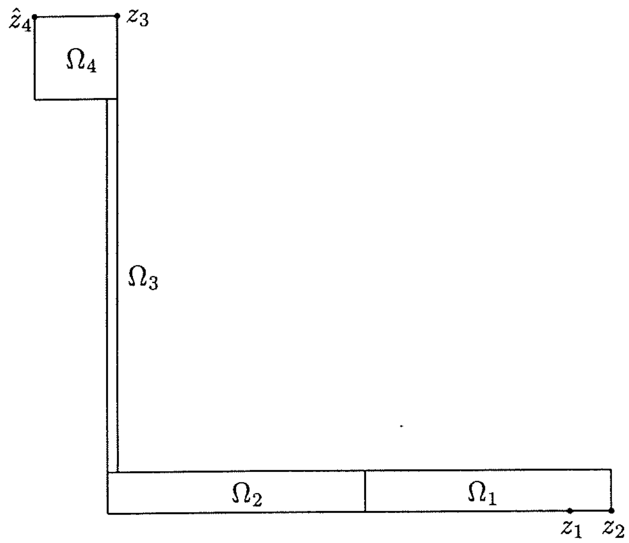


Figure 14.

For the quadrilateral $Q := \{\Omega; z_1, z_2, z_3, \hat{z}_4\}$ illustrated in Figure 8 the methodology described in [11] is equivalent to decomposing Q as shown in Figure 14 and approximating the modules of the component quadrilaterals $R_{(6.0,1.0)}$, $R_{(6.2,0.2)}$ and $S_{(1.0,0.1)}$ by means of (3.7) and (3.8). That is, the resulting approximation to $m(Q)$ is

$$\tilde{m}(Q) = \tilde{m}(R_{(6.0,1.0)}) + \tilde{m}(R_{(6.2,0.2)}) + 35 + \tilde{m}(S_{(1.0,0.1)}),$$

where, from (3.7) and (3.8),

$$\tilde{m}(R_{(6.0,1.0)}) = 5.5, \quad \tilde{m}(R_{(6.2,0.2)}) = 7.3, \quad \text{and} \quad \tilde{m}(S_{(1.0,0.1)}) = 2.869\,23\ldots.$$

Thus, to three decimal places,

$$\tilde{m}(Q) = 50.669,$$

while, according to (2.12), the true value is given correct to three decimal places by

$$m(Q) = 49.659.$$

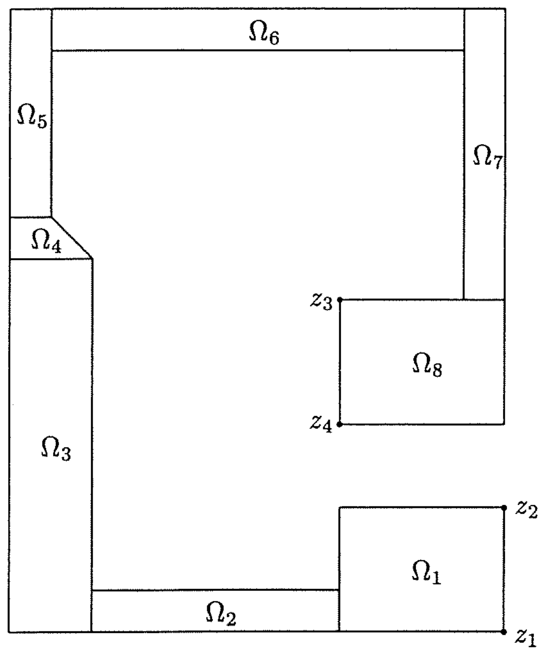


Figure 15.

Finally, for the quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ of Figure 9 the methodology of [11] is equivalent to decomposing Q as shown in Figure 15 and approximating the modules of the component quadrilaterals $S_{(4/3,1/3)}$, $R_{(4.5,0.5)}$, $R_{(5.0,1.0)}$, $R_{(7.0,1.0)}$ and $R_{(4/3,1/3)}$ by means of (3.7), (3.8) and of T_2' by means of (3.1). Here, T_2' denotes the reciprocal of the trapezium quadrilateral T_2, *i.e.*

$$m(T_2') = 1/m(T_2).$$

Therefore, the resulting approximation to $m(Q)$ is

$$\tilde{m}(Q) = \tilde{m}(S_{(\frac{4}{3},\frac{1}{3})}) + 6 + \tilde{m}(R_{(4.5,0.5)}) + 1/\tilde{m}(T_2) + \tilde{m}(R_{(5.0,1.0)}) + 10 + \tilde{m}(R_{(7.0,1.0)}) + \tilde{m}(R_{(\frac{4}{3},\frac{1}{3})}).$$

That is, to three decimal places,

$$\tilde{m}(Q) = 36.183, \tag{3.24}$$

while, according to (2.14), the correct value is given to three decimal places by

$$m(Q) = 36.169.$$

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