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Zero Distributions for Polynomials Orthogonal with Weights over Certain Planar Regions

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Abstract. Let G be a bounded Jordan domain in $\mathbb{C}$ and let $w \not\equiv 0$ be an analytic function on G such that $\int_G |w|^2 dm < \infty$, where dm is the area measure. We investigate the zero distribution of the sequence of polynomials that are orthogonal on G with respect to $|w|^2 dm$. We find that such a distribution depends on the location of the singularities of the reproducing kernel $K_w(z,\zeta)$ of the space $\mathcal{L}^2_w(G) := \left\{ f \text{ analytic on } G : \int_G |f|^2|w|^2\, dm < \infty \right\}$. A fundamental theorem is given for the case when $K_w(\cdot,\zeta)$ has a singularity on ∂G for at least some $\zeta \in G$. To investigate the opposite case, we consider two examples in detail: first when G is the unit disk and w is meromorphic, and second when G is a lens-shaped domain and w is entire. Our analysis can also be applied for $w \equiv 1$ in the case when G is a rectangle or a special triangle. We also provide formulas for $K_w(\cdot,\zeta)$ that are of help for the determination of its singularities.

Keywords. Orthogonal polynomials, zeros of polynomials, kernel function, logarithmic potential, equilibrium measure.

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1. Introduction

Let G be the interior of a closed Jordan curve $L = \partial G$ in the complex plane, and let $dm = dxdy$ denote the two-dimensional Lebesgue measure. For a function $w\colon \mathbb{C} \to \mathbb{C}$, analytic and not identically zero on G that satisfies the integrability condition

$$\int_G |w(z)|^2\, dm(z) < \infty\,, \tag{1}$$

we consider the space

$$\mathcal{L}^2_w(G) := \left\{ f \text{ analytic on } G \,:\, \int_G |f(z)|^2 |w(z)|^2\, dm(z) < \infty \right\}, \tag{2}$$

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endowed with the inner product and corresponding norm

$$\langle f|g\rangle_w := \int_G f(z)\,\overline{g(z)}\,|w(z)|^2\,dm(z)\,, \qquad \|f\|_{\mathcal{L}^2_w(G)} := \sqrt{\langle f|f\rangle_w}\,. \tag{3}$$

Let $\{P_n(z;w)\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to the measure $|w|^2 dm|_G$. This is the sequence of polynomials,

$$P_n(z;w) = \kappa_n^w z^n + \cdots\,, \qquad \kappa_n^w > 0,\ n = 0,\,1,\,2,\ldots,$$

that are orthonormal with respect to the inner product $\langle\cdot|\cdot\rangle_w$. When $w \equiv 1$, these polynomials are often called *Carleman* or *Bergman polynomials* for G.

The aim of this paper is to investigate the zero distribution of the sequence of *weighted Bergman polynomials* $\{P_n(z;w)\}_{n=0}^{\infty}$. Namely, we address the following question: *given a domain G and a function w as described above, where do the zeros of the P_n's accumulate as $n \to \infty$?*

This question has been studied to some extent for the Bergman polynomials of G (see for e.g. [6], [5]). In [5], the authors found that the zero distribution of the polynomials $P_n(z;1)$ is related to the analytic continuation properties of a conformal mapping φ of G onto the unit disk $\mathbb{D}$. Some of their main results are particular cases of ours.

A key role in our investigation is played by the reproducing kernel of the space $\mathcal{L}^2_w(G)$, which is the unique function

$$K_w(z,\zeta)\colon G \times G \to \mathbb{C} \tag{4}$$

such that

$$K_w(\cdot,\zeta) \in \mathcal{L}^2_w(G), \quad \forall\,\zeta \in G, \quad \text{and} \quad f(\zeta) = \langle f|K_w(\cdot,\zeta)\rangle_w, \quad \forall\, f \in \mathcal{L}^2_w(G). \tag{5}$$

When w is a function as described above, we find that the zero distribution of the $P_n(\cdot\,;w)$'s depends on the analytic continuation properties of the family of functions $\{K_w(\cdot,\zeta) : \zeta \in G\}$. For example, Theorem 2.1 of Section 2 below, which extends [5, Thm. 2.1], can be roughly stated as follows.

If w is such that the polynomials are dense in $\mathcal{L}^2_w(G)$, and if for some $\zeta \in G$, $K_w(\cdot,\zeta)$ has a singularity on the boundary ∂G of G, then every point of ∂G attracts zeros of the P_n's (a converse of this statement is valid in some sense as well).

The relevance of this result is strengthened by the fact that we have formulas that express $K_w(z,\zeta)$ in terms of the weight w and a conformal mapping φ of G onto the unit disk $\mathbb{D}$, which help us to determine the singularities of $K_w(\cdot,\zeta)$, and in particular, whether or not this kernel has a singularity on ∂G. For instance, it is well-known that if $w(z) \neq 0$ for all $z \in G$, then (see [16, p. 37])

$$K_w(z,\zeta) = \frac{\varphi'(z)\overline{\varphi'(\zeta)}}{\pi\, w(z)\,\overline{w(\zeta)}\left[1 - \varphi(z)\overline{\varphi(\zeta)}\right]^2}\,.$$

It is clear from this formula that certain properties of φ and w will guarantee that $K_w(\cdot, \zeta)$ has a singularity on ∂G. Possibly the simplest is that w has a zero at a point $z_0 \in \partial G$ in a neighborhood of which $|\varphi'|$ is bounded below.

Much more interesting is the situation when w has zeros in G. In this paper we derive formulas for $K_w(z, \zeta)$ when the number of these zeros is finite. In particular, in Proposition 2.3 of Section 2 we describe an iterative procedure that, given a zero $a \in G$ of w, allows one to construct $K_w(z, \zeta)$ from the kernel corresponding to the weight $w(z)/(z-a)$. Applying this procedure we derive Lemma 3.5 of Section 3, which gives a representation of the kernel in terms of w and φ. If the zeros of w inside G are simple, then a simple determinant representation for $K_w(z, \zeta)$ is also valid (see Proposition 3.4).

To gain insight into what can happen in the less transparent situation where $K_w(\cdot, \zeta)$ can be analytically continued across ∂G for every $\zeta \in G$, we analyze in detail two specific cases. First, we let G be the unit disk, and take w to be meromorphic with no poles in $\overline{G}$. We prove that the zeros of the $P_n(\cdot\,; w)$ accumulate on a disk of radius $r \leq 1$, and each point of the boundary of this disk attracts zeros of the polynomials. The radius r is determined by the zeros and the poles of w.

In the second case, G is a domain bounded by two circular arcs that meet at $-i$ and i with opening angle π/N, $N \in \mathbb{N}$, $N \geq 2$. The weight w is taken to be an entire function. This case was studied in [5] for $N = 2$, $w \equiv 1$, and it was shown that the zeros of Bergman polynomials for these lens-shaped domains accumulate on an arc Γ that connects the vertices $-i$, i (see Figure 1(a)). The same result is true for $N > 2$. For a general entire function w, we find that the zeros of the

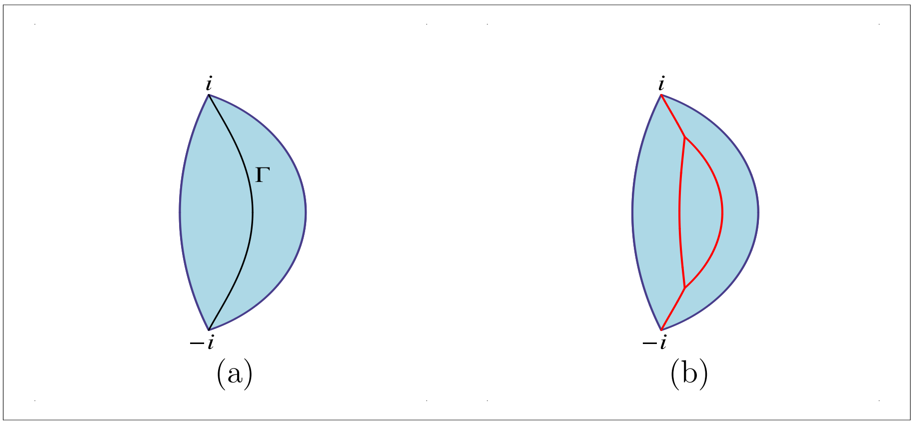


FIGURE 1. (a) Zeros accumulate on Γ in unweighted case and (b) on bubble with subarcs of Γ in weighted case for w entire.

$P_n(\cdot\,; w)$'s accumulate on a compact set consisting of two subarcs of the same curve Γ and a "bubble" connecting these two subarcs (see Figure 1(b)). This bubble is determined by the zeros of w, and each boundary point of it, as well as each point of the two subarcs, attracts zeros of the polynomials.

We remark that in both of the above cases one can consider more general functions w. As long as we are able to determine the singularity of $K_w(\cdot,\zeta)$ that is closest (in some sense) to G, our method of proof will yield similar results. For example, the same phenomenon is observed in the case of a lens if one considers meromorphic weights. However, we restrict ourselves to the case of w entire for the sake of simplicity. We also state, without proof, some results that can be obtained by applying the same ideas and methods of the present paper to Bergman polynomials for rectangles as well as for special triangles G whose interior conformal mapping φ has no singularities on ∂G.

The rest of this paper is organized as follows. In Section 2 we introduce some notation and present the main results. In Section 3 we establish the existence of the kernel function as well as some of its properties and formulas. In Section 4 we derive a basic relation between the orthogonal polynomials and the kernel function (Corollary 4.2), and give (in Lemma 4.3) the general argument that is employed in Section 5 to prove the zero distribution results.

2. Main results

Throughout this paper, (G,w) will denote a pair formed by a bounded Jordan domain G and a function $w:\mathbb{C}\to\mathbb{C}$ that is analytic and not identically zero on G, and that satisfies (1). In each theorem, it will be clearly stated whether any other property of G or w is assumed, and $P_n(z):=P_n(z;w)$ will denote the n-orthonormal polynomial with respect to the measure $|w|^2dm|_G$ corresponding to the domain G and weight w so specified. The letter $\mathbb{D}$ will stand for the open unit disk and $\mathbb{D}_r$ for the open disk $\{z:|z|<r\}$.

For any G under consideration,

$$\Phi\colon \overline{\mathbb{C}}\setminus\overline{G}\to\overline{\mathbb{C}}\setminus\overline{\mathbb{D}} \tag{6}$$

will denote the exterior conformal map from $\overline{\mathbb{C}}\setminus\overline{G}$ onto $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$, normalized so that $\Phi(\infty)=\infty$ and $\Phi'(\infty)>0$. This map Φ can be naturally extended to a homeomorphism (also denoted by Φ) between $L:=\partial G$ and the unit circle $\mathbb{T}:=\partial\mathbb{D}$. Then, the *equilibrium measure* μ_L of the compact set L can be defined as the preimage by Φ of the normalized arclength measure $|dz|/2\pi$ on $\mathbb{T}$, that is,

$$\mu_L(A):=\frac{1}{2\pi}\int_{\Phi(A)}|dz|$$

for any Borel set $A\subset L$. We refer the reader to [12] or [14] for the definition of the equilibrium measure of more general compact sets and also for the related notion of *logarithmic capacity* of a set E, which we denote by $\operatorname{cap}(E)$.

If Q is a polynomial of degree n with zeros $z_1, z_2, \dots, z_n$ (listed according to multiplicity), *the normalized counting measure of the zeros of* Q is denoted by ν_Q and defined by

$$\nu_Q := \frac{1}{n}\sum_{k=1}^{n} \delta_{z_k}\,,$$

where δ_z denotes the unit mass at the point z.

We say that the sequence of Borel measures $\{\sigma_n\}$ *converges in the weak*-sense* to a measure σ, symbolically $\sigma_n \xrightarrow{*} \sigma$, if

$$\lim_{n\to\infty} \int_{\overline{\mathbb{C}}} f\, d\sigma_n = \int_{\overline{\mathbb{C}}} f\, d\sigma\,,$$

for every function f continuous on the extended complex plane $\overline{\mathbb{C}}$.

Recall that $K_w(z,\zeta)$, defined by (4) and (5), is the reproducing kernel of the space $\mathcal{L}^2_w(G)$ introduced in (2). The existence of this kernel, as well as some of its properties, will be established in Section 3. With this notation, we have the following basic theorem:

Theorem 2.1. *For any* (G, w) *as above, if*

(a) *there exists a subsequence* $\mathcal{N} \subset \mathbb{N}$ *such that*

$$\nu_{P_n} \xrightarrow{*} \mu_L \qquad \text{as } n \to \infty\,,\ n \in \mathcal{N}\,,$$

then

(b) *there exists a point* $\zeta \in G$ *for which* $K_w(\cdot,\zeta)$ *has a singularity on the boundary* L *of* G.

Moreover, if w *is such that the polynomials are dense in* $\mathcal{L}^2_w(G)$*, then* (b)⇒(a)*; that is,* (a) *and* (b) *are equivalent.*

Remark 2.2. There are several results giving conditions that ensure the completeness of the system of polynomials in Banach spaces of analytic functions on a domain G whose norm is given by an integral over G with respect to a weight function. For example, see the survey [7] and the papers [2], [3], as well as the references therein. Here, we just mention that when w is analytic in $\overline{G}$ (which is the case in Theorems 2.5 and 2.9 below), the polynomials are dense in $\mathcal{L}^2_w(G)$. This assertion is easy to verify with the help of [2, Thm. 2].

