

# On the Numerical Performance of a Domain Decomposition Method for Conformal Mapping

N. Papamichael and N.S. Stylianopoulos

Department of Mathematics and Statistics, Brunel University  
 Uxbridge, Middlesex UB8 3PH, U.K.

## 1. Introduction

This paper is a sequel to a recent paper [14], concerning a domain decomposition method (hereafter referred to as *DDM*) for the conformal mapping of a certain class of quadrilaterals. For the description of the *DDM* we proceed exactly as in [14:§1], by introducing the following terminology and notations.

Let  $G$  be a simply-connected Jordan domain in the complex  $z$ -plane ( $z = x + iy$ ), and consider a system consisting of  $G$  and four distinct points  $z_1, z_2, z_3, z_4$  in counterclockwise order on its boundary  $\partial G$ . Such a system is said to be a quadrilateral  $Q$  and is denoted by  $Q = \{G; z_1, z_2, z_3, z_4\}$ . The conformal module  $m(Q)$  of  $Q$  is defined as follows:

Let  $R$  be a rectangle of the form

$$(1.1) \quad R := \{(\xi, \eta) : a < \xi < b, c < \eta < d\},$$

in the  $w$ -plane ( $w = \xi + i\eta$ ), and let  $h$  denote its aspect ratio, i.e.  $h := (d - c)/(b - a)$ . Then  $m(Q)$  is the unique value of  $h$  for which  $Q$  is conformally equivalent to a rectangle of the form (1.1), in the sense that for  $h = m(Q)$  and for this value only there exists a unique conformal map  $R \rightarrow G$  which takes the four corners  $a + ic$ ,  $b + ic$ ,  $b + id$ , and  $a + id$ , of  $R$  respectively onto the four points  $z_1, z_2, z_3, z_4$ . In particular,  $h = m(Q)$  is the only value of  $h$  for which  $Q$  is conformally equivalent to a rectangle of the form

$$(1.2) \quad R_h\{\alpha\} := \{(\xi, \eta) : 0 < \xi < 1, \alpha < \eta < \alpha + h\}.$$

The *DDM* is a method for computing approximations to the conformal modules and associated conformal maps of quadrilaterals of the form illustrated in Figure 1.1(b). That is, the method is concerned with the mapping of quadrilaterals

$$(1.3a) \quad Q := \{G; z_1, z_2, z_3, z_4\},$$

where:

• The domain  $G$  is bounded by the straight lines  $x = 0$  and  $x = 1$  and two Jordan arcs with cartesian equations  $y = -\tau_1(x)$  and  $y = \tau_2(x)$ , where  $\tau_j$ ;  $j = 1, 2$ , are positive in  $[0, 1]$ , i.e.

$$(1.3b) \quad G := \{(x, y) : 0 < x < 1, -\tau_1(x) < y < \tau_2(x)\}.$$

• The points  $z_1, z_2, z_3, z_4$  are the corners where the arcs intersect the straight lines, i.e.

$$(1.3c) \quad z_1 = -i\tau_1(0), \quad z_2 = 1 - i\tau_1(1), \quad z_3 = 1 + i\tau_2(1), \quad z_4 = i\tau_2(0).$$

Let  $Q$  be of the form (1.3) and let

$$(1.4a) \quad G_1 := \{(x, y) : 0 < x < 1, -\tau_1(x) < y < 0\},$$

and

$$(1.4b) \quad G_2 := \{(x, y) : 0 < x < 1, 0 < y < \tau_2(x)\},$$

so that  $\overline{G} = \overline{G}_1 \cup \overline{G}_2$ . Also, let  $Q_1$  and  $Q_2$  denote the quadrilaterals

$$(1.4c) \quad Q_1 := \{G_1; z_1, z_2, 1, 0\} \quad \text{and} \quad Q_2 := \{G_2; 0, 1, z_3, z_4\},$$

and let  $h := m(Q)$  and  $h_j := m(Q_j)$ ;  $j = 1, 2$ ; see Figures 1.2(b) and 1.3(b). Finally, let  $g$  and  $g_j$ ;  $j = 1, 2$ , denote the conformal maps

$$(1.5) \quad g : R_h\{-h_1\} \rightarrow G,$$

$$(1.6) \quad g_1 : R_{h_1}\{-h_1\} \rightarrow G_1 \quad \text{and} \quad g_2 : R_{h_2}\{0\} \rightarrow G_2,$$

where, with the notation (1.2),

$$R_h\{-h_1\} := \{(\xi, \eta) : 0 < \xi < 1, -h_1 < \eta < h - h_1\},$$

$$R_{h_1}\{-h_1\} := \{(\xi, \eta) : 0 < \xi < 1, -h_1 < \eta < 0\},$$

$$R_{h_2}\{0\} := \{(\xi, \eta) : 0 < \xi < 1, 0 < \eta < h_2\};$$

see Figures 1.1-1.3. Then, the DDM consists of the following:

(a) Subdividing the quadrilateral  $Q$ , given by (1.3), into the two smaller quadrilaterals  $Q_1$  and  $Q_2$ , given by (1.4).

