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On Finite-Term Recurrence Relations for Bergman and Szegő Polynomials

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Dedicated to Nicolas Papamichael, a great mentor, collaborator, and friend.

Abstract. With the aid of Havin's Lemma (which we generalize) we prove that polynomials orthogonal over the unit disk with respect to certain weighted area measures (Bergman polynomials) cannot satisfy a finite-term recurrence relation unless the weight is radial, in which case the polynomials are simply monomials. For polynomials orthogonal over the unit circle (Szegő polynomials) we provide a simple argument to show that the existence of a finite-term recurrence implies that the weight must be the reciprocal of the square modulus of a polynomial.

Keywords. Szegő polynomials, Bergman orthogonal polynomials, recurrence relations, weights.

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1. Introduction

Let μ denote a finite positive Borel measure with compact support $S_\mu := \operatorname{supp}(\mu)$ consisting of infinitely many points in the complex plane $\mathbb{C}$ and consider the inner product defined by μ

$$\langle f, g\rangle_\mu := \int f(z)\overline{g(z)}\, d\mu(z). \tag{1.1}$$

Then there exist a unique sequence of polynomials

$$p_n(z;\mu) = p_n(z) = \lambda_n z^n + \cdots, \qquad \lambda_n > 0, \quad n = 0, 1, \dots, \tag{1.2}$$

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that are orthonormal with respect to $d\mu$; that is, $\langle p_m, p_n\rangle_\mu = \delta_{m,n}$. The Fourier expansion of $zp_n(z)$ in terms of the polynomial system $\{p_n\}_0^\infty$ yields the recurrence relation

$$zp_n(z) = \sum_{k=0}^{n+1} b_{k,n} p_k(z), \qquad n = 0, 1, \dots, \tag{1.3}$$

where the Fourier coefficients $b_{k,n}$ are given by

$$b_{k,n} = \langle zp_n, p_k\rangle_\mu, \qquad n \geq 0,\ k = 0, 1, \dots, n+1.$$

The coefficients $b_{k,n}$ constitute the entries of the (infinite) upper Hessenberg matrix

$$M = \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} & \cdots \\ b_{10} & b_{11} & b_{12} & b_{13} & \cdots \\ 0 & b_{21} & b_{22} & b_{23} & \cdots \\ 0 & 0 & b_{32} & b_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \tag{1.4}$$

which provides a representation of the "multiplication by z" operator in terms of the p_n's. We note that the entries $b_{n+1,n}$ along the first subdiagonal of (1.4) are simply the ratios of the leading coefficients:

$$b_{n+1,n} = \frac{\lambda_n}{\lambda_{n+1}}, \qquad n = 0, 1, \dots.$$

We say that the upper Hessenberg matrix is *banded* or, equivalently, that the orthonormal polynomials p_n satisfy a *finite-term recurrence* if there exists a positive integer d such that

$$b_{k,n} = 0, \qquad \text{for } 0 \leq k \leq n - d + 1. \tag{1.5}$$

For the case when μ is supported on the real axis, it is well-known that (1.3) is a finite-term recurrence (in fact, a 3-term recurrence) since $b_{k,n} = 0$ for $0 \leq k \leq n-2$, and the associated matrix M is a tridiagonal matrix that is called a Jacobi matrix. If, however, μ is not supported on a line, then the recurrence (1.3) may not terminate after a fixed number of terms as the following simple result for Szegő polynomials implies.