We drop the subscript w and write $K(z,\zeta)$ for the kernel corresponding to $w \equiv 1$, which is the so-called *Bergman kernel function* of G. Notice that the possibility of continuing $K(\cdot,\zeta)$ analytically across L is independent of ζ since, as easily follows from (22), $K(\cdot,\zeta)$ has a singularity on L if and only if an interior conformal map φ has a singularity on L.

For the practical determination of the singularities of $K_w(\cdot,\zeta)$, one can use formula (22) of Section 3 for a weight $w \neq 0$. When w has finitely many zeros on G, the iterative procedure given in Proposition 2.3 below can be used to find $K_w(z,\zeta)$ in terms of the weight w and a conformal map φ of G onto $\mathbb{D}$ (see Lemma 3.5 of Section 3). Such iterative representation for $K_w(z,\zeta)$ follows along the same lines as the formulas in [10, ex. 11, p. 262] and [4, p. 58]. As usual, any empty product of the form $\prod_{i=1}^{0}\cdots$ is understood to equal 1.

Proposition 2.3. *Let (G,w) be such that w has exactly $n\geq 0$ zeros in G, counting multiplicity. Write w as $w(z)=h(z)\prod_{i=1}^{n}(z-a_i)$, with $\{a_1,a_2,\dots,a_n\}\subset G$ and $h(z)\neq 0$ for $z\in G$ (the a_i's not necessarily distinct). Then*

$$K_w(z,\zeta)=\frac{H_n(z,\zeta)}{h(z)\,\overline{h(\zeta)}}\,, \tag{7}$$

where $H_n(z,\zeta)$ is constructed from the sequence $\{a_1,a_2,\dots,a_n\}$ by using the following iterative procedure:

$$H_0(z,\zeta):=K(z,\zeta)\,;$$

if $H_i(z,\zeta)$ is already defined for all $z,\zeta\in G$, put

$$H_{i+1}(z,\zeta):=\frac{H_i(z,\zeta)-\dfrac{H_i(a_{i+1},\zeta)H_i(z,a_{i+1})}{H_i(a_{i+1},a_{i+1})}}{(z-a_{i+1})\overline{(\zeta-a_{i+1})}}\,,\qquad \forall\, z,\,\zeta\in G\setminus\{a_{i+1}\}\,,$$

and

$$H_{i+1}(a_{i+1},\zeta):=\lim_{z\to a_{i+1}}H_{i+1}(z,\zeta)\,,\qquad \forall\,\zeta\in G\setminus\{a_{i+1}\}\,,$$
$$H_{i+1}(z,a_{i+1}):=\lim_{\zeta\to a_{i+1}}H_{i+1}(z,\zeta)\,,\qquad \forall\,z\in G\,.$$

We shall see that when $K_w(\cdot,\zeta)$ can be analytically continued across $L=\partial G$ for every $\zeta\in G$, very different situations may arise. A simple application of Carleman's strong asymptotic formula (see [1, Thm. 2, p. 12]) together with Lemma 4.3 yield the following proposition, which shows that in the case when $w\equiv 1$ and G is bounded by an analytic Jordan curve, the zeros of the Bergman polynomials for G can accumulate deeply inside G, on rather arbitrary compact sets.

Proposition 2.4. *Suppose that $E\subset\mathbb{C}$ is a continuum, not a single point, and let Ω be the unbounded component of $\overline{\mathbb{C}}\setminus E$. Let Φ_E be the conformal map from Ω onto $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$ such that $\Phi_E(\infty)=\infty$ and $\Phi_E'(\infty)>0$. For a fixed number $r>1$, define*

$$l_r:=\{z:\,|\Phi_E(z)|=r\}\qquad\text{and}\qquad G:=\operatorname{int}(l_r),$$

and let P_n be the n-th Bergman polynomial for G. Then for any closed set $K\subset\Omega$, there exists $n_0\in\mathbb{N}$ such that for all $n>n_0$, the zeros of P_n lie outside K. Moreover, if E has empty interior and does not separate the plane, then

$$\nu_{P_n}\xrightarrow{\;*\;}\mu_E\qquad\text{as } n\to\infty\,,$$

where μ_E is the equilibrium measure of the compact set E.

In the simplest case of Proposition 2.4, when $E = \{z : |z| \le 1/r\}$ and G is the unit disk, the Bergman polynomials for G are $P_n(z) = \sqrt{(n+1)/\pi}\, z^n$, so that all their zeros coincide with 0. For $w \not\equiv 1$, we have the following result:

Theorem 2.5. *Let $w \not\equiv 0$ be a meromorphic function in $\mathbb{C}$ which is analytic in $\overline{\mathbb{D}}$. Let*

$$\begin{aligned} \{a_1, \ldots, a_\ell\} &= \textit{set of zeros of } w \textit{ in } \mathbb{D}\,, \\ \{b_1, b_2, \ldots\} &= \textit{set of zeros of } w \textit{ in } \mathbb{C}\setminus\mathbb{D}\,, \\ \{c_1, c_2, \ldots\} &= \textit{set of poles of } w\,; \end{aligned}$$

and let

$$\mathcal{A} := \{|a_i| : 1/\overline{a}_i = c_j \textit{ for some } j \textit{ and } \operatorname{mult}(c_j) \ge \operatorname{mult}(a_i) + 1\}\,,$$

where $\operatorname{mult}(c_j)$ *and* $\operatorname{mult}(a_i)$ *denote the respective orders of the pole c_j and the zero a_i. Set*

$$r := \max\left(\{0, |a_1|, \ldots, |a_\ell|, |b_1|^{-1}, |b_2|^{-1}, \ldots\} \setminus \mathcal{A}\right)\,.$$

Then, for all but countably many $z \in \mathbb{D}$,

$$\limsup_{n\to\infty} |P_n(z)|^{1/n} = \begin{cases} |z| & \textit{if } |z| > r \\ r & \textit{if } |z| \le r \end{cases}\,, \tag{8}$$

which implies that

(a) *if $r = 0$, then*

$$\nu_{P_n} \xrightarrow{*} \delta_0 \qquad \textit{as } n \to \infty\,,$$

where δ_0 denotes the unit point mass at 0;

(b) *if $r > 0$, then any measure that is a weak*-limit point of the sequence $\{\nu_{P_n}\}$ is supported in $\overline{\mathbb{D}}_r := \{z : |z| \le r\}$. Let $\mathcal{N} \subset \mathbb{N}$ be a subsequence (which indeed exists) such that the* lim sup *in* (8) *is realized for some $z \in \mathbb{D}_r$. Then*

$$\nu_{P_n} \xrightarrow{*} \mu_r \qquad \textit{as } n \to \infty\,,\ n \in \mathcal{N}\,,$$

where $\mu_r := |dz|/2\pi r$ is the normalized arclength measure on the circle $\mathbb{T}_r := \{z : |z| = r\}$.

Example 2.6. Let $w(z) := (z-a)^v/(1-z\overline{a})^\lambda$, where $0 < |a| < 1$ and $v \ge 1$, $\lambda \ge 0$ are integers. Then, according to Theorem 2.5, when $\lambda < v+1$, $\{\nu_{P_n}\}$ has at least a subsequence converging weakly* to $\mu_{|a|}$. However, if $\lambda \ge v+1$, the entire sequence $\{\nu_{P_n}\}$ converges weakly* to δ_0. Figure 2 illustrates the case $\lambda = 0$, $v = 1$. Figure 3 illustrates the case $\lambda = 2$, $v = 1$. Another example is discussed after the proof of Theorem 2.5 in Section 5.

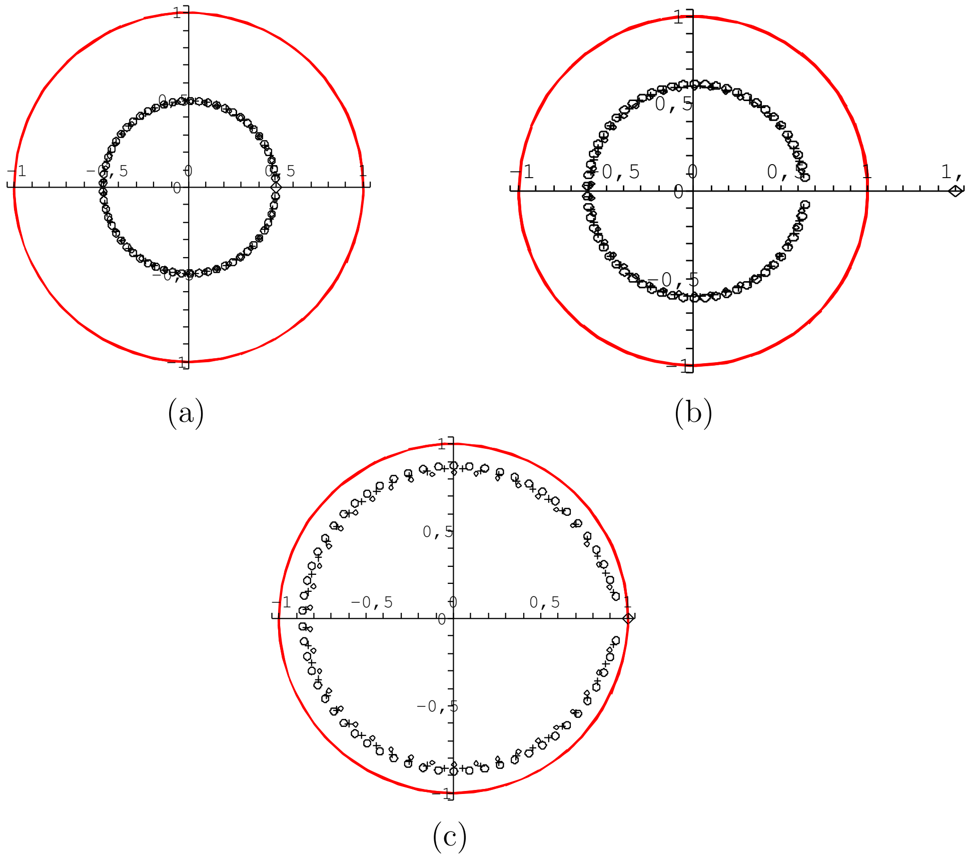


FIGURE 2. Zeros of P_n, $n = 40(\diamond)$, $50(+)$, $60(\circ)$, for $G = \mathbb{D}$ and (a) $w(z) = z - 1/2$, (b) $w(z) = z - 3/2$, (c) $w(z) = z - 1$.

Remark 2.7. The ideas involved in the proof of Theorem 2.5 can be applied to other functions w not necessarily meromorphic. For example, the function

$$w(z) = \prod_{i=1}^{\ell}(z - a_i)\prod_{j=1}^{m} e^{1/(z-d_j)}\,, \qquad a_i \in \mathbb{D}\,,\ d_j \in \mathbb{C}\setminus\mathbb{D}\,,$$

has essential singularities at each d_j, and for this function the conclusions of Theorem 2.5 hold with

$$r := \max\left(\{0, |a_1|, \dots, |a_\ell|, |d_1|^{-1}, \dots, |d_m|^{-1}\}\right).$$

Remark 2.8. We note that a result similar to Theorem 2.5 is known for orthogonal polynomials on the unit circle $\mathbb{T}$. Let ψ_n be the n-th orthonormal polynomial with respect to a measure σ in the Szegő class of $\mathbb{T}$, and let $0 \le \rho \le 1$ be the

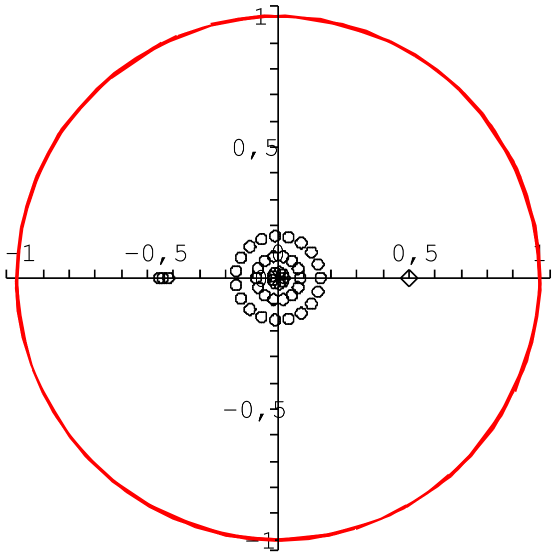


FIGURE 3. Zeros of P_n, $n = 10, 15, 20$, for $w(z) = (z - \frac{1}{2})/(2-z)^2$ and $G = \mathbb{D}$.

smallest number such that the reciprocal of the interior Szegő function for σ'

$$D(\sigma', z)^{-1} := \exp\left\{-\frac{1}{4\pi}\int_0^{2\pi} \log \sigma'(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right\}$$

is analytic in $\mathbb{D}_{1/\rho} := \{z : |z| < 1/\rho\}$. In [8] it was shown that for some subsequence $\mathcal{N} \subset \mathbb{N}$,

$$\nu_{\psi_n} \xrightarrow{*} \mu_\rho\,, \qquad \text{as } n \to \infty\,, n \in \mathcal{N}\,,$$

where μ_ρ is the arc-measure $|dz|/2\pi\rho$ on $\mathbb{T}_\rho$ if $\rho > 0$, or $\mu_\rho = \delta_0$ if $\rho = 0$. Hence, if $w(z)$ is as in Theorem 2.5 then

$$D(|w|^2, z)^{-1} = \frac{h(0)}{|h(0)|h(z)(1 - z\overline{a}_1)^{v_1}\cdots(1 - z\overline{a}_\ell)^{v_\ell}}\,,$$

where

$$v_i = \operatorname{mult}(a_i) \qquad \text{and} \qquad h(z) = w(z)/[(z-a_1)^{v_1}\cdots(z-a_\ell)^{v_\ell}].$$

Thus

$$\rho = \max\left(\{0, |a_1|, \ldots, |a_\ell|, |b_1|^{-1}, |b_2|^{-1}, \ldots\} \setminus \mathcal{A}^*\right)\,,$$

where

$$\mathcal{A}^* := \{|a_i| : 1/\overline{a}_i = c_j \text{ for some } j \text{ and } \operatorname{mult}(c_j) \geq \operatorname{mult}(a_i)\}\,.$$

So, if w is as in Theorem 2.5, the zeros of the ψ_n's and the zeros of the P_n's accumulate on the same circle, except possibly when $\mathcal{A} \neq \mathcal{A}^*$. Indeed, for $w(z) = (z - 1/2)/(2 - z)$ the Szegő polynomials have all zeros at the origin (since $|w(z)| \equiv 2$ for $|z| = 1$), while the weighted Bergman polynomials have zeros accumulating on $|z| = 1/2$.