(b) Approximating the conformal module of  $Q$  by the sum of the conformal modules of  $Q_1$  and  $Q_2$ , i.e. approximating  $h$  by

$$(1.7a) \quad \tilde{h} := h_1 + h_2.$$

(c) Approximating the rectangle  $R_h\{-h_1\}$  and the conformal map  $g : R_h\{-h_1\} \rightarrow G$  respectively by

$$(1.7b) \quad R_{\tilde{h}}\{-h_1\} := \{(\xi, \eta) : 0 < \xi < 1, -h_1 < \eta < h_2\},$$

and

$$(1.7c) \quad \tilde{g}(w) := \begin{cases} g_2(w) : R_{h_2}\{0\} \rightarrow G_2, & \text{for } w \in R_{h_2}\{0\}, \\ g_1(w) : R_{h_1}\{-h_1\} \rightarrow G_1, & \text{for } w \in R_{h_1}\{-h_1\}. \end{cases}$$

The initial motivation for considering the above method came from: (a) The intuitive observation that if the constituent quadrilaterals  $Q_1$  and  $Q_2$  are both "long" then  $\tilde{h}$  is close to  $h$ . (b) Experimental evidence indicating that  $\tilde{h}$  is close to  $h$  even when  $Q_1$  and  $Q_2$  are only moderately long; see [12:§5] and [14:§1]. (It is important to note that  $h \geq h_1 + h_2$  and equality occurs only in the trivial cases where  $G$  is a rectangle or  $\tau_1(x) = \tau_2(x)$ ,  $x \in [0, 1]$ ; see e.g. [9: p. 437].)

The treatment of the DDM contained in [14] is a theoretical investigation leading to estimates of the errors in the approximations (1.7). These error estimates are derived by assuming that the functions  $\tau_j$ ;  $j = 1, 2$ , satisfy the following:

(i)  $\tau_j$ ;  $j = 1, 2$ , are absolutely continuous in  $[0, 1]$ , and

$$(1.8) \quad d_j := \operatorname{ess\,sup}_{0 \leq x \leq 1} |\tau'_j(x)| < \infty.$$

(ii) If

$$(1.9a) \quad m_j := \max_{0 \leq x \leq 1} \{\exp(-\pi \tau_j(x))\}; \quad j := 1, 2,$$

then

$$(1.9b) \quad \varepsilon_j := d_j \{(1 + m_j)/(1 - m_j)\} < 1; \quad j = 1, 2.$$

In addition to the theory, [14] also contains two numerical examples comparing the actual errors in the DDM approximations with those predicted by the theoretical estimates.

The present paper is concerned with the numerical performance of the DDM and, in particular, with the performance of the method in cases where the functions  $\tau_j$ ;  $j = 1, 2$ , do not fulfil the rather restrictive condition (1.9) needed for the theory of [14]. More specifically, the main purpose of this paper is to show by means of numerical examples that some of the theoretical results of [14] remain valid even when the condition (1.9) is violated, and thus to provide experimental support for certain conjectures made in [14].

We end this introductory section by making the following remarks concerning the DDM and related matters:

- A survey of available methods for computing approximations to the conformal modules and the associated conformal maps of general quadrilaterals is given in [12], where also several areas of application of the conformal maps are discussed; see also [4]-[7] and [9:§16.11].

- Although this is not considered here, the DDM can also be applied to quadrilaterals of the form illustrated in Figure 1.4, provided that the crosscut  $c$  of subdivision is taken