Proposition 1.1. *Suppose that the polynomials p_n are orthonormal with respect to a finite positive Borel measure μ with infinite support on the unit circle $\mathbb{T} := \{z\colon |z| = 1\}$. Then the Hessenberg matrix associated with μ is banded if and only if $d\mu = |dz|/|Q(z)|^2$ for some polynomial $Q(z)$ having all its zeros in the open unit disk $\mathbb{D} := \{z\colon |z| < 1\}$.*

Proof. If $Q(z) = \gamma z^m + \cdots$ is a polynomial of degree m all of whose zeros lie in $\mathbb{D}$, then it is readily verified (see also [10, Sect. 11.2]) that, for $n \geq m$, the orthonormal polynomials p_n with respect to $|dz|/|Q(z)|^2$ on $\mathbb{T}$ are

$$p_n(z) = \frac{e^{-i\arg\gamma}}{\sqrt{2\pi}} z^{n-m} Q(z). \tag{1.6}$$

Clearly then, the full sequence $\{p_n\}_0^\infty$ satisfies a finite-term recurrence.

Conversely, suppose that the orthonormal polynomials p_n corresponding to a measure μ with infinite support $S_\mu \subset \mathbb{T}$ satisfy a finite-term recurrence. Without loss of generality, we assume that μ is a probability measure. For orthonormal polynomials on $\mathbb{T}$, a result originally due to Geronimus [5] (see also [9, Sect. 4.1 and Prop. 1.5.9]) asserts that the coefficients $b_{k,n}$ in (1.3) have the following special form:

$$b_{k,n} = \begin{cases} -\dfrac{\lambda_k}{\lambda_n} P_{n+1}(0)\overline{P_k(0)} & k = 0, 1, ..., n, \\ \dfrac{\lambda_n}{\lambda_{n+1}}, & k = n+1, \end{cases} \tag{1.7}$$

where $P_k(z) := p_k(z)/\lambda_k$ are the corresponding monic orthogonal polynomials.

The assumption of bandedness in particular implies that $b_{0,n} = 0$ for each n sufficiently large. Let $m-1$ denote the largest non-negative integer n such that $b_{0,n} \neq 0$; if no such integer exists, we put $m = 0$. Then from (1.7) we must have $P_j(0) = 0$ for $j \geq m+1$ and consequently, from (1.7) and (1.3),

$$zp_n(z) = \frac{\lambda_n}{\lambda_{n+1}} p_{n+1}(z) \qquad \text{for all } n \geq m.$$

By orthonormality, it then follows that $\lambda_m/\lambda_{m+j} = 1$ and

$$p_{m+j}(z) = z^j p_m(z) \qquad \text{for all } j \geq 0. \tag{1.8}$$

As is well known, all the zeros of p_m lie in $\mathbb{D}$. Thus, from the first part of the proof, we see that the probability measures $\mu_1 := \mu$ and $\mu_2 := |dz|/(2\pi|p_m(z)|^2)$ generate the same orthonormal polynomials p_n for all $n \geq m$. But then, from the following recurrences [10, Sect. 11.4] which hold for any sequence of orthonormal polynomials $\phi_n(z) = \kappa_n z^n + \cdots$ on $\mathbb{T}$:

$$\kappa_n{}^2 = \kappa_{n+1}{}^2 - |\phi_{n+1}(0)|^2, \qquad \kappa_n z\phi_n(z) = \kappa_{n+1}\phi_{n+1}(z) - \phi_{n+1}(0)\phi^*_{n+1}(z),$$

where

$$\phi^*_k(z) = z^k\, \overline{\phi_k\left(\frac{1}{\overline{z}}\right)},$$

it follows that the orthonormal polynomials generated by μ_1 and μ_2 are also identical for $n = m-1, m-2, \ldots, 0$. Finally, from Favard's Theorem for the circle, also called Verblunsky's Theorem, we deduce that $\mu_1 = \mu_2$ (see also [9, Cor. 1.7.6]). ∎

Remark 1.1. Note that we have actually proved something considerably stronger; namely that the conclusion of Proposition 1.1 holds if we assume merely that the first row of the Hessenberg matrix has finitely many non-zero entries.