We now consider a special class of domains bounded by a piecewise analytic Jordan curve. Let $N \geq 2$ be a natural number and let G be a lens-shaped domain whose boundary L consists of two circular arcs L_α and L_β (L_α being to the left of L_β) meeting at i and $-i$ with opening angle π/N. Let α and β be the angles formed by L_α and L_β with the segment $[-i, i]$, respectively. Notice that L_α and L_β are arcs of circles centered, respectively, at $a := \cot\alpha$, $b := -\cot\beta$, with corresponding radii $\rho_\alpha := 1/\sin\alpha$, $\rho_\beta := 1/\sin\beta$. In the limit case when either α or $\beta = 0$, one of these circles becomes the imaginary axis.

For any point $z \in \overline{G}$, let

$$z_\alpha = \frac{a\,\overline{z} + 1}{\overline{z} - a}\,, \qquad z_\beta = \frac{b\,\overline{z} + 1}{\overline{z} - b} \tag{9}$$

be the reflections of z with respect to L_α and L_β, respectively. The following facts are stated without proof, since they can be obtained by using the method employed for $N = 2$ in [5, Section 4].

The set

$$\Gamma := \left\{ z \in \overline{G} : \, |\Phi(z_\alpha)| = |\Phi(z_\beta)| \right\} . \tag{10}$$

is an analytic Jordan arc that lies on G, except for its two endpoints i, $-i$. Define

$$G_\alpha := \operatorname{int}(L_\alpha \cup \Gamma)\,, \qquad G_\beta := \operatorname{int}(L_\beta \cup \Gamma)\,.$$

Then, by the reflection principle, the function

$$\widehat{\Phi}(z) := \begin{cases} \Phi(z) & \text{if } z \in \overline{\mathbb{C}} \setminus G \\ 1\big/\overline{\Phi(z_\alpha)} & \text{if } z \in G_\alpha \cup \Gamma \\ 1\big/\overline{\Phi(z_\beta)} & \text{if } z \in G_\beta \end{cases} \tag{11}$$

is analytic in $\mathbb{C} \setminus \Gamma$, and $|\widehat{\Phi}|$ is continuous in $\overline{\mathbb{C}}$. If $p_\Gamma := \Gamma \cap \{\operatorname{Im} z = 0\}$ is the midpoint of Γ, then

$$0 < R_\Gamma := |\widehat{\Phi}(p_\Gamma)| < |\widehat{\Phi}(z)|, \qquad \forall\, z \neq p_\Gamma\,. \tag{12}$$

For any $R_\Gamma \leq r < \infty$, consider the level set

$$\gamma_r := \{ z : \, |\widehat{\Phi}(z)| = r \}\,. \tag{13}$$

When $R_\Gamma < r < 1$, γ_r is a Jordan curve that intersects Γ at two conjugate points, and it is such that $\gamma_r \setminus \Gamma$ consists of two analytic simple arcs, one contained in G_α, the other in G_β. Notice that $\gamma_{R_\Gamma} = \{p_\Gamma\}$, $\gamma_1 = L$, and that for $r > 1$, γ_r is a standard level curve of the exterior mapping Φ (see Figure 4).

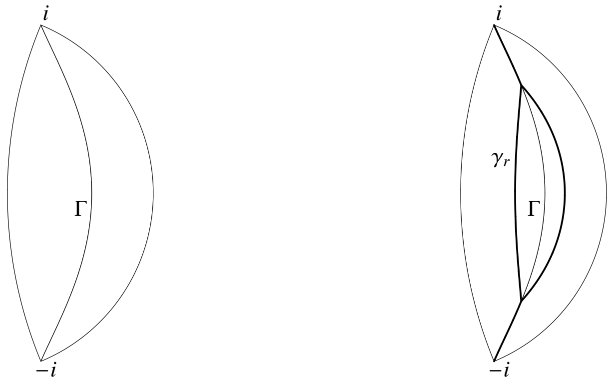


FIGURE 4. Curves Γ and γ_r for $N = 2$, $\alpha = \pi/8$, $r = 5/7$. Here, $R_\Gamma \approx 0.58731$.

Theorem 2.9. *Let G be a lens-shaped domain with opening angle π/N, and let $w \not\equiv 0$ be an entire function. Let $\{a_1, \ldots, a_\ell\}$ and $\{b_1, b_2, \ldots\}$ be the sets of zeros of w in G and $\mathbb{C} \setminus G$, respectively, and define r as the largest number of the set*

$$\left\{R_\Gamma, |\widehat{\Phi}(a_1)|, \ldots, |\widehat{\Phi}(a_\ell)|\right\} \cup \left\{|\widehat{\Phi}(b_k)|^{-1} : \ b_k \notin \{-i, i\} \ \text{or} \ \operatorname{mult}(b_k) > N-1\right\} .$$

Then, for all but countably many $z \in G$,

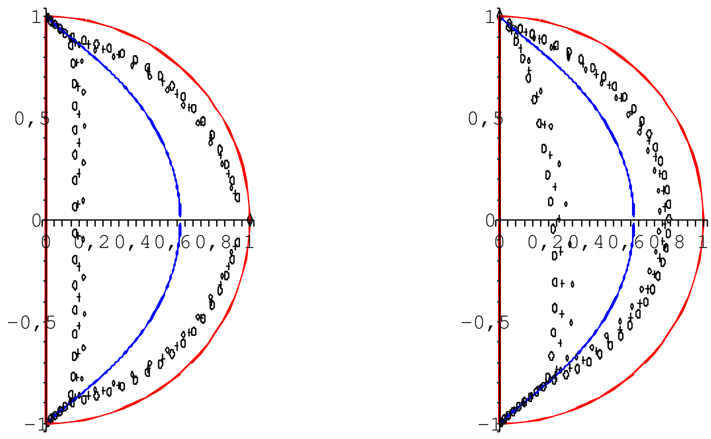
$$\limsup_{n\to\infty} |P_n(z)|^{1/n} = \begin{cases} |\widehat{\Phi}(z)| & \text{if } z \in \operatorname{ext}(\gamma_r) \\ r & \text{if } z \in \gamma_r \cup \operatorname{int}(\gamma_r) \end{cases}, \tag{14}$$

which implies that any weak-limit point σ of the measures ν_{P_n} is supported in $\Gamma \cup \gamma_r \cup \operatorname{int}(\gamma_r)$, and every point of $\Gamma \setminus \operatorname{int}(\gamma_r)$ belongs to* $\operatorname{supp}(\sigma)$. *Moreover, there is a measure μ_r whose support coincides with $(\Gamma \cup \gamma_r) \setminus \operatorname{int}(\gamma_r)$ such that*

(a) *if $r = R_\Gamma$ (i.e. if $\gamma_r = \{p_\Gamma\}$), then $\nu_{P_n} \xrightarrow{*} \mu_r$ as $n \to \infty$;*
(b) *if $r > R_\Gamma$ and for some $z \in \operatorname{int}(\gamma_r)$ the* $\limsup$ *in* (14) *is realized through a subsequence $\mathcal{N} \subset \mathbb{N}$, then*

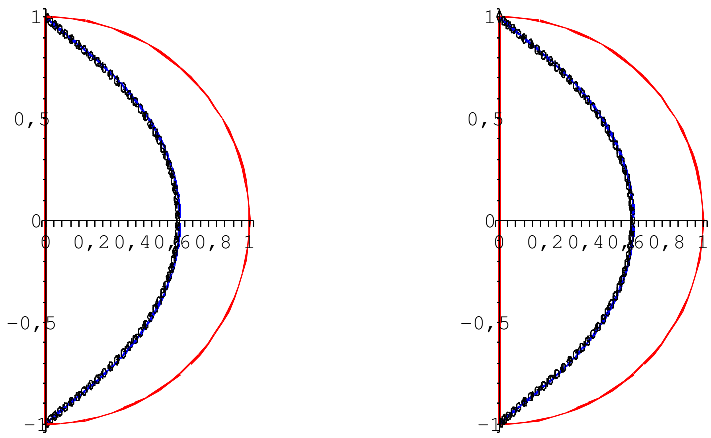
$$\nu_{P_n} \xrightarrow{*} \mu_r \qquad \text{as } n \to \infty \,,\ n \in \mathcal{N} \,.$$

It is of help to discuss Theorem 2.9 for the simplest case when $w(z) = (z-a)^v$, $v \in \mathbb{N}$, has a zero in a single point. If $a \in L\setminus\{-i, i\}$, or $a \in \{-i, i\}$ and $v > N-1$, we see that γ_r coincides with the boundary L of G, and every point of L attracts zeros of the P_n's (see Figure 5). If $a \in \{-i, i\}$ but $v \le N-1$, or $a \in \mathbb{C} \setminus \overline{G}$ is sufficiently far from the lens (in the sense $|\Phi(a)| \ge 1/R_\Gamma$), or a coincides with the midpoint p_Γ of Γ, then γ_r shrinks to the point p_Γ and the zeros of the P_n's accumulate on the whole of Γ (see Figure 6). If none of these things happen, then a "proper" bubble bounded by γ_r and joining two subarcs of Γ is formed, and every point of γ_r, as well as of the subarcs, attracts zeros of the polynomials (see Figure 7).



(a) (b)

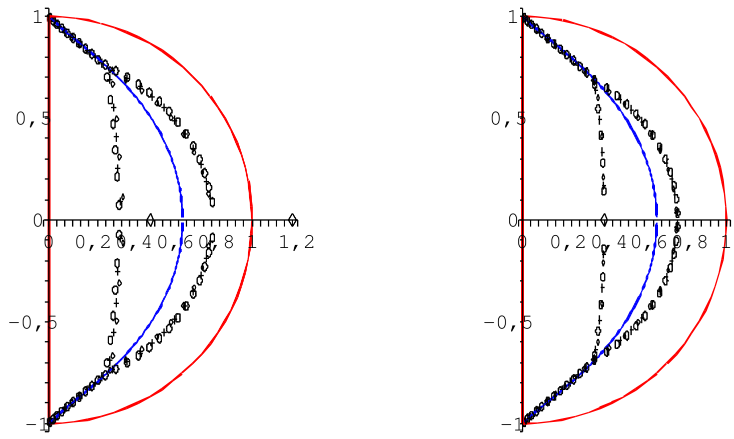
FIGURE 5. Zeros of P_n, $n = 40, 50, 60$, for lens parameters $N = 2$, $\alpha = 0$, and (a) $w(z) = z - 1$, (b) $w(z) = (z - i)^2$.



(a) (b)

FIGURE 6. Zeros of P_n, $n = 40, 50, 60$, for lens parameters $N = 2$, $\alpha = 0$, and (a) $w(z) = z - 4$, (b) $w(z) = z - i$.

From the proof of Theorem 2.9 one can see that the measure μ_r in that theorem can be characterized in different ways. For example, if μ_{R_Γ} is the limiting measure corresponding to the value $r = R_\Gamma$, which is supported on Γ, then for any other $R_\Gamma < r \leq 1$, μ_r is the measure supported on $(\Gamma \cup \gamma_r) \setminus \mathrm{int}(\gamma_r)$ that coincides



(a) (b)

FIGURE 7. Zeros of P_n, $n = 40, 50, 60$, for lens parameters $N = 2$, $\alpha = 0$, and (a) $w(z) = (z - 1.2)^2$, (b) $w(z) = z - 0.4$.

with μ_{R_Γ} on the two subarcs $\Gamma \setminus \operatorname{int}(\gamma_r)$, and that equals the balayage of the restriction of μ_{R_Γ} to $\Gamma \cap \operatorname{int}(\gamma_r)$ onto γ_r. Alternatively, μ_r can also be characterized as the unique measure whose logarithmic potential U^{μ_r} is

$$U^{\mu_r}(z) = \begin{cases} -\log|\operatorname{cap}(L)\widehat{\Phi}(z)| & \text{if } z \in \operatorname{ext}(\gamma_r) \\ -\log[\operatorname{cap}(L)r] & \text{if } z \in \gamma_r \cup \operatorname{int}(\gamma_r) \end{cases}. \tag{15}$$

In fact, we can recover μ_r from its potential (15) by using [14, Thm. II.1.5]. As an example we present the following result, but we omit its proof since it relies upon tedious computations (for details see [9]).