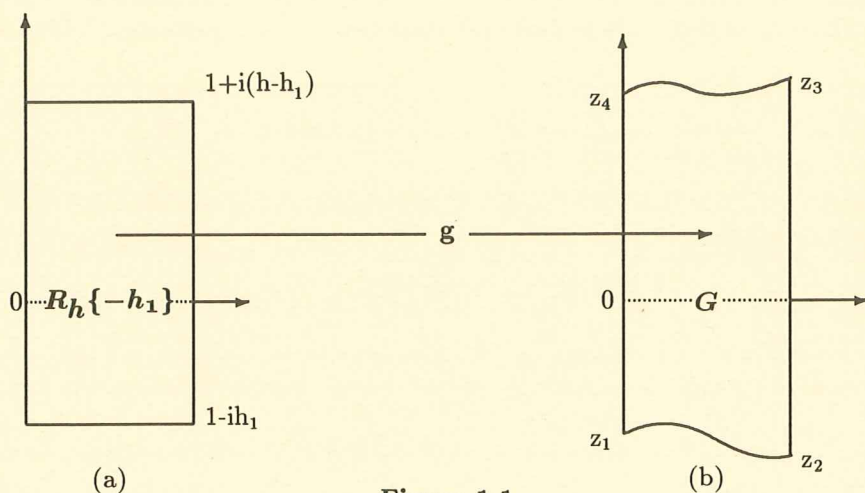


Figure 1.1

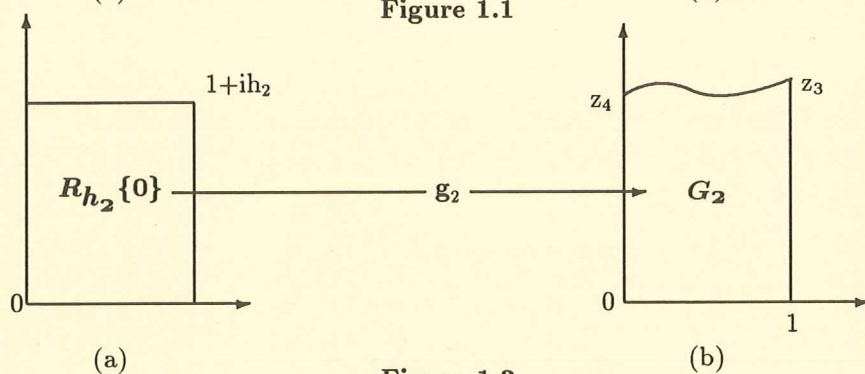


Figure 1.2

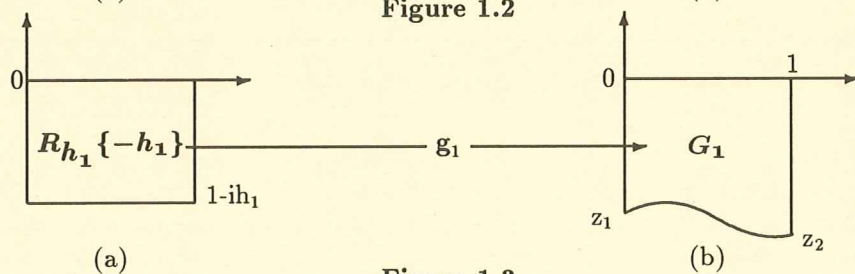


Figure 1.3

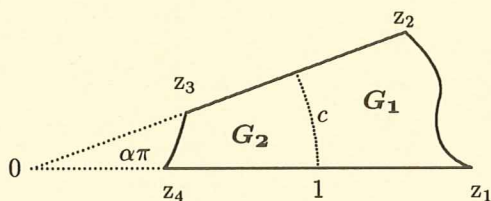


Figure 1.4

to be a circular arc; see [14: Remark 4.7]. In other words, the application of the DDM is restricted to quadrilaterals that have one of the two special forms illustrated in Figures 1.1 and 1.4. We note however that the mapping of such quadrilaterals has received considerable attention recently; see e.g. [2], [8], [11], [12], [15] and [17].

• A more general form of the DDM involves subdividing the original quadrilateral  $Q$  into two quadrilaterals  $Q_1$  and  $Q_2$  of the form (1.4) at the lower and upper ends, and a rectangle in the middle. This can be described more precisely as follows:

Let

$$G := \{(x, y) : 0 < x < 1, -\tau_1(x) < y < \tau_2(x) + c\},$$

where  $c > 0$ , let

$$G_1 := \{(x, y) : 0 < x < 1, -\tau_1(x) < y < 0\},$$

and

$$G_2 := \{(x, y) : 0 < x < 1, c < y < \tau_2(x) + c\},$$

so that  $\overline{G} = \overline{G_1} \cup \overline{R_c\{0\}} \cup \overline{G_2}$ , and let

$$z_1 = -i\tau_1(0), \quad z_2 = 1 - i\tau_1(1), \quad z_3 = 1 + i(\tau_2(1) + c), \quad z_4 = i(\tau_2(0) + c).$$

Then the general form of the DDM consists of the following:

(a) Subdividing the quadrilateral  $Q := \{G; z_1, z_2, z_3, z_4\}$  into three smaller quadrilaterals, i.e. the quadrilaterals

$$Q_1 := \{G_1; z_1, z_2, 1, 0\} \quad \text{and} \quad Q_2 := \{G_2; ic, 1 + ic, z_3, z_4\},$$

at the lower and top ends, and the rectangular quadrilateral

$$\{R_c\{0\}; 0, 1, 1 + ic, ic\},$$

in the middle.

(b) Approximating the conformal module  $h := m(Q)$  by

$$(1.10) \quad \tilde{h} := h_1 + h_2 + c,$$

where  $h_j := m(Q_j)$ ;  $j = 1, 2$ .

(c) Approximating the rectangle  $R_h\{-h_1\}$  and the conformal map  $g : R_h\{-h_1\} \rightarrow G$  respectively by  $R_{\tilde{h}}\{-h_1\}$  and

$$(1.11) \quad \tilde{g}(w) := \begin{cases} g_2(w) : R_{h_2}\{c\} \rightarrow G_2, & \text{for } w \in R_{h_2}\{c\}, \\ w, & \text{for } w \in R_c\{0\}, \\ g_1(w) : R_{h_1}\{-h_1\} \rightarrow G_1, & \text{for } w \in R_{h_1}\{-h_1\}. \end{cases}$$

• The DDM is of practical interest for the following reasons:

(i) Given the conformal modules and associated conformal maps of two quadrilaterals  $Q_1$  and  $Q_2$  of the form (1.4), the method provides approximations to the conformal module and associated conformal map of any quadrilateral consisting of  $Q_1$  and  $Q_2$ , at the lower and top ends, and a rectangle of any height in the middle.

(ii) The method can be used to overcome the "crowding" difficulties associated with the numerical conformal mapping of "long" quadrilaterals of the form (1.3). (Full details of the crowding phenomenon and its damaging effects on numerical procedures for the mapping of "long" quadrilaterals can be found in [12], [13], and [10]; see also [4:p.179], [9:p.428] and [16:p.4].)

(iii) Numerical methods for approximating the conformal maps of quadrilaterals of the form (1.4) are often substantially simpler than those for quadrilaterals of the more general form (1.3); see e.g. [8], and [12,§3.4].

• Of the two conditions involved in the assumptions (1.8)-(1.9), only (1.9) is restrictive from the practical point of view. This condition is more or less equivalent to requiring that the slopes of the two curves  $y = \tau_j(x)$ ;  $j = 1, 2$ , are numerically less than unity in  $[0,1]$ . This is so because the values  $m_j$ ;  $j = 1, 2$ , given by (1.9a) are "small", even when the quadrilaterals  $Q_j$ ;  $j = 1, 2$ , are only moderately "long".