In contrast to Proposition 1.1, it is known that polynomials orthogonal with respect to any measure supported on $\mathbb{T}$ satisfy a 3-term recurrence (see e.g. [3]) if we allow one of the coefficients on the right-hand side of (1.3) to be replaced by a multiple of z. More precisely, there holds for suitable constants $c_{k,n}$:

$$zp_n(z) = c_{1,n}p_{n+1}(z) + c_{2,n}p_n(z) + zc_{3,n}p_{n-1}(z), \qquad n > 0.$$

2. Banded Hessenberg matrices for Bergman polynomials

We now turn to the primary purpose of this article, which concerns the much more difficult problem of characterizing banded Hessenberg matrices arising from *Bergman orthonormal polynomials* in the unit disk; i.e. polynomials orthogonal with respect to $d\mu(z) = w(z)dA(z)$, where dA denotes the differential of the area measure and $w(z)$ is a non-negative weight function on the unit disk $\mathbb{D}$ satisfying

$$0 < \int_{\mathbb{D}} w\,dA < \infty. \tag{2.1}$$

Note that if w is a *radial weight function*; i.e. $w(z) = w(|z|)$ for $z \in \mathbb{D}$, we find that

$$p_n(z) = \lambda_n z^n, \quad n \geq 0, \qquad \text{where } {\lambda_n}^{-2} = 2\pi \int_0^1 r^{2n+1}w(r)\,dr, \tag{2.2}$$

and clearly the p_n's satisfy a trivial finite-term recurrence relation.

For the study of Bergman polynomials we consider the inner product

$$\langle f, g\rangle_{L^2(\mathbb{D},w)} := \int_{\mathbb{D}} f(z)\overline{g(z)}w(z)\,dA(z), \tag{2.3}$$

and denote by $L^2(\mathbb{D}, w)$ the set of Borel measurable functions f on $\mathbb{D}$ for which

$$\|f\|^2_{L^2(\mathbb{D},w)} := \langle f, f\rangle_{L^2(\mathbb{D},w)} < \infty.$$

Furthermore, by $L^2_a(\mathbb{D}, w)$ we denote the subspace of functions $f \in L^2(\mathbb{D}, w)$ that are analytic in $\mathbb{D}$. In the unweighted case $w \equiv 1$ we will omit the argument w in the above notation.

Acting on the space $L^2_a(\mathbb{D}, w)$ we consider the "multiplication by z" operator $\mathcal{M}\colon f \to zf$, which is sometimes referred to as the *Bergman operator* (cf. [4]). This operator is related to the upper Hessenberg matrix (1.4) in the sense that $\mathcal{M}$ can be represented with respect to the basis $\{p_n\}_0^\infty$ by M.

The question that we consider is the following: For which weights w on $\mathbb{D}$ is the infinite matrix M banded? As earlier remarked, this is clearly the case for radial weights. We surmise that these are the only such instances when there is a finite-term recurrence for the Bergman polynomials p_n. More precisely, we propose the following.

Conjecture 2.1. Assume that the weight w on $\mathbb{D}$ is positive a.e. in an annulus of the form $\{z\colon r < |z| < 1\}$. If the Hessenberg matrix M for the Bergman polynomials associated with w is banded, then w is a radial weight.

This conjecture is motivated by results of Putinar and Stylianopoulos [8] and Khavinson and Stylianopoulos [7] who studied a similar question but for the unweighted case of Bergman polynomials orthogonal with respect to dA over a finite Jordan region. They showed, in particular, that under mild conditions on the boundary of the Jordan region, the associated Hessenberg matrix is banded only in the case when the region is bounded by an ellipse.

Here we shall prove that the above conjecture holds true for a special class of weight functions w.

Theorem 2.1. *Suppose that $w(z) = |h(z)|^2$, $h \in L^2_a(\mathbb{D})$ with $h(z) \neq 0$ in $\mathbb{D}$, and that polynomials are complete in $L^2_a(\mathbb{D}, w)$. If the Hessenberg matrix associated with w is banded, then w is constant in $\mathbb{D}$.*

We remark that if h is zero-free and analytic on the closed disk $\overline{\mathbb{D}}$, then the hypotheses on h in Theorem 2.1 are satisfied.