Proposition 2.10. *Suppose G is the symmetric lens-shaped domain with parameters $N = 2$ and $\alpha = \beta = \pi/4$. Let μ_{R_Γ} be the limiting measure in Theorem 2.9 corresponding to the value $r = R_\Gamma$, whose support is $\Gamma = [-i, i]$. Then for any Lebesgue measurable set $E \subset [-1, 1]$,*

$$\mu_{R_\Gamma}(iE) = \int_E \frac{4\sqrt{3}}{3\pi(1-y^2)^{1/3}} \cdot \frac{(1+y)^{8/3} - (1-y)^{8/3}}{(1+y)^4 - (1-y)^4}\, dy\,.$$

Suppose now that G is a convex polygon with m sides having the property that an interior conformal map φ of G onto $\mathbb{D}$ (or equivalently, $K(\cdot,\zeta)$) can be analytically continued across ∂G. Then necessarily G is either an equilateral triangle, or an isosceles right triangle, or a triangle with angles $\pi/2, \pi/3, \pi/6$, or G is a rectangle. When G is either an equilateral triangle or a square, it has been shown in [6] that the zeros of the Bergman polynomials lie on the segments joining the center of G with its vertices, and every point of these segments is an accumulation point of the zeros. We now state (without proof) a result for the other three cases and $w \equiv 1$, which can be obtained by applying the analysis of the present paper. Further results of this type will appear in a future paper.

Suppose that G is either an isosceles right triangle or a triangle with angles $\pi/2, \pi/3, \pi/6$ ($m = 3$) or that G is a rectangle ($m = 4$). Let $v_1, \ldots, v_m$ and $l_1, \ldots, l_m$ be, respectively, the vertices and sides of G. For any $z \in \overline{G}$, let z_i denote the reflection of z with respect to l_i. Then, for every $1 \le i \le m$, we can extend the exterior conformal map Φ to all of $\mathbb{C}$ by reflection across the side l_i, so that the function

$$\widehat{\Phi}_i(z) := \begin{cases} \Phi(z) & \text{if } z \in \mathbb{C} \setminus \overline{G} \\ 1\big/\overline{\Phi(z_i)} & \text{if } z \in \overline{G} \end{cases}$$

is analytic in $\mathbb{C} \setminus \left(\bigcup_{\substack{j=1 \\ j \neq i}}^m l_j \right)$. Then, the set $\mathcal{S} \subset \overline{G}$ defined by

(16)
$$\mathcal{S} := \left\{ z \in \overline{G} : \, |\widehat{\Phi}_i(z)| = |\widehat{\Phi}_j(z)| \ge \max_{1 \le k \le m} |\widehat{\Phi}_k(z)| \text{ for some } 1 \le i < j \le m \right\}$$

consists of a finite union of analytic and simple Jordan arcs with the following properties:

1. $\partial G \cap \mathcal{S} = \{v_1, \ldots, v_m\}$;
2. $G \setminus \mathcal{S} = G_1 \cup \cdots \cup G_m$, $G_i \cap G_j = \emptyset$, $1 \le i \ne j \le m$, where the G_i's are connected open sets such that $\partial G_i \cap \partial G = l_i$ for all $1 \le i \le m$;
3. the function

$$\widehat{\Phi}(z) := \begin{cases} \Phi(z) & \text{if } z \in \overline{\mathbb{C}} \setminus G \\ 1\big/\overline{\Phi(z_i)} & \text{if } z \in G_i\,, \ 1 \le i \le m \end{cases}$$

   is analytic in $\mathbb{C} \setminus \mathcal{S}$, and $|\widehat{\Phi}|$ can be extended continuously to $\overline{\mathbb{C}}$.

Then, with this notation, we have the following result.

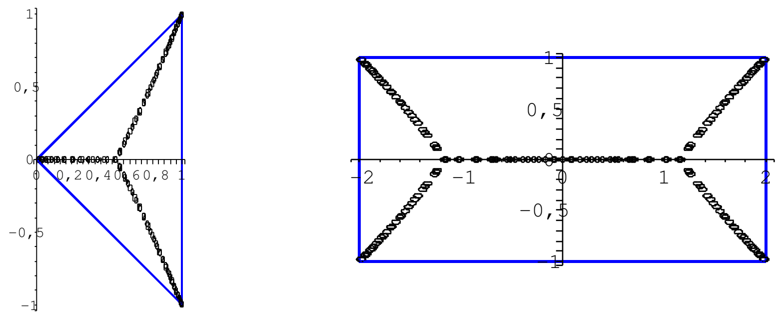
Proposition 2.11. *If G is a rectangle or one of the two triangles described above and $w \equiv 1$, then*

$$\limsup_{n\to\infty} |P_n(z)|^{1/n} = |\widehat{\Phi}(z)|, \qquad \forall\, z \in \mathbb{C}\,,$$

and there exists a positive unit measure $\mu_{\mathcal{S}}$ whose support coincides with $\mathcal{S}$ and such that

$$\nu_{P_n} \xrightarrow{\ *\ } \mu_{\mathcal{S}} \qquad \text{as } n \to \infty\,.$$

(See Figure 8.)



(a) (b)

FIGURE 8. Zeros of Bergman polynomials for (a) G an isosceles right triangle, $n = 40, 45, 50$, (b) G a 2×1-rectangle, n=50, 55, 60.

3. The reproducing kernel $K_w(z, \zeta)$

For any (G, w), we have introduced in (2) and (3) the space $\mathcal{L}^2_w(G)$ together with its inner product $\langle\cdot|\cdot\rangle_w$ and norm $\|\cdot\|_{\mathcal{L}^2_w(G)}$. When $w \equiv 1$, we simply write $\mathcal{L}^2(G)$. Although the notation (G, w) assumes that $L = \partial G$ is a Jordan curve, all the results stated in this section are also valid for any bounded simply-connected domain G. Here, we establish the existence of the kernel function $K_w(z, \zeta)$, state some of its basic properties, and give some formulas for it.

Lemma 3.1. *Let $z \in G$ be such that $w(z) \neq 0$. Then, for every $f \in \mathcal{L}^2_w(G)$ we have*

$$|f(z)| \le \frac{\|f\|_{\mathcal{L}^2_w(G)}}{\sqrt{\pi}\,|w(z)|\,d_z}\,, \tag{17}$$

where

$$d_z := \operatorname{dist}(z, L) = \inf_{\zeta\in L} |\zeta - z|\,.$$

Consequently, for any compact set $K \subset G$, we can find a constant C_K such that

$$|f(z)| \le C_K \|f\|_{\mathcal{L}^2_w(G)}, \qquad \forall\ f \in \mathcal{L}^2_w(G)\,,\ z \in K\,. \tag{18}$$

Proof. Inequality (17) follows at once by applying [1, Lemma 1, p. 4] to fw. Now, given any compact set $K \subset G$, one can find a Jordan curve $\Gamma_K \subset G$ surrounding K on which w has no zeros. Then from (17) we get that (18) holds for all $f \in \mathcal{L}^2_w(G)$ and $z \in \Gamma_K$, where

$$C_K^{-1} = \sqrt{\pi} \times \min\{|w| \text{ on } \Gamma_K\} \times \min\{d_z : z \in \Gamma_K\} > 0\,.$$

Then, by the maximum modulus principle for analytic functions, the same estimate holds for all $z \in K$. ∎

With the help of (18) one can easily extend some results that are already known to be valid for the Bergman case $w \equiv 1$. For example, paraphrasing the proof of [1, Thm. 1, p. 5], we get

Lemma 3.2. *The space $\mathcal{L}^2_w(G)$ is a Hilbert space with respect to the inner product $\langle\cdot|\cdot\rangle_w$. Moreover, if $\{f_n\}_{n=0}^{\infty} \subset \mathcal{L}^2_w(G)$ and $\lim_{n\to\infty}\|f_n - f\|_{\mathcal{L}^2_w(G)} = 0$ for some $f \in \mathcal{L}^2_w(G)$, then $f_n(z) \to f(z)$ uniformly on compact subsets of G.*

Inequality (18) shows that for every $\zeta \in G$, the linear functional that assigns to each $f \in \mathcal{L}^2_w(G)$ the value $f(\zeta)$ is bounded. Therefore, by the Riesz representation theorem, there is a unique function $K_w(\cdot,\zeta) \in \mathcal{L}^2_w(G)$ having the reproducing property

$$f(\zeta) = \int_G \overline{K_w(z,\zeta)}\, f(z)\, |w(z)|^2\, dm(z) = \langle f|K_w(\cdot,\zeta)\rangle_w\,, \quad \forall f \in \mathcal{L}^2_w(G)\,. \tag{19}$$

That is, $K_w(z,\zeta)$ is the *kernel function* for the space $\mathcal{L}^2_w(G)$. For $w \equiv 1$, we write $K(z,\zeta)$, which is the so-called *Bergman kernel function* for G. The following basic properties of $K_w(z,\zeta)$, which we state without proof, are consequences of its reproducing property (19).

Lemma 3.3. (i) *For all z, ζ, $a \in G$,*

$$K_w(z,\zeta) = \overline{K_w(\zeta,z)} \qquad \text{and} \qquad K_w(a,a) = \|K_w(\cdot,a)\|^2_{\mathcal{L}^2_w(G)} > 0\,;$$

(ii) *If $\{S_n\}_{n=1}^{\infty}$ is an orthonormal system of functions in the space $\mathcal{L}^2_w(G)$, then $\{S_n\}_{n=1}^{\infty}$ is complete if and only if for every $\zeta \in G$,*

$$K_w(\cdot,\zeta) = \sum_{n=1}^{\infty} \overline{S_n(\zeta)} S_n(\cdot)$$

in the $\mathcal{L}^2_w(G)$-norm.

Let $\varphi(z)$ be any conformal mapping of G onto the unit disk $\mathbb{D}$. Then it is well-known (see [1, p. 33]) that the Bergman kernel function for the space $\mathcal{L}^2(G)$ is given by

$$K(z,\zeta) = \frac{\varphi'(z)\overline{\varphi'(\zeta)}}{\pi[1-\varphi(z)\overline{\varphi(\zeta)}]^2}\,. \tag{20}$$

It is straightforward to check that if $h(z)$ is analytic and never zero in G, and such that $h(z)w(z)|_G \in \mathcal{L}^2(G)$, then

$$K_{wh}(z,\zeta) = \frac{K_w(z,\zeta)}{h(z)\,\overline{h(\zeta)}}\,. \tag{21}$$

In particular, if $w(z) \neq 0\ \forall\, z \in G$, then

$$K_w(z,\zeta) = \frac{K(z,\zeta)}{w(z)\,\overline{w(\zeta)}} = \frac{\varphi'(z)\overline{\varphi'(\zeta)}}{\pi\, w(z)\,\overline{w(\zeta)}\left[1-\varphi(z)\overline{\varphi(\zeta)}\right]^2}\,. \tag{22}$$

Now, if for $h\colon \mathbb{C} \to \mathbb{C}$ with $h|_G \in \mathcal{L}^2(G)$ and $a \in G$, we put $h_a(z) := (z-a)h(z)$, then it is straightforward to check that (compare with [10, ex. 11, p. 262])

$$K_{h_a}(z,\zeta) = \frac{K_h(z,\zeta) - \dfrac{K_h(a,\zeta)K_h(z,a)}{K_h(a,a)}}{(z-a)\overline{(\zeta-a)}}\,. \tag{23}$$

As mentioned in Section 2, by reiterating this formula as similarly done in [4, p. 58], we arrive at Proposition 2.3. For completeness, we provide a brief proof.

Proof of Proposition 2.3. In light of (21), it suffices to show that

$$K_{Q_n}(z,\zeta) = H_n(z,\zeta), \tag{24}$$

with $Q_n(z) = \prod_{i=1}^n (z-a_i)$. For $n=0$, relation (24) is exactly the definition of $H_0(z,\zeta)$. Suppose (24) holds for some n. By Lemma 3.3(i) we have $H_n(a_{n+1},a_{n+1}) = K_{Q_n}(a_{n+1},a_{n+1}) > 0$, so that

$$H_{n+1}(z,\zeta) := \frac{H_n(z,\zeta) - \dfrac{H_n(a_{n+1},\zeta)H_n(z,a_{n+1})}{H_n(a_{n+1},a_{n+1})}}{(z-a_{n+1})\overline{(\zeta-a_{n+1})}}$$

is an analytic function of z on G, for any fixed $\zeta \in G \setminus \{a_{n+1}\}$.

Fix $\zeta \in G \setminus \{a_{n+1}\}$. Observe that $H_{n+1}(\cdot,\zeta) \in \mathcal{L}^2_{Q_{n+1}}(G)$ since $H_n(\cdot,\zeta)$ and $H_n(\cdot,a_{n+1})$ belong to $\mathcal{L}^2_{Q_n}(G)$. If $f \in \mathcal{L}^2_{Q_{n+1}}(G)$, then $f(z)(z-a_{n+1}) \in \mathcal{L}^2_{Q_n}(G)$, and so

$$\begin{aligned}
&\int_G \overline{H_{n+1}(z,\zeta)}\, f(z)|Q_{n+1}(z)|^2\, dm(z) \\
&\quad = \int_G \left(\overline{H_n(z,\zeta)} - \frac{\overline{H_n(a_{n+1},\zeta)}\,\overline{H_n(z,a_{n+1})}}{H_n(a_{n+1},a_{n+1})} \right) \frac{f(z)(z-a_{n+1})|Q_n(z)|^2 dm(z)}{(\zeta-a_{n+1})} \\
&\quad = \frac{f(\zeta)(\zeta-a_{n+1})}{(\zeta-a_{n+1})} - \frac{\overline{H_n(a_{n+1},\zeta)}}{H_n(a_{n+1},a_{n+1})} \frac{f(a_{n+1})(a_{n+1}-a_{n+1})}{(\zeta-a_{n+1})} = f(\zeta).
\end{aligned}$$

This shows that $H_{n+1}(z,\zeta) = K_{Q_{n+1}}(z,\zeta)$ if $\zeta \neq a_{n+1}$. But then by continuity, the same is true for all $\zeta \in G$. ∎

When the zeros a_i's of w are simple we have the following determinant representation.