## 2. Theoretical error estimates

As in Section 1, let  $Q$  and  $Q_j$ ;  $j = 1, 2$ , denote the three quadrilaterals defined by (1.3) and (1.4), let  $h := m(Q)$ ,  $h_j := m(Q_j)$ ;  $j = 1, 2$ , and let  $g, g_j$ ;  $j = 1, 2$ , be the associated conformal maps (1.5) and (1.6). Also, let

$$X(\xi) := \operatorname{Re} g(\xi - ih_1), \quad \hat{X}(\xi) := \operatorname{Re} g(\xi + i(h - h_1)),$$

and

$$X_1(\xi) := \operatorname{Re} g_1(\xi - ih_1), \quad \hat{X}_2(\xi) := \operatorname{Re} g_2(\xi + ih_2),$$

and let  $E_h$  and  $E_g\{j\}$ ,  $E_X\{j\}$ ,  $E_\tau\{j\}$ ;  $j = 1, 2$ , denote the following domain decomposition errors:

$$(2.1) \quad E_h := h - (h_1 + h_2),$$

$$(2.2a) \quad E_g\{1\} := \max\{|g(w) - g_1(w)| : w \in \overline{R_{h_1}}\{-h_1\}\},$$

$$(2.2b) \quad E_g\{2\} := \max\{|g(w + iE_h) - g_2(w)| : w \in \overline{R_{h_2}}\{0\}\},$$

$$(2.3) \quad E_X\{1\} := \max_{0 \leq \xi \leq 1} |X(\xi) - X_1(\xi)|,$$

$$E_X\{2\} := \max_{0 \leq \xi \leq 1} |\hat{X}(\xi) - \hat{X}_2(\xi)|,$$

and

$$(2.4) \quad E_\tau\{1\} := \max_{0 \leq \xi \leq 1} |\tau_1(X(\xi)) - \tau_1(X_1(\xi))|, \quad E_\tau\{2\} := \max_{0 \leq \xi \leq 1} |\tau_2(\hat{X}(\xi)) - \tau_2(\hat{X}_2(\xi))|.$$

Finally, assume that the functions  $\tau_j$ ;  $j = 1, 2$ , satisfy the assumptions (1.8)-(1.9), and let

$$(2.5a) \quad \alpha(\varepsilon_j, \varepsilon) := 2/\{(1 - \varepsilon_j)(1 - \varepsilon^2)^{\frac{1}{2}}\}; \quad j = 1, 2,$$

and

$$(2.5b) \quad \beta(\varepsilon_j, \varepsilon) := \sqrt{8}/\{(1 - \varepsilon_j)(1 - \varepsilon^2)\}^{\frac{1}{2}}; \quad j = 1, 2,$$

where the  $\varepsilon_j$ ;  $j = 1, 2$ , are given by (1.9), and  $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$ . Then, the main results of [14] are the following estimates of the errors  $E_h$  and  $E_g\{j\}$ ;  $j = 1, 2$ , in the DDM approximations (1.7):

$$(2.6) \quad E_h \leq \pi^{-1} d_1 \alpha(\varepsilon_1, \varepsilon) \{\varepsilon_1 e^{-2\pi h_1} + \varepsilon_2 e^{-\pi h}\} + \pi^{-1} d_2 \alpha(\varepsilon_2, \varepsilon) \{\varepsilon_2 e^{-2\pi h_2} + \varepsilon_1 e^{-\pi h}\}$$

and

$$(2.7a) \quad E_g\{j\} \leq \max\{M_j, N_j\}; \quad j = 1, 2,$$

where

$$(2.7b) \quad M_j := \pi^{-\frac{1}{2}}(1 + d_j^2)^{\frac{1}{2}} \beta(\varepsilon_j, \varepsilon) \{\varepsilon_j + \varepsilon_{3-j} e^{-\pi h}\}^{\frac{1}{2}} \{\varepsilon_j e^{-2\pi h_j} + \varepsilon_{3-j} e^{-\pi h}\}^{\frac{1}{2}},$$

$$(2.7c) \quad N_j := \frac{1}{2} \pi^{-\frac{1}{2}} \beta(0, \varepsilon) \{5\varepsilon_j e^{-\pi h_j} + 3\varepsilon_{3-j} e^{-\pi(h-h_j)}\} \\ + \pi^{-1} d_j \alpha(\varepsilon_j, \varepsilon) \{\varepsilon_j e^{-2\pi h_j} + \varepsilon_{3-j} e^{-\pi h}\};$$

see [14:Thms 4.1, 4.4]. Since  $h \geq h_1 + h_2$ , the above estimates show that if the functions  $\tau_j$ ;  $j = 1, 2$ , satisfy the assumptions (1.8)-(1.9), then

$$(2.8) \quad E_h = O\{\exp(-2\pi h^*)\},$$

and

$$(2.9) \quad E_g\{j\} = O\{\exp(-\pi h^*)\}; \quad j = 1, 2,$$

where  $h^* := \min(h_1, h_2)$ .