The proof of Theorem 2.1, which is given in the next section, is motivated by the argument in [8]. It relies on a lemma of Havin [6]. Although not needed for the proof of Theorem 2.1, but because of its possible independent interest, we provide here an extension of this lemma for an arbitrary open set Ω in $\mathbb{C}$ of *bounded width*; that is, a set that lies between two parallel straight lines.

For w a non-negative weight function on Ω, we consider the Hilbert spaces $L^2(\Omega, w)$ and $L^2_a(\Omega, w)$ that are defined as we did above for $\mathbb{D}$; in particular, for $f \in L^2(\Omega, w)$,

$$\|f\|_{L^2(\Omega,w)} := \left(\int_\Omega |f(z)|^2 w(z)\,dA(z)\right)^{1/2} < \infty.$$

The Sobolev space $W^{1,2}(\Omega)$ consists of complex functions in the unweighted space $L^2(\Omega)$ whose distributional derivatives of the first-order again belong to $L^2(\Omega)$. It is a Hilbert space endowed with the norm

$$\|f\|_{W^{1,2}(\Omega)} := \left(\|f\|^2_{L^2(\Omega)} + \|\partial_z f\|^2_{L^2(\Omega)} + \|\partial_{\overline{z}} f\|^2_{L^2(\Omega)}\right)^{1/2},$$

where, for $z = x + iy$, we set

$$\partial_z f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \partial_{\overline{z}} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

We note for later use the obvious relation $\overline{\partial_z f} = \partial_{\overline{z}}\overline{f}$. The closure in $W^{1,2}(\Omega)$ of the space $C_0^\infty(\Omega)$ of infinitely smooth complex-valued functions with compact support in Ω is denoted by $W_0^{1,2}(\Omega)$.

For $1/w$ locally integrable in Ω, it is an easy consequence of the Cauchy formula that $L_a^2(\Omega, w)$ is closed in $L^2(\Omega, w)$. Therefore, in this case, we have an orthogonal decomposition

$$L^2(\Omega, w) = L_a^2(\Omega, w) \oplus \left(L_a^2(\Omega, w)\right)^\perp .$$

In the particular case where Ω has bounded width and $w(z) = |h(z)|^2$, with h analytic and non-zero in Ω, the orthogonal complement of $L_a^2(\Omega, w)$ is easy to describe and gives rise to the following simple extension of Havin's Lemma.

Lemma 2.1. *If Ω has bounded width and h is analytic and non-vanishing in Ω, then it holds for $w(z) = |h(z)|^2$ that*

$$L^2(\Omega, w) = L_a^2(\Omega, w) \oplus \frac{1}{h}\partial_z W_0^{1,2}(\Omega). \tag{2.4}$$

3. Proofs of Lemma 2.1 and Theorem 2.1

Proof of Lemma 2.1. Assume $f \in L_a^2(\Omega, w)$ and $F \in C_0^\infty(\Omega)$. Since

$$\overline{h}\partial_z F = \partial_z(\overline{h}F),$$

it is straightforward from the definitions above that

$$\left\langle f, \frac{1}{h}\partial_z F\right\rangle_{L^2(\Omega,w)} = \left\langle f, \partial_z(\overline{h}F)\right\rangle_{L^2(\Omega)} . \tag{3.1}$$

Let Γ be a positively oriented system of smooth curves included in Ω, encompassing the support of F. By the Green formula we get, since f is analytic, that

$$\begin{aligned}\langle f, \partial_z(\overline{h}F)\rangle_{L^2(\Omega)} &= \frac{i}{2}\int_\Omega f(z)\partial_{\overline{z}}[h(z)\overline{F}(z)]\, dz \wedge d\overline{z} \\ &= \frac{i}{2}\int_\Omega \partial_{\overline{z}}[f(z)h(z)\overline{F}(z)]\, dz \wedge d\overline{z} \\ &= -\frac{i}{2}\int_\Gamma f(z)h(z)\overline{F}(z)\, dz = 0.\end{aligned}$$