Proposition 3.4. *If in Proposition 2.3, the a_i's are all distinct, then the kernel function $K_w(z,\zeta)$ for the space $\mathcal{L}^2_w(G)$ is given by*

$$K_w(z,\zeta)=\frac{\begin{vmatrix} K(a_1,a_1) & K(a_2,a_1) & \cdots & K(a_n,a_1) & K(z,a_1)\\ K(a_1,a_2) & K(a_2,a_2) & \cdots & K(a_n,a_2) & K(z,a_2)\\ \vdots & \vdots & & \vdots & \vdots\\ K(a_1,a_n) & K(a_2,a_n) & \cdots & K(a_n,a_n) & K(z,a_n)\\ K(a_1,\zeta) & K(a_2,\zeta) & \cdots & K(a_n,\zeta) & K(z,\zeta)\end{vmatrix}}{h(z)\overline{h(\zeta)}Q_n(z)\overline{Q_n(\zeta)}\,A_n} \tag{25}$$

where $Q_n(z)=\prod_{i=1}^n(z-a_i)$ and $A_n>0$ is the $n\times n$ principal minor of the determinant above.

Proof. The proposition can be derived by using Proposition 2.3 and Silvester's determinant identity. However, here we give a more straightforward proof. Again by (21), it suffices to prove the proposition for the case $h(z)\equiv 1$, that is, when $w(z)=Q_n(z)=\prod_{i=1}^n(z-a_i)$. It is easy to see that the system of functions $\{K(z,a_1),\dots,K(z,a_n)\}$ is linearly independent. The Grammian of this system is precisely A_n. Hence $A_n>0$.

Let us denote by $D_{Q_n}(z,\zeta)$ the right-hand side of (25) and let $\zeta\notin\{a_i\}_{i=1}^n$ be fixed. Then $D_{Q_n}(z,\zeta)$ is well defined for all $z\in G$. Moreover, if we develop the determinant in (25) by its last column, we see that $D_{Q_n}(\cdot,\zeta)\in\mathcal{L}^2_{Q_n}(G)$ and, for certain constants C_i,

$$\begin{aligned}\int_G &\overline{D_{Q_n}(z,\zeta)}f(z)|Q_n(z)|^2dm(z)\\ &=\int_G\left(\frac{\overline{K(z,\zeta)}+\sum_{i=1}^n\overline{C}_i\,\overline{K(z,a_i)}}{\overline{Q_n(z)}Q_n(\zeta)}\right)f(z)|Q_n(z)|^2dm(z)\\ &=\frac{f(\zeta)Q_n(\zeta)}{Q_n(\zeta)}+\sum_{i=1}^n\overline{C}_i\frac{f(a_i)Q_n(a_i)}{Q_n(\zeta)}=f(\zeta)\,.\end{aligned}$$

Therefore, for $\zeta\notin\{a_i\}_{i=1}^n$, $D_{Q_n}(z,\zeta)=K_{Q_n}(z,\zeta)$. But then for $1\le i\le n$,

$$D_{Q_n}(z,a_i):=\lim_{\zeta\to a_i}K_{Q_n}(z,\zeta)=K_{Q_n}(z,a_i)\,. \qquad \blacksquare$$

In order to find the singularities of $K_w(\cdot,\zeta)$ we need a description of these formulas that reflects the dependence of the kernel on the conformal mapping φ and the weight w. For this purpose we provide a useful lemma. Suppose that

$$K(z,\zeta)=\frac{f(z)\overline{g(\zeta)}}{\pi\left[1-t(z)\overline{s(\zeta)}\right]^2}\,,\qquad z,\zeta\in G\,, \tag{26}$$

where f,t,g,s are analytic functions in G, and moreover, that t and s are one-to-one in G and

$$1-t(z)\overline{s(\zeta)}\neq 0\,,\qquad \forall\, z,\,\zeta\in G\,. \tag{27}$$

In view of (20), a representation like (26) is always possible. Notice that, from Lemma 3.3(i), $f(z)\overline{g(\zeta)} \neq 0$ for all $z, \zeta \in G$.

Lemma 3.5. *With the above notation we have*

(a) *for* $w(z) = \omega_a^v(z) := (z-a)^v$, $v \in \mathbb{N} \cup \{0\}$,

$$(28) \quad \begin{aligned} K_w(z,\zeta) = K(z,\zeta) &\times \frac{[t(z)-t(a)]^v \left[\overline{s(\zeta)} - \overline{s(a)}\right]^v}{(z-a)^v(\overline{\zeta}-\overline{a})^v} \\ &\times \frac{\left[1-t(a)\overline{s(\zeta)}\right]\left[1-t(z)\overline{s(a)}\right] + v\left[1-t(a)\overline{s(a)}\right]\left[1-t(z)\overline{s(\zeta)}\right]}{\left[1-t(a)\overline{s(\zeta)}\right]^{v+1}\left[1-t(z)\overline{s(a)}\right]^{v+1}}; \end{aligned}$$

(b) *for* $w(z) = (z-a_1)^{v_1}(z-a_2)^{v_2}\cdots(z-a_n)^{v_n}$, $v_i \in \mathbb{N}$, $1 \le i \le n$, a_i*'s distinct,*

$$(29) \quad \begin{aligned} K_w(z,\zeta) = K(z,\zeta) &\times \frac{\prod_{i=1}^n [t(z)-t(a_i)]^{v_i} \left[\overline{s(\zeta)} - \overline{s(a_i)}\right]^{v_i}}{\prod_{i=1}^n (z-a_i)^{v_i}(\overline{\zeta}-\overline{a}_i)^{v_i}} \\ &\times \frac{Q_w\left(t(z), \overline{s(\zeta)}\right)}{\prod_{i=1}^n \left[1-t(a_i)\overline{s(\zeta)}\right]^{v_i+1}\left[1-t(z)\overline{s(a_i)}\right]^{v_i+1}}, \end{aligned}$$

where $Q_w(\tau,\xi)$ *is a polynomial in the two variables* τ *and* ξ *(of degree at most* n *in each independent variable) satisfying:*

(i) $Q_w\left(t(a), \overline{s(a)}\right) \neq 0 \quad \forall\, a \in G$*;*

(ii) *if* $\xi \neq 0$*, then*

$$Q_w(1/\xi, \xi) \neq 0 \Leftrightarrow \xi \notin \left\{1/t(a_1), \ldots, 1/t(a_n), \overline{s(a_1)}, \ldots, \overline{s(a_n)}\right\};$$

(iii) *for every* $1 \le i \le n$,

$$Q_w(1/\overline{s(a_i)}\,, \cdot) \not\equiv 0 \quad \text{if } \overline{s(a_i)} \neq 0\,,$$
$$Q_w(\cdot\,, 1/t(a_i)) \not\equiv 0 \quad \text{if } t(a_i) \neq 0;$$

(iv) *for every* $1 \le i \le n$,

$$Q_w(\tau, \overline{s(a_i)}) = \left[1 - \tau\overline{s(a_i)}\right] S_i^w(\tau)$$

and

$$Q_w(t(a_i), \xi) = [1 - t(a_i)\xi]\; T_i^w(\xi),$$

with

$$S_i^w(1/\overline{s(a_i)}) \neq 0 \quad \text{if } \overline{s(a_i)} \neq 0$$

and

$$T_i^w(1/t(a_i)) \neq 0 \quad \text{if } t(a_i) \neq 0.$$

Consequently, from (i), $S_i^w(t(a_i)) = T_i^w(\overline{s(a_i)}) \neq 0$ *for all* $1 \le i \le n$.

Proof. Given a point a, let us define the iteration I_a by

$$I_a\left(H(z,\zeta)\right) := H(z,\zeta) - \frac{H(a,\zeta)H(z,a)}{H(a,a)}\,, \tag{30}$$

which applies to any function $H(z,\zeta)$ for which (30) makes sense. Then, (a) follows without major complications by induction on the number v, since by Proposition 2.3,

$$K_{\omega_a^{v+1}}(z,\zeta) = \frac{I_a\left(K_{\omega_a^v}(z,\zeta)\right)}{(z-a)(\overline{\zeta}-\overline{a})}\,.$$

The computations involved can be simplified by observing the following fact: if $H(z,\zeta) = r(z)l(\zeta)H_1(z,\zeta)$ with $r(a)l(a) \neq 0$, then

$$I_a(H(z,\zeta)) := r(z)l(\zeta)I_a(H_1(z,\zeta))\,. \tag{31}$$

We now prove (b). If $w(z) = \omega_{a_1}^{v_1}(z) = (z-a_1)^{v_1}$, $v_1 \geq 1$, then $K_w(z,\zeta)$ is given by formula (28), so that in this case

$$Q_w(\tau,\xi) = Q_{\omega_{a_1}^{v_1}}(\tau,\xi) = [1-t(a_1)\xi]\left[1-\tau\overline{s(a_1)}\right] + v_1\left[1-t(a_1)\overline{s(a_1)}\right][1-\tau\xi]$$

and properties (i)-(iv) are trivially satisfied (property (i) is a consequence of Lemma 3.3(i)). Thus, we only need prove (b) for a w that has zeros in $n \geq 2$ points. We proceed by induction. Let $w(z) = (z-a_1)^{v_1}(z-a_2)^{v_2}\cdots(z-a_n)^{v_n}$ be such that $n \geq 2$, and let $m := v_1 + \cdots + v_n$. Assume that (b) holds for any other w such that the sum of the multiplicities of its zeros is $\leq m-1$. For all $1 \leq i \leq n$, define

$$w_i(z) := (z-a_i)^{v_i-1}\prod_{\substack{1\leq j\leq n\\ j\neq i}}(z-a_j)^{v_j},$$

so that by the induction hypothesis $K_{w_i}(z,\zeta)$ has the following form:

$$\frac{f(z)\overline{g(\zeta)}\,[t(z)-t(a_i)]^{v_i-1}\left[\overline{s(\zeta)}-\overline{s(a_i)}\right]^{v_i-1}\prod_{j\neq i}^n[t(z)-t(a_j)]^{v_j}\left[\overline{s(\zeta)}-\overline{s(a_j)}\right]^{v_j}}{\pi\left[1-t(z)\overline{s(\zeta)}\right]^2(z-a_i)^{v_i-1}(\overline{\zeta}-\overline{a}_i)^{v_i-1}\prod_{j\neq i}^n(z-a_j)^{v_j}(\overline{\zeta}-\overline{a}_j)^{v_j}}$$

$$\times\frac{\widehat{Q}_{w_i}\left(t(z),\overline{s(\zeta)}\right)}{\left[1-t(a_i)\overline{s(\zeta)}\right]^{v_i}\left[1-t(z)\overline{s(a_i)}\right]^{v_i}\prod_{j\neq i}^n\left[1-t(a_j)\overline{s(\zeta)}\right]^{v_j+1}\left[1-t(z)\overline{s(a_j)}\right]^{v_j+1}},$$

where

$$\widehat{Q}_{w_i}(\tau,\xi) = \begin{cases} Q_{w_i}(\tau,\xi) & \text{if } v_i \geq 2 \\ Q_{w_i}(\tau,\xi)\,[1-t(a_i)\xi]\left[1-\tau\overline{s(a_i)}\right] & \text{if } v_i = 1 \end{cases}$$

is a polynomial in the two variables τ and ξ (of degree at most n in each independent variable) that satisfies (i') $\widehat{Q}_{w_i}\left(t(a),\overline{s(a)}\right) \neq 0$ for all $a \in G$;

(ii') if $\xi \neq 0$, then

$$\widehat{Q}_{w_i}\,(1/\xi,\xi) \neq 0 \Leftrightarrow \xi \notin \left\{1/t(a_1),\ldots,1/t(a_n),\overline{s(a_1)},\ldots,\overline{s(a_n)}\right\};$$

(iv') for every $1 \leq j \leq n$,

$$\widehat{Q}_{w_i}\left(\tau,\overline{s(a_j)}\right) = \left[1-\tau\overline{s(a_j)}\right]\widehat{S}_j^{w_i}(\tau)$$

and

$$\widehat{Q}_{w_i}\,(t(a_j),\xi) = [1-t(a_j)\xi]\,\widehat{T}_j^{w_i}(\xi),$$

with $\widehat{S}_j^{w_i}\left(1/\overline{s(a_j)}\right) \neq 0$ if $\overline{s(a_j)} \neq 0$, and $\widehat{T}_j^{w_i}\,(1/t(a_i)) \neq 0$ if $t(a_j) \neq 0$. (It then follows from (i') that $\widehat{S}_j^{w_i}(t(a_j)) = \widehat{T}_j^{w_i}\left(\overline{s(a_j)}\right) \neq 0$ for all $1 \leq i \leq n$.)