In addition to (2.6) and (2.7), [14] also contains estimates of the errors  $E_X\{j\}$  and  $E_\tau\{j\}$ ;  $j = 1, 2$ ; see Theorem 4.2 and Remark 4.2 in [14]. These estimates show that under the assumptions (1.8)-(1.9),

$$(2.10) \quad E_X\{j\} = O\{\exp(-\pi h^*)\} \quad \text{and} \quad E_\tau\{j\} = O\{\exp(-\pi h^*)\}; \quad j = 1, 2,$$

where as before  $h^* := \min(h_1, h_2)$ . Finally, [14] contains a theorem (Theorem 4.3), which shows that if  $Q_1$  and  $Q_2$  are both "long" quadrilaterals, then at points sufficiently far from the two sides  $\eta = -h_1$  and  $\eta = h - h_1$  of  $R_h\{-h_1\}$ , the conformal map  $g$  can be approximated closely by the identity map. In particular, this theorem of [14] shows that

$$(2.11) \quad \max_{0 \leq \xi \leq 1} |g(\xi + i0) - \xi| = O\{\exp(-\pi h^*)\}.$$



The method of analysis used in [14] for deriving the above results makes extensive use of the theory given in [3: Kap.V,§3], in connection with the integral equation method of Garrick, for the conformal mapping of doubly-connected domains. This involves expressing the three problems for the conformal maps  $g$  and  $g_j$ ;  $j = 1, 2$ , as equivalent problems for the conformal maps of three symmetric doubly-connected domains; see [8], [12:§3.2,3.4] and [13:§3]. (With reference to the above comment, readers who are familiar with the method of Garrick will recognize the very close similarity between the condition (1.9), which is needed for the analysis used in [14], and the so-called  $\varepsilon\delta$ -condition needed for the theory of the Garrick method.)

We end this section by observing that the results of [14] simplify considerably in the case where one of the two subdomains  $G_1$  of  $G_2$  is a rectangle. For example, let  $\tau_1(x) = c > 0$ ,  $x \in [0, 1]$ , i.e. let

$$(2.12) \quad G_1 := \{(x, y) : 0 < x < 1, -c < y < 0\} = R_c\{-c\}.$$

Then,  $g_1(w) = w$ ,  $h_1 = c$ ,  $d_1 = \varepsilon_1 = 0$  and, for any value  $c > 0$ , the results (2.8)-(2.10) simplify as follows:

$$(2.13) \quad E_h := h - (c + h_2) = O\{\exp(-2\pi h_2)\},$$

$$(2.14) \quad E_g\{j\} = O\{\exp(-\pi h_2)\}; \quad j = 1, 2,$$

$$(2.15) \quad E_x\{j\} = O\{\exp(-\pi h_2)\}, \quad E_r\{j\} = O\{\exp(-\pi h_2)\}; \quad j = 1, 2.$$

Also, in place of (2.11) we now have for any point  $w := \xi + i\eta \in R_c\{-c\}$ ,

$$(2.16) \quad |g(w) - w| = O\{\exp(-\pi(h_2 - \eta))\}.$$

Furthermore, all the above simplified results hold under the less restrictive assumptions obtained by replacing the inequalities (1.9) by

$$(2.17) \quad \varepsilon_2 := d_2\{(1 + m_2^2)/(1 - m_2^2)\} < 1;$$

see [14:Remark 4.5].

### 3. Numerical results and discussion

In addition to the theoretical estimates summarized in Section 2, [14:§5] contains two numerical examples in which the quadrilaterals are chosen so that the functions  $\tau_j$ ;  $j = 1, 2$ , satisfy the assumptions (1.8)-(1.9). The numerical results of these examples confirm the theory of [14], and indicate that the DDM is capable of producing approximations of high accuracy, even when the quadrilaterals under consideration are only moderately long.

In this section we study further the numerical performance of the DDM, but here we consider its application to quadrilaterals that do not satisfy the conditions (1.9). That



is, we are concerned with cases for which the theory of [14] does not apply. Our main purpose is to provide experimental evidence supporting the following two conjectures made in [14]:

**Conjecture 3.1.** *The results (2.8)-(2.9) hold even when the condition (1.9) is not fulfilled; see [14:Remark 5.4]. More specifically, the claim here is that*

$$(3.1) \quad E_h = O\{\exp(-2\pi h^*)\},$$

and

$$(3.2) \quad E_g\{j\} = O\{\exp(-\pi h^*)\}; \quad j = 1, 2,$$

with  $h^* := \min(h_1, h_2)$  even when  $d_j \geq 1$ ;  $j = 1, 2$ , where  $d_j$  are the values given by (1.8).

**Conjecture 3.2.** *The errors  $E_X\{j\}$  and  $E_\tau\{j\}$ ;  $j = 1, 2$ , are  $O\{\exp(-2\pi h^*)\}$ , rather than  $O\{\exp(-\pi h^*)\}$  as predicted by the theory of [14]; see (2.10) and [14:Remark 5.2]. That is, the claim here is that*

$$(3.3) \quad E_X\{j\} = O\{\exp(-2\pi h^*)\} \quad \text{and} \quad E_\tau\{j\} = O\{\exp(-2\pi h^*)\}; \quad j = 1, 2,$$

and that the above results hold even when the condition (1.9) is violated.