Therefore,

$$L_a^2(\Omega, w) \subset \left(\frac{1}{h}\partial_z C_0^\infty(\Omega)\right)^\perp . \tag{3.2}$$

Conversely, assume that $f \in L^2(\Omega, w)$ is orthogonal to $\frac{1}{h}\partial_z C_0^\infty(\Omega)$. As F ranges over $C_0^\infty(\Omega)$, so does $\overline{h}F$, since h does not vanish in Ω. Therefore, in view of (3.1), which is valid even if f is not analytic, it holds that

$$0=\int_\Omega f\,\overline{\partial_z G}\,dA=\int_\Omega f\,\partial_{\overline{z}}\overline{G}\,dA, \qquad \text{for all } G\in C_0^\infty(\Omega),$$

showing that $\partial_{\overline{z}}f=0$, in the sense of distributions. Thus, by Weyl's Lemma f is analytic in Ω. Hence, by taking (3.2) into account we obtain

$$L_a^2(\Omega,w)=\left(\frac{1}{h}\partial_z C_0^\infty(\Omega)\right)^\perp. \tag{3.3}$$

Passing to orthogonal complements this yields

$$\big(L_a^2(\Omega,w)\big)^\perp=\overline{\left(\frac{1}{h}\partial_z C_0^\infty(\Omega)\right)}, \tag{3.4}$$

where the bar over the space on the right refers to the closure in $L^2(\Omega,w)$.

To prove that the right-hand side of (3.4) is equal to $\frac{1}{h}\partial_z W_0^{1,2}(\Omega)$, it is equivalent to show that the closure of $\partial_z C_0^\infty(\Omega)$ in $L^2(\Omega)$ is $\partial_z W_0^{1,2}(\Omega)$. For this, let $\{\varphi_n\}$ be a sequence of functions in $C_0^\infty(\Omega)$ such that $\partial_z\varphi_n\to\psi$ in $L^2(\Omega)$. We claim that

$$\overline{\varphi_n(z)}=\frac{1}{2\pi i}\int_\Omega\frac{\overline{\partial_z\varphi_n(z)}}{\zeta-z}\,d\zeta\wedge d\overline{\zeta}, \qquad z\in\mathbb{C}, \tag{3.5}$$

where it is understood that φ_n extends as the zero function in $\mathbb{C}\setminus\Omega$. Indeed, if we call $g(z)$ the right-hand side of (3.5), we observe, since $1/z$ is a fundamental solution to $\partial_{\overline{z}}$, that

$$\partial_{\overline{z}}g(z)=\overline{\partial_z\varphi_n(z)}=\partial_{\overline{z}}\overline{\varphi_n(z)}.$$

Hence $\overline{\varphi_n}-g$ is analytic in $\mathbb{C}$, and since φ_n has compact support while g vanishes at infinity, it must be the zero function. This proves the claim.

From (3.5), it follows that

$$\overline{\partial_{\overline{z}}\varphi_n(z)}=\partial_z\overline{\varphi_n(z)}=\frac{1}{2\pi i}\int_\Omega\frac{\overline{\partial_z\varphi_n(z)}}{(\zeta-z)^2}\,d\zeta\wedge d\overline{\zeta}, \qquad z\in\mathbb{C}, \tag{3.6}$$