Properties (i') and (ii') are obvious. As for (iv'), notice that if $v_i = 1$ then

$$\widehat{S}_j^{w_i}(\tau) = \begin{cases} \left[1-t(a_i)\overline{s(a_j)}\right]\left[1-\tau\overline{s(a_i)}\right] S_j^{w_i}(\tau) & \text{if } j \neq i \\ \left[1-t(a_i)\overline{s(a_i)}\right] Q_{w_i}\left(\tau,\overline{s(a_i)}\right) & \text{if } j = i \end{cases},$$

and

$$\widehat{T}_j^{w_i}(\xi) = \begin{cases} \left[1-t(a_j)\overline{s(a_i)}\right][1-t(a_i)\xi]\,T_j^{w_i}(\xi) & \text{if } j \neq i \\ \left[1-t(a_i)\overline{s(a_i)}\right] Q_{w_i}(t(a_i),\xi) & \text{if } j = i \end{cases},$$

so that $\widehat{S}_j^{w_i}\left(1/\overline{s(a_j)}\right) \neq 0$ and $\widehat{T}_j^{w_i}\,(1/t(a_j)) \neq 0$ for all $\overline{s(a_j)}$, $t(a_j) \neq 0$, $1 \leq j \leq n$, since $t(z)$ and $s(\zeta)$ are one-to-one and $Q_{w_i}\,(\tau,\xi)$ satisfies (ii) and (iv). Observe that the degrees of $\widehat{S}_j^{w_i}\,(\cdot)$ and $\widehat{T}_j^{w_i}\,(\cdot)$ are $\leq n-1$.

Thus, according to Proposition 2.3 and taking (31) into account, we have that, for all $1 \leq i \leq n$,

$$\begin{aligned} K_w(z,\zeta) &= \frac{I_{a_i}\left(K_{w_i}(z,\zeta)\right)}{(z-a_i)\left(\overline{\zeta}-\overline{a}_i\right)} \\ &= \frac{f(z)\overline{g(\zeta)}\,\prod_{j=1}^n[t(z)-t(a_j)]^{v_j}\left[\overline{s(\zeta)}-\overline{s(a_j)}\right]^{v_j}}{\pi\left[1-t(z)\overline{s(\zeta)}\right]^2\prod_{j=1}^n(z-a_j)^{v_j}(\overline{\zeta}-\overline{a}_j)^{v_j}} \\ &\quad\times\frac{Q_w\left(t(z),\overline{s(\zeta)}\right)}{\prod_{j=1}^n\left[1-t(a_j)\overline{s(\zeta)}\right]^{v_j+1}\left[1-t(z)\overline{s(a_j)}\right]^{v_j+1}}, \end{aligned} \tag{32}$$

where

$$Q_w\left(t(z),\overline{s(\zeta)}\right) := \frac{\left[1-t(z)\overline{s(\zeta)}\right]^2\left[1-t(a_i)\overline{s(\zeta)}\right]\left[1-t(z)\overline{s(a_i)}\right]}{[t(z)-t(a_i)]\left[\overline{s(\zeta)}-\overline{s(a_i)}\right]}$$
$$\times I_{a_i}\left(\widehat{Q}_{w_i}\left(t(z),\overline{s(\zeta)}\right)\left[1-t(z)\overline{s(\zeta)}\right]^{-2}\right).$$

On expanding the last term and replacing $t(z)$ by τ and $s(\zeta)$ by ξ, we get

$$\begin{aligned} &Q_w(\tau,\xi)\,\widehat{Q}_{w_i}\left(t(a_i),\overline{s(a_i)}\right)[\tau-t(a_i)]\left[\xi-\overline{s(a_i)}\right] \\ &\quad=[1-t(a_i)\xi]\left[1-\tau\overline{s(a_i)}\right]\widehat{Q}_{w_i}\left(t(a_i),\overline{s(a_i)}\right)\widehat{Q}_{w_i}(\tau,\xi) \\ &\quad\quad-[1-\tau\xi]^2\left[1-t(a_i)\overline{s(a_i)}\right]^2\widehat{S}_i^{w_i}(\tau)\widehat{T}_i^{w_i}(\xi). \end{aligned} \tag{33}$$

This shows that $Q_w(\tau,\xi)$ is a polynomial with the degree in each independent variable no greater than n, and so we see from (32) that $K_w(z,\zeta)$ has the form (29). Notice that the representation for $Q_w(\tau,\xi)$ given by (33) is valid for every $1\le i\le n$. Also, since $K_w(a,a)>0$ for all $a\in G$ (see Lemma 3.3(i)), we must have $Q_w\left(t(a),\overline{s(a)}\right)\ne 0$ for all $a\in G$. Hence, property (i) holds.

Further, it follows from (33) that for every $\xi\ne 0$

$$Q_w(1/\xi,\xi)=\frac{\widehat{Q}_{w_i}(1/\xi,\xi)\,[1-t(a_i)\xi]\left[1-\overline{s(a_i)}/\xi\right]}{[1/\xi-t(a_i)]\left[\xi-\overline{s(a_i)}\right]}=\widehat{Q}_{w_i}(1/\xi,\xi)\,.$$

Thus, in view of (ii'), property (ii) also holds.

To prove (iii), suppose that $1\le i\le n$ is such that $\overline{s(a_i)}\ne 0$. Then by (33) and (iv'),

$$\begin{aligned} &\lim_{\xi\to\overline{s(a_i)}}\frac{Q_w\left(1/\overline{s(a_i)},\xi\right)}{\xi-\overline{s(a_i)}} \\ &\quad=\lim_{\xi\to\overline{s(a_i)}}\frac{-\left[1-\xi/\overline{s(a_i)}\right]^2\left[1-t(a_i)\overline{s(a_i)}\right]^2\widehat{S}_i^{w_i}\left(1/\overline{s(a_i)}\right)\widehat{T}_i^{w_i}(\xi)}{\widehat{Q}_{w_i}\left(t(a_i),\overline{s(a_i)}\right)\left[1/\overline{s(a_i)}-t(a_i)\right]\left[\xi-\overline{s(a_i)}\right]^2} \\ &\quad=\frac{-\widehat{S}_i^{w_i}\left(1/\overline{s(a_i)}\right)}{\overline{s(a_i)}}\ne 0\,. \end{aligned}$$

Similarly, we find for $t(a_i)\ne 0\,$,

$$\lim_{\tau\to t(a_i)}\frac{Q_w(\tau,1/t(a_i))}{\tau-t(a_i)}=\frac{-\widehat{T}_i^{w_i}(1/t(a_i))}{t(a_i)}\ne 0\,,$$

from which (iii) follows.

Finally, we prove (iv). For any $1 \le j \le n$, choose $a_i \neq a_j$ (this is possible because $n \ge 2$). Then, with the notation of (iv'), we have

$$\widehat{Q}_{w_i}\left(\tau, \overline{s(a_j)}\right) = \left[1 - \tau\overline{s(a_j)}\right] \widehat{S}_j^{w_i}(\tau),$$

and therefore we get from (33)

$$Q_w\left(\tau, \overline{s(a_j)}\right) = \left[1 - \tau\overline{s(a_j)}\right] S_j^{w}(\tau),$$

where

$$\begin{aligned} S_j^w(\tau)\, \widehat{Q}_{w_i}\left(t(a_i), \overline{s(a_i)}\right) [\tau - t(a_i)] \left[\overline{s(a_j)} - \overline{s(a_i)}\right] \\ = \left[1 - t(a_i)\overline{s(a_j)}\right] \left[1 - \tau\overline{s(a_i)}\right] \widehat{Q}_{w_i}\left(t(a_i), \overline{s(a_i)}\right) \widehat{S}_j^{w_i}(\tau) \\ - \left[1 - \tau\overline{s(a_j)}\right] \left[1 - t(a_i)\overline{s(a_i)}\right]^2 \widehat{S}_i^{w_i}(\tau)\widehat{T}_i^{w_i}\left(\overline{s(a_j)}\right). \end{aligned}$$

Hence, if $\overline{s(a_j)} \neq 0$,

$$\begin{aligned} S_j^w\left(1/\overline{s(a_j)}\right) &= \frac{\left[1 - t(a_i)\overline{s(a_j)}\right] \left[1 - \overline{s(a_i)}/\overline{s(a_j)}\right] \widehat{S}_j^{w_i}\left(1/\overline{s(a_j)}\right)}{\left[1/\overline{s(a_j)} - t(a_i)\right] \left[\overline{s(a_j)} - \overline{s(a_i)}\right]} \\ &= \widehat{S}_j^{w_i}\left(1/\overline{s(a_j)}\right) \neq 0, \end{aligned}$$

by (iv'). Similarly, we find that

$$T_j^w(1/t(a_j)) = \widehat{T}_j^{w_i}\left(1/\overline{s(a_j)}\right) \neq 0.$$

■

4. Orthogonal polynomials and the kernel function

Recall that for any (G, w), $P_n(z) := P_n(z; w) = \kappa_n^w z^n + \cdots$ denotes the polynomial of degree n and positive leading coefficient κ_n^w that is orthonormal with respect to the measure $|w|^2 dm|_G$. It is well-known that the logarithmic capacity $\operatorname{cap}(L)$ of $L = \partial G$ is given by

$$\operatorname{cap}(L) = 1/\Phi'(\infty)\,, \tag{34}$$

where, as before,

$$\Phi\colon \overline{\mathbb{C}} \setminus \overline{G} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \tag{35}$$

is the exterior conformal map associated with G, normalized so that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$.

In the sense of [15, Definition 3.1.2], the measure $|w|^2 dm|_G$ belongs to the **Reg** class, that is,

$$\lim_{n\to\infty} (\kappa_n^w)^{1/n} = [\mathrm{cap}(L)]^{-1}\,. \tag{36}$$

To see that this is true, first notice that since L is a regular set with respect to the Dirichlet problem in $\mathbb{C}\setminus\overline{G}$, (36) is equivalent to (see [15, Thm. 3.2.3])

$$\lim_{n\to\infty} \|P_n\|_{L_\infty(\overline{G})}^{1/n} = 1\,. \tag{37}$$

To show that (37) holds, one can proceed as in the proof of the corresponding result [11, Lemma 4.3] for the case $w \equiv 1$, using (18) instead of inequality (4.4) of [11].

We say that a property $\mathcal{P}$ holds for *quasi-every* $z \in \Omega$, or that $\mathcal{P}$ holds *quasi-everywhere* on Ω (briefly, $\mathcal{P}$ *q.e.* $z \in \Omega$), if

$$\mathrm{cap}(\{z \in \Omega : \mathcal{P} \text{ does not hold for } z\}) = 0.$$

Another relation that is equivalent to (36) and that will be used in this paper is the following (see [15, Thm. 3.1.1]):

$$\limsup_{n\to\infty} |P_n(z)|^{1/n} = 1 \qquad q.e.\ z \in L. \tag{38}$$

For each $r > 1$, set

$$l_r := \{z : |\Phi(z)| = r\} \tag{39}$$

and $l_1 := L = \partial G$. If g is an analytic function on G, define

$$\rho(g) := \sup\{r : g \text{ is analytic on } \mathrm{int}(l_r)\}\,. \tag{40}$$

Then $1 \le \rho(g) \le \infty$, and if $\mathcal{P}_w^2(G)$ denotes the closure of the set of polynomials in $\mathcal{L}_w^2(G)$, we have

Lemma 4.1. *Let $g \in \mathcal{L}_w^2(G)$ and let $a_n := \langle g|P_n(\cdot\,;w)\rangle_w$, $n = 0, 1, \ldots$. Then*

$$\limsup_{n\to\infty} |a_n|^{1/n} \le \frac{1}{\rho(g)}. \tag{41}$$

Moreover, if $g \in \mathcal{P}_w^2(G)$, then equality holds in (41) and

$$g(z) = \sum_{n=0}^{\infty} a_n P_n(z)$$

locally uniformly on $\mathrm{int}\left(l_{\rho(g)}\right)$.

With Lemma 3.2 and (37) at hand, the proof of Lemma 4.1 is essentially the same as that given by J. L. Walsh in [17, pp. 130–131] (see also [11, p. 336]).

We can apply the above lemma to estimate $|P_n(\zeta)|$ for $\zeta \in G$. Indeed, since by (19), $\overline{P_n(\zeta)} = \langle K_w(\cdot,\zeta)|P_n\rangle_w$, it follows that for each $\zeta \in G$ fixed,

$$\sum_{n=1}^{\infty} \overline{P_n(\zeta)} P_n(\cdot) =: L_w(\cdot,\zeta)$$

represents a function of the space $\mathcal{L}^2_w(G)$. By Lemma 3.3(ii), $\mathcal{P}^2_w(G) = \mathcal{L}^2_w(G)$ if and only if

$$L_w(\cdot,\zeta) = K_w(\cdot,\zeta)\,, \qquad \forall\, \zeta \in G\,.$$

Of course, we also have $\overline{P_n(\zeta)} = \langle L_w(\cdot,\zeta)|P_n\rangle_w$, so that by applying Lemma 4.1 to $g = L_w(\cdot,\zeta)$ and $g = K_w(\cdot,\zeta)$ we get

Corollary 4.2. *For every $\zeta \in G$,*

$$\limsup_{n\to\infty} |P_n(\zeta)|^{1/n} = \frac{1}{\rho\,(L_w(\cdot,\zeta))} \le \frac{1}{\rho\,(K_w(\cdot,\zeta))}\,. \tag{42}$$

Furthermore, if $\mathcal{P}^2_w(G) = \mathcal{L}^2_w(G)$, then equality holds in (42) *and, therefore,*

$$\limsup_{n\to\infty} |P_n(\zeta)|^{1/n} = 1$$

if and only if $K_w(\cdot,\zeta)$ has a singularity on $L = \partial G$.