Each of the three examples considered below involves the mapping of a quadrilateral  $Q$  of the form (1.3) and, in each case, the decomposition is performed by subdividing  $Q$  into two quadrilaterals  $Q_j$ ;  $j = 1, 2$ , of the form (1.4). In presenting the numerical results we use the following notations:

- $E_h$  and  $E_g\{j\}$ ,  $E_X\{j\}$ ,  $E_\tau\{j\}$ ;  $j = 1, 2$ : As before these denote the actual DDM errors (2.1)-(2.4). More precisely, the values listed in the examples are reliable estimates of the errors (2.1)-(2.4). They are determined, as in [14:§5], from accurate approximations to  $h$ ,  $h_j$ ,  $j = 1, 2$ , and  $g$ ,  $g_j$ ,  $j = 1, 2$ , which are computed by using the iterative algorithms of [8]. In particular,  $E_g\{j\}$ ;  $j = 1, 2$ , are the maxima of two sets of values which are obtained by sampling respectively the approximations to the functions  $g(w) - g_1(w)$  and  $g(w + iE_h) - g_2(w)$  at a number of test points on the boundary segments  $\eta = -h_1, 0$  of  $R_{h_1}\{-h_1\}$  and  $\eta = 0, h_2$  of  $R_{h_2}\{0\}$ . The values  $E_X\{j\}$  and  $E_\tau\{j\}$  are determined in a similar manner, by sampling the approximations to the functions  $X(\xi) - X_1(\xi)$ ,  $\hat{X}(\xi) - \hat{X}_2(\xi)$ , etc. at a number of test points in  $0 \leq \xi \leq 1$ . (The only exception to the above are the values of  $E_h$  given in Example 3.1, in which  $Q$  is a trapezium and the subdivision consists of a smaller trapezium  $Q_2$  and a rectangle  $Q_1 := R_c\{-c\}$ . In this case,  $h := m(Q)$  and  $h_2 := m(Q_2)$  are known exactly in terms of elliptic integrals. Hence  $E_h := h - (c + h_2)$  is also known exactly.)

- $\delta_h$  and  $\delta_g\{j\}$ ,  $\delta_X\{j\}$ ,  $\delta_\tau\{j\}$ ;  $j = 1, 2$ : These denote the values used for testing the validity of (3.1)-(3.3). They are determined from the computed values of the errors  $E_h$  and  $E_g\{j\}$ ,  $E_X\{j\}$ ,  $E_\tau\{j\}$ ;  $j = 1, 2$ , as follows:

In each of the Examples 3.2 and 3.3, the functions  $\tau_j$ ;  $j = 1, 2$ , are of the form

$$\tau_j(x) := \sigma_j(x) + l; \quad j = 1, 2,$$

where  $l \geq 0$ , and in each case the values of the conformal modules and the errors in the DDM approximations are computed for several values of the parameter  $l$ . Let  $h_1(l)$  and  $h_2(l)$  denote the conformal modules of  $Q_1$  and  $Q_2$  corresponding to the value  $l$ , and let  $h^*(l) := \min(h_1(l), h_2(l))$ . Also, let  $E$  denote any of the errors  $E_h$ ,  $E_g\{j\}$ ,  $E_X\{j\}$  or  $E_\tau\{j\}$ , and let  $E(l)$  be the value of  $E$  corresponding to  $l$ . Then, the validity of (3.1)-(3.3) is checked by assuming that

$$E = O\{\exp(-\delta\pi h^*)\}; \quad h^* := \min(h_1, h_2),$$

and computing various values of  $\delta$  (i.e. of  $\delta_h$ ,  $\delta_g\{j\}$ ,  $\delta_X\{j\}$  or  $\delta_\tau\{j\}$ ) by means of the formula

$$\delta = -\{\log[E(l_1)/E(l_2)]\}/\{\pi[h^*(l_1) - h^*(l_2)]\}.$$

(In the examples,  $l_1$  and  $l_2$  are taken to be successive values of the parameter  $l$  for which numerical results are listed.)

In the first example, i.e. in Example 3.1, we consider only the error  $E_h$  and, because of the form of  $Q$ , we check the validity of (2.13) rather than (3.1). That is we assume that

$$E_h = O\{\exp(-\delta_h\pi h_2)\},$$

and determine  $\delta_h$  from the listed values of  $E_h$ , by modifying in an obvious manner the procedure described above.

**Example 3.1** (See also [12:§5])  $Q$  is the trapezium illustrated in Figure 3.1. That is,  $Q$  is defined by the functions

$$\tau_1(x) = c \quad \text{and} \quad \tau_2(x) = x - 1 + l,$$

where  $c > 0$  and  $l > 1$ . Here,  $d_2 = 1$  and, because of this, the theory of [14] does not apply.

As was previously remarked, in this case the conformal modules  $h := m(Q)$  and  $h_2 := m(Q_2)$  are known in terms of elliptic integrals. Thus, Table 3.1 contains the exact values of  $h$  and  $h_2$  corresponding to the parameters  $l = 1.25, 2.00, 2.50, 4.00, 5.00$ , and  $c = 0.75, 0.50, 1.50, 1.00, 5.00$ . (These were determined correct to twelve decimal places, by using the formulae of Bowman [1:p.104].) The table also contains the values of the error  $E_h := h - (h_2 + c)$  and, where possible, the corresponding values of  $\delta_h$ . These values of  $\delta_h$  indicate clearly that, for any  $c > 0$ ,  $E_h = O\{\exp(-2\pi h_2)\}$ .