where the integral in the right-hand side of (3.6), to be understood in the sense of principal value, is the so-called Beurling transform of $\overline{\partial_z\varphi_n(z)}$ [2, Ch. 4, p. 95]. Because the Beurling transform is isometric on $L^2(\mathbb{C})$, our assumption that $\partial_z\varphi_n\to\psi$ now implies that $\partial_{\overline{z}}\varphi_n$ also converges in $L^2(\Omega)$. Since Ω has finite width, we conclude from the Poincaré inequality that φ_n converges to some function φ in $W_0^{1,2}(\Omega)$; see [1, Thm. 6.30]. Moreover, by continuity of differentiation from $W^{1,2}(\Omega)$ into $L^2(\Omega)$, we get that $\partial_z\varphi=\psi$. Altogether, we have shown that the closure of $\partial_z C_0^\infty(\Omega)$ in $L^2(\Omega)$ is included in $\partial_z W_0^{1,2}(\Omega)$, but the reverse inclusion is trivial by the density of $C_0^\infty(\Omega)$ in $W_0^{1,2}(\Omega)$ and the continuity of differentiation alluded to above, so the lemma follows. ∎

Next for $F \in L^2(\Omega, w)$, we denote by $P_w(F)$ the projection of F onto the space $L^2_a(\Omega, w)$; that is,

$$\|F - P_w(F)\|_{L^2(\Omega,w)} = \inf_{p\in L^2_a(\Omega,w)} \|F - p\|_{L^2(\Omega,w)}. \tag{3.7}$$

Observe that for $f, g \in L^2_a(\Omega, w)$, we have

$$\langle \mathcal{M}g, f\rangle_{L^2(\Omega,w)} = \langle zg, f\rangle_{L^2(\Omega,w)} = \langle g, \overline{z}f\rangle_{L^2(\Omega,w)} = \langle g, P_w(\overline{z}f)\rangle_{L^2(\Omega,w)},$$

since

$$\langle g, \overline{z}f - P_w(\overline{z}f)\rangle_{L^2(\Omega,w)} = 0.$$

Hence the adjoint of the operator $\mathcal{M}\colon L^2_a(\Omega, w) \to L^2_a(\Omega, w)$ is given by

$$\mathcal{M}^* f = P_w(\overline{z}f), \qquad f \in L^2_a(\Omega, w). \tag{3.8}$$

Proof of Theorem 2.1. Taking $\Omega = \mathbb{D}$ and $f(z) = z^k$, $k = 0, 1, 2, \ldots$, we have from Havin's Lemma (the original version suffices) that

$$\begin{aligned}\overline{z}z^k &= P_w(\overline{z}z^k) + \frac{1}{h(z)}\partial_z g_k(z)\\ &= \mathcal{M}^* z^k + \frac{1}{h(z)}\partial_z g_k(z),\end{aligned} \tag{3.9}$$

for some function $g_k \in W^{1,2}_0(\mathbb{D})$. Now assume that the Hessenberg matrix M is d-banded; that is, (1.5) holds. Since the Bergman polynomials form a complete orthonormal system in $L^2_a(\mathbb{D}, w)$, we can use the matrix representation of $\mathcal{M}^*$ to deduce that

$$\mathcal{M}^* z^k = q_k(z),$$

where $q_k(z)$ is a polynomial of degree at most $k+d-2$. Consequently, from (3.9) we have

$$\overline{z}z^k h(z) = q_k(z)h(z) + \partial_z g_k(z).$$

and integration with respect to z yields

$$\overline{z}H_k(z) = Q_k(z) + g_k(z) + \overline{F_k}(z), \qquad z \in \mathbb{D}, \tag{3.10}$$

where $H'_k(z) = z^k h(z)$, $Q'_k(z) = q_k(z)h(z)$, and $F_k(z) = \sum_{j=0}^{\infty} a_{j,k}z^j$ is analytic in $\mathbb{D}$. Since $h \in L^2_a(\mathbb{D})$, it is easily verified that H_k and Q_k belong to the Hardy space $\mathcal{H}^2$ on the unit circle $\mathbb{T}$. Thus, as g_k has bounded L^2 circle means on $\mathbb{D}$, we see from (3.10) that the same is true for F_k; that is $F_k \in \mathcal{H}^2$. But g_k has trace zero on $\mathbb{T}$ (cf. [1, Ch. 5]), and so after multiplication by z we obtain

$$H_k(z) = zQ_k(z) + z\overline{F_k(z)}, \qquad \text{a.e. on } \mathbb{T}. \tag{3.11}$$