Corollary 4.2 describes a basic relationship between the orthogonal polynomials and the kernel function which will play an essential role in deriving our zero distribution results. We shall also apply the next lemma which involves the logarithmic potential of a measure, as well as the notion of harmonic majorant. While somewhat more general, it is similar to results of Walsh (see Remark 4.5 below).

For any finite, positive Borel measure σ with compact support $\operatorname{supp}(\sigma) \subset \mathbb{C}$, we denote by U^σ its *logarithmic potential* defined by

$$U^\sigma(z) := \int_{\mathbb{C}} \log \frac{1}{|z-t|}\, d\sigma(t)\,, \qquad z \in \mathbb{C}\,.$$

Notice that if q_n is a monic polynomial of degree n, then the logarithmic potential of the counting measure ν_{q_n} is

$$U^{\nu_{q_n}}(z) = n^{-1} \log |q_n(z)|^{-1}.$$

Lemma 4.3. *Let $E \neq \emptyset$ be a compact subset of $\mathbb{C}$ such that both $\overline{\mathbb{C}} \setminus E$ and $\mathring{E} := \operatorname{int}(E)$ are connected (see Figure 9). Let $g\colon \overline{\mathbb{C}} \setminus \mathring{E} \to \overline{\mathbb{C}}$ be such that g is analytic in $\mathbb{C} \setminus E$, $|g|$ is continuous and never zero in $\overline{\mathbb{C}} \setminus \mathring{E}$, $g(\infty) = \infty$ and $g'(\infty) = 1$. Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of monic polynomials of respective degrees $n = 1, 2, \ldots$, such that ∞ is not an accumulation point of the set of zeros of the q_n's. Further, assume that*

$$\limsup_{n\to\infty} |q_n(z)|^{1/n} \le |g(z)| \qquad \text{q.e. } z \in \partial E\,. \tag{43}$$

Then, any measure σ that is a weak-limit point of the sequence $\{\nu_{q_n}\}_{n=1}^{\infty}$ is supported on E and*

$$U^{\sigma}(z) = \log|g(z)|^{-1}, \qquad \forall\, z \in \mathbb{C} \setminus \overset{\circ}{E}\,. \tag{44}$$

Moreover, there is a unique measure μ_g supported on ∂E such that (44) *holds with $\sigma = \mu_g$. For such a measure, we have*

$$\limsup_{n\to\infty} |q_n(z)|^{1/n} \le e^{-U^{\mu_g}(z)}, \qquad \forall\, z \in \mathbb{C}\,, \tag{45}$$

and

(a) *if $\overset{\circ}{E} = \emptyset$, then $\nu_{q_n} \xrightarrow{*} \mu_g$ as $n \to \infty$;*

(b) *if $\overset{\circ}{E} \neq \emptyset$ and for some $z_0 \in \overset{\circ}{E}$ and a subsequence $\mathcal{N} \subset \mathbb{N}$*

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} |q_n(z_0)|^{1/n} = e^{-U^{\mu_g}(z_0)}, \tag{46}$$

then

$$\nu_{q_n} \xrightarrow{*} \mu_g \qquad \text{as } n \to \infty,\ n \in \mathcal{N}\,. \tag{47}$$

Conversely, if μ_g is a weak-star limit point of the sequence $\{\nu_{q_n}\}$, then equality holds in* (45) *for* quasi-every $z \in \mathbb{C}$.

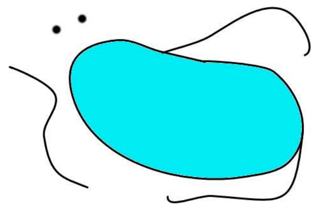


FIGURE 9. A set E satisfying the hypotheses of Lemma 4.3.

Proof. Observe that (43) is equivalent to

$$\liminf_{n\to\infty} U^{\nu_{q_n}}(z) \ge \log|g(z)|^{-1} \qquad q.e.\ z \in \partial E\,. \tag{48}$$

Let σ be a weak*-limit point of the sequence $\{\nu_{q_n}\}_{n=1}^{\infty}$, so that for some subsequence $\mathcal{N} \subset \mathbb{N}$

$$\nu_{q_n} \xrightarrow{*} \sigma \qquad \text{as } n \to \infty\,,\ n \in \mathcal{N}\,.$$

Then σ is a probability measure and, by (48) and the Lower Envelope Theorem (see [14, Thm. I.6.9]), we have

$$U^{\sigma}(z) = \liminf_{\substack{n\to\infty\\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z) \ge \liminf_{n\to\infty} U^{\nu_{q_n}}(z) \ge \log|g(z)|^{-1} \qquad q.e.\ z \in \partial E\,. \tag{49}$$

By the assumptions on g, the function

$$F^{\sigma}(z) := U^{\sigma}(z) - \log|g(z)|^{-1}\,, \qquad z \in \mathbb{C} \setminus E\,,$$

is superharmonic and lower bounded in $\mathbb{C}\setminus E$, harmonic and equal to zero at ∞, and in view of (49) and the lower semicontinuity of U^σ, it also satisfies for *quasi-every* $z' \in \partial E$

$$\liminf_{\substack{z\to z'\\ z\in\mathbb{C}\setminus E}} F^\sigma(z) \geq \liminf_{z\to z'} U^\sigma(z) - \lim_{\substack{z\to z'\\ z\in\mathbb{C}\setminus E}} \log|g(z)|^{-1} \geq U^\sigma(z') - \log|g(z')|^{-1} \geq 0\,.$$

Then, by the generalized minimum principle for superharmonic functions (see [14, Thm. I.2.4]) we conclude that $F^\sigma \equiv 0$, which implies that (44) holds in $\mathbb{C}\setminus E$. It also implies that U^σ is harmonic in $\mathbb{C}\setminus E$ and therefore, in view of the unicity theorem (see for e.g. [14, Thm. II.2.1]), $\operatorname{supp}(\sigma)$ must be contained in E. Since the boundary of the domain $\mathbb{C}\setminus E$ in the fine topology (i.e. the coarsest topology that makes every logarithmic potential continuous) coincides with its boundary in the Euclidean topology (see [14, Cor. I.5.6]), we see that (44) is also valid in $\mathbb{C}\setminus \overset{\circ}{E}$.

It is a direct consequence of Carleson's Unicity Theorem (see [14, Thm. II.4.13]) that there can be at most one measure μ_g supported on ∂E that satisfies (44) with $\sigma = \mu_g$. To see that such a μ_g actually exists, choose any measure σ that is a weak*-star limit point of the sequence $\{\nu_{q_n}\}_{n=1}^\infty$. (This is possible in view of Helly's Theorem ([14, Thm. 0.1.3]) because, by assumption, all the zeros of the q_n's lie in a fixed compact subset of $\mathbb{C}$.) Let σ_1 be the restriction of σ to $\overset{\circ}{E}$, and let $\widehat{\sigma}_1$ be the balayage of σ_1 onto $\partial\overset{\circ}{E}$. Then, $\mu_g := \sigma - \sigma_1 + \widehat{\sigma}_1$ is the measure we are looking for, since it easily follows from the properties of balayage measures (see [14, Thm. II.4.1]) that this μ_g satisfies

$$U^{\mu_g}(z) = U^\sigma(z), \quad \forall\, z \in \mathbb{C}\setminus E, \qquad U^\sigma(z) \geq U^{\mu_g}(z), \quad \forall\, z\in\mathbb{C}\,. \tag{50}$$

Accordingly, when $\overset{\circ}{E} = \emptyset$, the measure μ_g is the unique weak*-limit point of $\{\nu_{q_n}\}$, so that (a) takes place.

Now, for any $z \in \mathbb{C}$ fixed, choose a subsequence $\mathcal{N} \subset \mathbb{N}$ through which the lim sup in (45) is realized. We can assume that also $\nu_{q_n} \xrightarrow{*} \sigma$ as $n\to\infty$, $n\in\mathcal{N}$. Then, by the principle of descent (see [14, Thm. I.6.8]) and (50),

$$\liminf_{n\to\infty} U^{\nu_{q_n}}(z) = \lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z) \geq U^\sigma(z) \geq U^{\mu_g}(z)\,,$$

which proves (45).

Let us now prove (b). Suppose (46) holds, and let σ_0 be an arbitrary weak*-limit point of $\{\nu_{q_n}\}_{n\in\mathcal{N}}$. Because U^{μ_g} is harmonic in $\overset{\circ}{E}$, we get from (50) and the minimum principle for superharmonic functions that $U^{\sigma_0}(z) > U^{\mu_g}(z)$ for all $z \in \overset{\circ}{E}$, unless $U^{\sigma_0} \equiv U^{\mu_g}$ on $\overset{\circ}{E}$. But from (46) and the principle of descent, we have that

$$U^{\mu_g}(z_0) = \lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z_0) = \liminf_{\substack{n\to\infty\\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z_0) \geq U^{\sigma_0}(z_0)\,.$$

Therefore, $U^{\sigma_0} \equiv U^{\mu_g}$ is harmonic in $\overset{\circ}{E}$, and consequently $\operatorname{supp}(\sigma_0) \subset \partial E$. By the uniqueness of μ_g, $\sigma_0 = \mu_g$, and since σ_0 is arbitrary, (47) must hold.

Finally, suppose that conversely, (47) takes place for some subsequence $\mathcal{N} \subset \mathbb{N}$. Then, by the Lower Envelope Theorem, we have for *quasi-every* $z \in \mathbb{C}$

$$U^{\mu_g}(z) = \liminf_{\substack{n\to\infty \\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z) \geq \liminf_{n\to\infty} U^{\nu_{q_n}}(z) \geq U^{\mu_g}(z);$$

that is, we have equality in (45) *quasi-everywhere* on $\mathbb{C}$. ∎

Remark 4.4. (i) By arguing as in the proof of Lemma 4.3, one readily sees that if the inequality in (43) is satisfied *quasi-everywhere* on $\mathbb{C} \setminus E$, then the conclusions of that lemma remain true, even if g has zeros on ∂E. One can also verify that if $z_0 \in \mathbb{C} \setminus E$ has a neighborhood on which q_n has no zeros for n large enough, then

$$\lim_{n\to\infty} |q_n(z_0)|^{1/n} = |g(z_0)|\,. \tag{51}$$

Hence, equality holds in (43) *quasi-everywhere* on $\mathbb{C} \setminus E$.

(ii) A well-known result by Fejér asserts that the zeros of orthogonal polynomials with respect to a compactly supported measure σ are contained in the closed convex hull of $\operatorname{supp}(\sigma)$ (see for e.g. [13]). Thus, if the q_n's in Lemma 4.3 are orthogonal, it is already guaranteed that all their zeros are uniformly bounded in $\mathbb{C}$. We will be using this fact in all the applications of Lemma 4.3.

Remark 4.5. The fact that a condition like (43) has consequences on the zero distribution of the sequence $\{q_n\}_{n=1}^{\infty}$ is well-known. For example, from (51) (see [19, Thm. 1]) it follows that for every continuum $Q \subset \mathbb{C} \setminus E$ containing more that one point

$$\limsup_{n\to\infty} \|q_n\|_{L_\infty(Q)}^{1/n} = \|g\|_{L_\infty(Q)}. \tag{52}$$

In the terminology of [19] (see also [18, p. 635]), this is expressed by saying that $\log|g(z)|$ is an *exact harmonic majorant* of the sequence $\{q_n^{1/n}\}_{n=1}^{\infty}$ in $\mathbb{C} \setminus E$. We refer the reader to [19] for earlier results on the behavior of zeros of functions having an exact harmonic majorant. More recent results of a similar nature to that of Lemma 4.3 can be found in [14, Section III.4].

5. Proofs of the zero distribution results

Proof of Theorem 2.1. Define $E := \overline{G}$, $q_n(z) := P_n(z)/\kappa_n^w$ and

$$g(z) := \operatorname{cap}(L)\Phi(z), \qquad z \in \mathbb{C} \setminus G\,.$$

With the help of (34), (36) and (38), it is easily seen that E, g and $\{q_n\}_{n=1}^{\infty}$ so defined satisfy the hypotheses of Lemma 4.3. Then, with the notation of that lemma, we have $\mu_g = \mu_L$ (the equilibrium measure of L), since it is well-known that μ_L is supported on L and satisfies (44). Hence, Theorem 2.1 is a direct consequence of (45), Lemma 4.3(b), and Corollary 4.2. ∎

Proof of Proposition 2.4. The exterior mapping of $G = \mathrm{int}(l_r)$ is $\Phi = \Phi_E/r$, so that Carleman's formula applied to this situation yields (see [1, Thm. 2, p. 12])

$$\lim_{n\to\infty} \frac{P_n(z)}{\sqrt{n+1}\,[\Phi_E(z)/r]^n} = \frac{\Phi_E'(z)}{r\sqrt{\pi}}\,, \tag{53}$$

locally uniformly on $\overline{\mathbb{C}} \setminus E$. Hence,

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = \frac{|\Phi_E(z)|}{r}\,, \qquad z \in \mathbb{C} \setminus E.$$

The first assertion of the proposition is a direct consequence of Hurwitz's Theorem and (53). The second assertion follows by applying Lemma 4.3(a) (see also Remark 4.4(i)) with $q_n := P_n/\kappa_n$, $g := \mathrm{cap}(E)\Phi_E$, since by (34) and (36)

$$\lim_{n\to\infty} (\kappa_n)^{1/n} = [\mathrm{cap}(l_r)]^{-1} = \Phi_E'(\infty)/r = [\mathrm{cap}(E)r]^{-1}\,,$$

and (44) is satisfied for this g with $\sigma = \mu_E$ the equilibrium measure of E. ∎

Proof of Theorem 2.5. Suppose for the moment that (8) holds. To prove (b), define $E := \overline{\mathbb{D}}_r$, $q_n := P_n/\kappa_n^w$ and $g(z) := z$ for all $|z| \geq r$. Since the capacity of the unit circle is 1, it follows from (36) and (8) that E, q_n and g so chosen satisfy the hypotheses of Lemma 4.3. It is well-known that $\mu_r := |dz|/2\pi r$ is the equilibrium measure of the circle $\mathbb{T}_r$, and that its potential is given by

$$U^{\mu_r}(z) = \begin{cases} \log(1/|z|) & \text{if } |z| > r \\ \log(1/r) & \text{if } |z| \leq r \end{cases}.$$

This implies, with the notations of Lemma 4.3, that $\mu_g = \mu_r$, and hence Theorem 2.5(b) is just a consequence of statement (b) of that lemma (cf. also the paragraph preceding (44)).