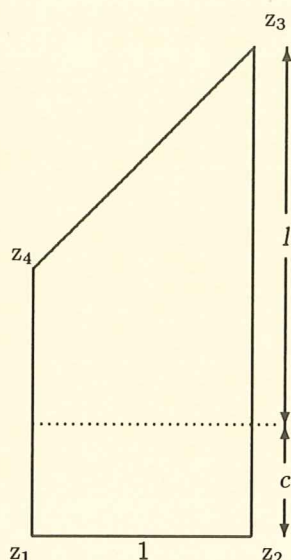


Figure 3.1

$l$	$c$	$h_2$	$h$	$E_h$	$\delta_h$
1.25	0.75	0.516 810 878 029	1.297 261 571 171	1.2 E-02	-
2.00	0.50	1.279 261 571 171	1.779 359 959 478	9.8 E-05	2.021
2.50	1.50	1.779 359 959 478	3.279 364 399 489	4.4 E-06	1.972
4.00	1.00	3.279 364 399 489	4.279 364 399 847	3.6 E-10	1.999
5.00	5.00	4.279 364 399 847	9.279 364 399 847	-	-

Table 3.1

**Example 3.2.**  $Q$  and  $Q_j$ ;  $j = 1, 2$ , are defined by (1.3) and (1.4) with

$$\tau_1(x) = 1 + 0.25 \cos(2\pi x) + l \quad \text{and} \quad \tau_2(x) = 0.25x^4 - 0.5x^2 + 1 + l.$$

Since  $d_1 = \pi/2 > 1$ , the above two functions do not satisfy the condition (1.9) needed for the theory of [14].

The numerical results corresponding to the values  $l = 0.0(0.5)2.5$  are listed in Tables 3.2(a)-3.2(c). Table 3.2(a) contains the computed values of the conformal modules, together with the estimates of  $E_h := h - (h_1 + h_2)$  and the corresponding values of  $\delta_h$ . (The values of  $h$  and  $h_j$  listed in the table are expected to be correct to all the figures quoted. The algorithms of [8] achieve this remarkable accuracy, because the two curves  $y = \tau_j(x)$ ;  $j = 1, 2$ , intersect the straight lines  $x = 0$  and  $x = 1$  at right angles; see [8:§6]) Tables 3.2(b) and 3.2(c) contain respectively the estimates of the errors  $E_g\{j\}$ ;  $j = 1, 2$ , and  $E_X\{j\}$ ,  $E_\tau\{j\}$ ;  $j = 1, 2$ , together with the values  $\delta_g\{j\}$ ;  $j = 1, 2$ , and  $\delta_X\{j\}$ ,  $\delta_\tau\{j\}$ ;  $j = 1, 2$ .



$l$	$h_1$	$h_2$	$h$	$E_h$	$\delta_h$
0.0	0.864 086 763 083	0.859 360 128 944	1.723 659 400 858	2.1 E-04	-
0.5	1.364 089 626 994	1.359 560 053 306	2.723 658 669 419	9.0 E-06	2.013
1.0	1.864 089 632 342	1.859 568 647 619	3.723 658 668 053	3.9 E-07	2.001
1.5	2.364 089 632 352	2.359 569 018 929	4.723 658 668 050	1.7 E-08	2.000
2.0	2.864 089 632 352	2.859 569 034 974	5.723 658 668 050	7.2 E-10	2.000
2.5	3.364 089 632 352	3.359 569 035 668	6.723 658 668 050	3.1 E-11	2.000

Table 3.2(a)

$l$	$E_g\{1\}$	$\delta_g\{1\}$	$E_g\{2\}$	$\delta_g\{2\}$
0.0	1.2 E-02	-	1.2 E-02	-
0.5	2.4 E-03	1.022	2.4 E-03	1.018
1.0	5.0 E-04	1.004	5.0 E-04	1.003
1.5	1.0 E-04	1.000	1.0 E-04	1.000
2.0	2.1 E-05	1.000	2.1 E-05	1.000
2.5	4.5 E-06	1.000	4.5 E-06	1.000

Table 3.2(b)

$l$	$E_X\{1\}$	$\delta_X\{1\}$	$E_X\{2\}$	$\delta_X\{2\}$	$E_\tau\{1\}$	$\delta_\tau\{1\}$	$E_\tau\{2\}$	$\delta_\tau\{2\}$
0.0	4.9 E-04	-	1.1 E-03	-	5.2 E-04	-	4.1 E-04	-
0.5	2.1 E-05	1.999	4.6 E-05	2.001	2.2 E-05	2.005	1.7 E-05	2.002
1.0	9.1 E-07	2.000	2.0 E-06	2.000	9.6 E-07	2.000	7.6 E-07	2.000
1.5	3.9 E-08	2.000	8.6 E-08	2.000	4.2 E-08	2.000	3.3 E-08	2.000
2.0	1.7 E-09	2.000	3.7 E-09	2.000	1.8 E-09	2.000	1.4 E-09	2.000
2.5	7.3 E-11	2.000	1.6 E-10	2.000	7.8 E-11	2.000	6.1 E-11	2.000

Table 3.2(c)

**Example 3.3**  $Q$  and  $Q_j$ ;  $j = 1, 2$ , are defined by (1.3) and (1.4) with

$$\tau_1(x) = 0.75 + 0.2 \operatorname{sech}^2(2.5x) + l \quad \text{and} \quad \tau_2(x) = x(1-x) + 1 + l.$$

In this case, the condition (1.9) is not fulfilled because  $d_2 = 1$ .

The numerical results corresponding to the values  $l = 0.00(0.25)1.25$  are listed in Tables 3.3(a)-3.3(c). (In this example, the values of  $h$  and  $h_j$ ;  $j = 1, 2$ , listed in Table 3.3(a) are expected to be correct to eight significant figures.)