Hence, by the analytic properties of $H_k(z)$ and $zQ_k(z)$ it follows that

$$z\overline{F_k(z)} = z\overline{a_{0,k}} + \overline{a_{1,k}}$$

and consequently

$$H_k(z) = zQ_k(z) + z\overline{a_{0,k}} + \overline{a_{1,k}}, \qquad z \in \mathbb{D}.$$

Differentiating this last equation twice with respect to z yields for $k = 0, 1, \ldots,$

$$\frac{h'(z)}{h(z)} = \frac{\pi_k(z)}{z^k - zq_k(z)}, \qquad z \in \mathbb{D}, \tag{3.12}$$

where

$$\pi_k(z) := -kz^{k-1} + 2q_k(z) + zq_k'(z). \tag{3.13}$$

To prove that h is constant we apply the identity (3.12) for two consecutive integers k and $k+1$ and cross-multiply to obtain

$$\pi_{k+1}(z)(z^k - zq_k(z)) = \pi_k(z)(z^{k+1} - zq_{k+1}(z)). \tag{3.14}$$

When $k = 0$ we deduce on setting $z = 0$ in (3.14) that $q_1(0) = 1/2$. More generally it follows by induction that

$$q_k^{(j)}(0) = \begin{cases} 0 & \text{for } 0 \leq j \leq k-2, \\ \dfrac{k!}{k+1} & \text{for } j = k-1. \end{cases} \tag{3.15}$$

Indeed, suppose this is true for $k = m (\geq 1)$. Then from (3.13) we see that π_m has a zero of order at least m at $z = 0$. Next, we write (3.14) in the form

$$\pi_{m+1}(z)(z^{m-1} - q_m(z)) = \pi_m(z)(z^m - q_{m+1}(z)), \tag{3.16}$$

and observe from the induction hypothesis that $z^{m-1} - q_m(z)$ has a zero of exact order $m-1$ at $z = 0$. Consequently, $\pi_{m+1}(0) = 0$, which implies that $q_{m+1}(0) = 0$. But then the right-hand side of (3.16) has a zero of order at least $m+1$ which, in turn, implies that $\pi_{m+1}'(0) = 0$. Continuing in this manner, we deduce that

$$\pi_{m+1}^{(j)}(0) = 0, \qquad \text{for } j = 0, 1, \ldots, m,$$

from which it follows that (3.15) holds for $k = m+1$. The induction is complete.

Next observe that if, for any integer k, we have $\deg q_k = k-1$, then from (3.15) we deduce that $q_k(z) = kz^{k-1}/(k+1)$. But then $\pi_k(z) \equiv 0$ and so from the representation for $h'(z)/h(z)$ in (3.12) we find that $h'(z) \equiv 0$ and so h is constant on $\mathbb{D}$. Thus to complete the proof of Theorem 2.1 it suffices to assume that $\deg q_k > k-1$ for each $k \geq 0$. (Here we consider the identically zero polynomial to have degree minus infinity.)

Returning to the cross-multiplied formula (3.14), we replace π_{k+1} and π_k by their expressions in (3.13), and starting with $k = 0$ we compare coefficients of the highest power terms on both sides to deduce that

$$\deg q_0 = \deg q_1 = \ldots = \deg q_{d-1}. \tag{3.17}$$

But for the d-term recurrence we already observed that $\deg q_k \leq k + d - 2$, and so (3.17) implies that $\deg q_{d-1} \leq d-2$, which contradicts our assumption that $\deg q_{d-1} > d-2$. ∎

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