$l$	$h_1$	$h_2$	$h$	$E_h$	$\delta_h$
0.00	0.815 399 73	1.121 813 26	1.937 329 02	1.2 E-04	-
0.25	1.065 491 74	1.371 813 33	2.437 329 08	2.4 E-05	2.005
0.50	1.315 510 77	1.621 813 33	2.937 329 08	5.0 E-06	2.001
0.75	1.565 514 72	1.871 813 33	3.437 329 08	1.0 E-06	2.000
1.00	1.815 515 54	2.121 813 33	3.937 329 08	2.1 E-07	2.000
1.25	2.065 515 71	2.371 813 33	4.437 329 08	4.5 E-08	2.000

Table 3.3(a)

$l$	$E_g\{1\}$	$\delta_g\{1\}$	$E_g\{2\}$	$\delta_g\{2\}$
0.00	8.9 E-03	-	8.8 E-03	-
0.25	4.0 E-03	1.027	3.9 E-03	1.017
0.50	1.8 E-03	1.011	1.8 E-03	1.007
0.75	8.1 E-04	1.005	8.1 E-04	1.003
1.00	3.7 E-04	1.002	3.7 E-04	1.001
1.25	1.7 E-04	1.000	1.7 E-04	1.000

Table 3.3(b)

$l$	$E_X\{1\}$	$\delta_X\{1\}$	$E_X\{2\}$	$\delta_X\{2\}$	$E_\tau\{1\}$	$\delta_\tau\{1\}$	$E_\tau\{2\}$	$\delta_\tau\{2\}$
0.00	8.2 E-04	-	6.8 E-04	-	2.4 E-04	-	2.2 E-04	-
0.25	1.7 E-04	2.003	1.4 E-04	1.999	5.0 E-05	2.007	4.5 E-05	2.000
0.50	3.5 E-05	2.000	2.9 E-05	2.000	1.0 E-05	2.001	9.4 E-06	2.000
0.75	7.3 E-06	2.000	6.1 E-06	2.000	2.2 E-06	2.000	1.9 E-06	2.000
1.00	1.5 E-06	2.000	1.3 E-06	2.000	4.5 E-07	2.000	4.0 E-07	2.000
1.25	3.2 E-07	2.000	2.6 E-07	2.000	9.4 E-08	2.000	8.4 E-08	2.000

Table 3.3(c)

The numerical results of Tables 3.1-3.3 indicate clearly that in the three examples considered above

$$\delta_h = 2, \quad \delta_g\{j\} = 1; \quad j = 1, 2, \quad \text{and} \quad \delta_X\{j\}, \delta_\tau\{j\} = 2; \quad j = 1, 2.$$

Thus, the numerical results provide experimental support for the Conjectures 3.1 and 3.2, which were made in Remarks 5.2 and 5.4 of [14].

**Acknowledgement.** One of us (NSS) wishes to thank the State Scholarships Foundation of Greece for their financial support.

## References

- [1] F. Bowman, *Introduction to Elliptic Functions*, English University Press, London, 1953.
- [2] N.V. Challis and D.M. Burley, *A numerical method for conformal mapping*, IMA J. Numer. Anal. **2** (1982), 169-181.
- [3] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer, Berlin, 1964.
- [4] D. Gaier, *Ermittlung des konformen Moduls von Vierecken mit Differenzenmethoden*, Numer. Math. **19** (1972) 179-194.
- [5] D. Gaier, *Determination of conformal modules of ring domains and quadrilaterals*, Lecture Notes in Mathematics 399, Springer, New York, 1974, 180-188.
- [6] D. Gaier, *Capacitance and the conformal module of quadrilaterals*, J. Math. Anal. Appl. **70** (1979), 236-239.
- [7] D. Gaier, *On an area problem in conformal mapping*, Results in Mathematics, **10** (1986), 66-81.
- [8] D. Gaier and N. Papamichael, *On the comparison of two numerical methods for conformal mapping*, IMA J. Numer. Anal. **7** (1987), 261-282.
- [9] P. Henrici, *Applied and Computational Complex Analysis*, Vol. III, Wiley, New York, 1986.
- [10] L.H. Howell and L.N. Trefethen, *A modified Schwarz-Christoffel transformation for elongated regions*, Numerical Analysis Report 88-5, Dept. of Maths, Massachusetts Institute of Technology, Cambridge, Mass., 1988.
- [11] C.D. Mobley and R.J. Stewart, *On the numerical generation of boundary-fitted orthogonal curvilinear coordinate systems*, J. Comput. Phys. **34** (1980), 124-135.
- [12] N. Papamichael, *Numerical conformal mapping onto a rectangle with applications to the solution of Laplacian problems*, J. Comput. Appl. Math. **28** (1989) 63-83.
- [13] N. Papamichael, C.A. Kokkinos and M.K. Warby, *Numerical techniques for conformal mapping onto a rectangle*, J. Comput. Appl. Math. **20** (1987), 349-358.
- [14] N. Papamichael and N.S. Stylianopoulos, *A domain decomposition method for conformal mapping onto a rectangle*, Tech. Report TR/04/89, Dept. of Math. and Stat., Brunel University, 1989, (to appear in: Constr. Approx.).
- [15] A. Seidl and H. Klose, *Numerical conformal mapping of a towel-shaped region onto a rectangle*, SIAM J. Sci. Stat. Comput. **6** (1985), 833-842.
- [16] L.N. Trefethen, Ed., *Numerical Conformal Mapping*, North-Holland, Amsterdam, 1986; reprinted from: J. Comput. Appl. Math. **14** (1986).



- [17] J.J. Wanstrath, R.E. Whitaker, R.O. Reid, A.C. Vastano, *Storm surge simulation in transformed co-ordinates*, Tech. Report 76-3, U.S. Coastal Engineering Research Center, Fort Belvoir, Va. 1976.

Received: August 1, 1989.