Similarly, we prove (a). Define $E := \overline{\mathbb{D}}_\rho$, $1 > \rho > 0$, with q_n and g as above. Then (8) implies that (43) holds on $\mathbb{T}_\rho$, and so by Lemma 4.3 any weak*-limit of $\nu_{q_n} = \nu_{P_n}$ is supported on $\overline{\mathbb{D}}_\rho$. Letting ρ go to zero we deduce Theorem 2.5(a).

Thus, it remains to establish (8). Let us write the function w as

$$w(z) = h(z)\prod_{i=1}^{\ell}(z - a_i)^{v_i},$$

where $v_i = \mathrm{mult}(a_i)$. The exterior conformal mapping for $\mathbb{D}$ is simply $\Phi(z) = z$, so that in view of (20), Lemma 3.5(b) with $f \equiv g \equiv 1$, $t(z) := z$, $s(\zeta) := \zeta$, and

(21), the kernel function $K_w(z,\zeta)$ for the space $\mathcal{L}^2_w(\mathbb{D})$ has the form

$$K_w(z,\zeta) = \frac{Q_w\left(z,\overline{\zeta}\right)}{\pi\left(1-z\overline{\zeta}\right)^2\left[\prod_{i=1}^{\ell}\left(1-a_i\overline{\zeta}\right)^{v_i+1}\left(1-z\overline{a}_i\right)^{v_i+1}\right]h(z)\overline{h(\zeta)}}, \tag{54}$$

where $Q_w(\cdot,\cdot)$ is a polynomial in two variables satisfying

(ii) if $\xi \neq 0$, then $Q_w\left(1/\xi,\xi\right) \neq 0 \Leftrightarrow \xi \notin \{1/a_1,\dots,1/a_\ell,\overline{a}_1,\dots,\overline{a}_\ell\}$;
(iii) for every $1 \le i \le \ell$, $Q_w\left(1/\overline{a}_i,\cdot\right) \not\equiv 0$ if $\overline{a}_i \neq 0$.

It follows from (54) that for all $\zeta \in \mathbb{D}$, $K_w(\cdot,\zeta)$ is a meromorphic function in $\mathbb{C}$ whose possible poles are the elements of the set

$$\left\{1/\overline{\zeta}, 1/\overline{a}_1,\dots,1/\overline{a}_\ell, b_1, b_2,\dots\right\} \setminus \mathcal{A}^{-1}, \tag{55}$$

where (notice that each c_j is now a zero of $K_w(\cdot,\zeta)$)

$$\mathcal{A}^{-1} := \{1/\overline{a}_i : 1/\overline{a}_i = c_j \text{ for some } j \text{ and } \operatorname{mult}(c_j) \ge \operatorname{mult}(a_i)+1\}.$$

We shall show that for every $\zeta \in \mathbb{D}$ (except possibly countably many), the finite elements of the set (55) are, in fact, poles of $K_w(\cdot,\zeta)$.

First, we see from (ii) that if $\zeta \in \mathbb{D}$ and

$$\zeta \notin \{a_1,\dots,a_\ell,0\} \cup \{1/\overline{c}_j : \operatorname{mult}(c_j) \ge 2,\ j \ge 1\},$$

then $K_w(\cdot,\zeta)$ has a pole at $z = 1\big/\overline{\zeta}$. Second, it is a consequence of (iii) that for all but finitely many $\zeta \in \mathbb{D}$, $K_w(\cdot,\zeta)$ has a pole at $1/\overline{a}_i$ if $a_i \neq 0$ and $1/\overline{a}_i \notin \mathcal{A}^{-1}$. And finally, if $b_k \notin \{1/\overline{a}_i : a_i \neq 0,\ 1 \le i \le \ell\}$, then again by (ii), $Q_w\left(b_k, 1/b_k\right) \neq 0$, so that $Q_w\left(b_k,\xi\right)$ is a polynomial in ξ not identically zero, and consequently, for all but finitely many $\zeta \in \mathbb{D}$, $K_w(\cdot,\zeta)$ has a pole at b_k.

Thus, according to (39) and (40), for all but countably many $\zeta \in \mathbb{D}$,

$$\rho(K_w(\cdot,\zeta)) = \min\left(|z| : z \in \left\{1/\overline{\zeta}, 1/\overline{a}_1,\dots,1/\overline{a}_\ell, b_1, b_2,\dots\right\} \setminus \mathcal{A}^{-1}\right);$$

whence, by Corollary 4.2 (recall Remark 2.2), for all but countably many $\zeta \in \mathbb{D}$,

$$\begin{aligned}\limsup_{n\to\infty} |P_n(\zeta)|^{1/n} &= \max\left(\left\{0, |\zeta|, |a_1|,\dots,|a_\ell|, |b_1|^{-1}, |b_2|^{-1},\dots\right\} \setminus \mathcal{A}\right) \\ &= \max\{|\zeta|, r\} = \begin{cases} |\zeta| & \text{if } r < |\zeta| < 1 \\ r & \text{if } |\zeta| \le r \end{cases}\end{aligned}$$

where

$$\begin{aligned}\mathcal{A} &= \{|z|^{-1} : z \in \mathcal{A}^{-1}\}, \\ r &= \max\left(\left\{0, |a_1|,\dots,|a_\ell|, |b_1|^{-1}, |b_2|^{-1},\dots\right\} \setminus \mathcal{A}\right).\end{aligned}$$

■

Example 5.1. Let w be a meromorphic function on $\mathbb{C}$, that does not vanish, whose poles $c_1, c_2, \dots$ all lie in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and each of them has multiplicity no less than 2. Since in this case the kernel function has the form

$$K_w(z,\zeta) = \frac{1}{\pi\left(1-z\overline{\zeta}\right)^2 w(z)\overline{w(\zeta)}},$$

we can see that $K_w(\cdot, 1/\overline{c}_j)$ is an entire function for all $1 \le j < \infty$, and if $\zeta \notin \{1/\overline{c}_1, 1/\overline{c}_2, \ldots\}$, then $K_w(\cdot, \zeta)$ is a meromorphic function with a double pole at $1/\overline{\zeta}$. Consequently, we have for all $\zeta \in \mathbb{D}$

$$\limsup_{n\to\infty} |P_n(\zeta)|^{1/n} = \begin{cases} 0 & \text{if } \zeta \in \{1/\overline{c}_1, 1/\overline{c}_2, \ldots\} \\ |\zeta| & \text{otherwise} \end{cases}, \tag{56}$$

and according to Theorem 2.5(a), this implies that $\nu_{P_n} \xrightarrow{*} \delta_0$ as $n \to \infty$. However, each point $1/\overline{c}_j$ is a limit point of the zeros of the P_n's, because if, to the contrary, there is a neighborhood V of $1/\overline{c}_j$ and a subsequence $\mathcal{N} \subset \mathbb{N}$ such that P_n has no zeros on V for $n \in \mathcal{N}$, then by the continuity of $\log|t - 1/\overline{c}_j|^{-1}$ in $\mathbb{C} \setminus V$, we would have

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} U^{\nu_{P_n}}(1/\overline{c}_j) = U^{\delta_0}(1/\overline{c}_j) = \log|c_j|\,,$$

contradicting (56).

Proof of Theorem 2.9. Recall that the lens-shaped domain G, as well as its associated curves Γ, γ_r, and function $\widehat{\Phi}$ have been introduced in the paragraph preceding the statement of Theorem 2.9. Assume that (14) is true for some r with $R_\Gamma \le r \le 1$. Set $E := \Gamma \cup \gamma_r \cup \operatorname{int}(\gamma_r)$, $q_n = P_n/\kappa_n^w$, and $g(z) := \operatorname{cap}(L)\widehat{\Phi}(z)$ for all $z \in \overline{\mathbb{C}} \setminus \operatorname{int}(\gamma_r)$. We see from (36) and (14) that E, q_n and g so defined satisfy the assumptions of Lemma 4.3, and hence, any weak*-limit point σ of $\{\nu_{P_n}\} = \{\nu_{q_n}\}$ is supported in $\Gamma \cup \gamma_r \cup \operatorname{int}(\gamma_r)$. Let $\mu_r := \mu_g$ be the unique measure supported on $\partial E = (\Gamma \cup \gamma_r) \setminus \operatorname{int}(\gamma_r)$ that satisfies

$$U^{\mu_r}(z) = \log|g(z)|^{-1} = \log|\operatorname{cap}(L)\widehat{\Phi}(z)|^{-1}, \qquad \forall\, z \in \mathbb{C} \setminus \operatorname{int}(\gamma_r)\,.$$

Now, from the definition of γ_r in (13), and the lower semicontinuity of U^{μ_r}, we have that if $\operatorname{int}(\gamma_r) \neq \emptyset$, then

$$\liminf_{\substack{z\to z'\\ z\in\operatorname{int}(\gamma_r)}} U^{\mu_r}(z) \ge U^{\mu_r}(z') = \log[\operatorname{cap}(L) r]^{-1}, \qquad \forall\, z' \in \gamma_r\,, \tag{57}$$

and in view of (14) and (45), we have for some $z_0 \in \operatorname{int}(\gamma_r)$

$$\operatorname{cap}(L)\, r = \limsup_{n\to\infty} |q_n(z_0)|^{1/n} \le e^{-U^{\mu_r}(z_0)}\,. \tag{58}$$

Since (57), (58) and the minimum principle for superharmonic functions imply that

$$U^{\mu_r}(z) = \log\left[\operatorname{cap}(L)\, r\right]^{-1}, \qquad \forall\, z \in \operatorname{int}(\gamma_r),$$

the statements (a) and (b) of Theorem 2.9 follow directly from their corresponding ones in Lemma 4.3.

Let us now show that if σ is a weak*-limit point of the measures ν_{P_n}, then necessarily every point of $\Gamma \setminus \operatorname{int}(\gamma_r)$ belongs to $\operatorname{supp}(\sigma)$. Suppose that $z_0 \in \Gamma \setminus \overline{\operatorname{int}(\gamma_r)}$ is not in $\operatorname{supp}(\sigma)$ and let us derive a contradiction. Let $D_{z_0} \subset G \setminus \overline{\operatorname{int}(\gamma_r)}$ be a disk centered at z_0 and of radius so small that $\operatorname{supp}(\sigma)\cap D_{z_0}=\emptyset$. Then U^σ is harmonic in D_{z_0} and we have from (44) and (11)

$$U^\sigma(z)=\begin{cases}\log\left|\overline{\Phi(z_\alpha)}/\operatorname{cap}(L)\right| & z\in G_\alpha\cap D_{z_0}\\ \log\left|\overline{\Phi(z_\beta)}/\operatorname{cap}(L)\right| & z\in G_\beta\cap D_{z_0}\end{cases}. \tag{59}$$

But since the harmonic extension is unique, it follows that the first row of the righthand side of (59) also represents U^σ in $G_\beta\cap D_{z_0}$, contradicting the obvious fact that

$$G_\beta=\{z\in G:\,|\Phi(z_\alpha)|>|\Phi(z_\beta)|\}\ .$$

Analogously, one can show that $\operatorname{supp}(\mu_r)=(\Gamma\cup\gamma_r)\setminus\operatorname{int}(\gamma_r)$.

We now turn to the proof of (14). Similar to the case of the unit disk, the argument is based on Corollary 4.2. Therefore, our next task is to find the singularities of the kernel function $K_w(\cdot,\zeta)$ for the lens-shaped domain G and an entire weight function w. It is not difficult to see that for every $\zeta\in G$, the function

$$\varphi_\zeta(z):=\frac{\left(\dfrac{z-i}{z+i}\right)^N-\left(\dfrac{\zeta-i}{\zeta+i}\right)^N}{\left(\dfrac{z-i}{z+i}\right)^N-\left(\dfrac{\overline{\zeta}+i}{\overline{\zeta}-i}\right)^N\cdot e^{-2N\alpha i}} \tag{60}$$

maps G conformally onto $\mathbb{D}$ in such a way that $\varphi_\zeta(\zeta)=0$. Then, choosing $\varphi=\varphi_\zeta$ in formula (20) for each particular $\zeta\in G$, we obtain after some computation that

$$K(z,\zeta)=-\frac{4N^2}{\pi}\cdot\frac{\left[\left(\overline{\zeta}-i\right)\left(\overline{\zeta}+i\right)(z-i)(z+i)\right]^{N-1}}{\left[e^{N\alpha i}\left(\overline{\zeta}-i\right)^N(z-i)^N-e^{-N\alpha i}\left(\overline{\zeta}+i\right)^N(z+i)^N\right]^2}. \tag{61}$$

Let $\{a_1,\dots,a_\ell\}$ be the set of zeros of w lying on G, and let $\{b_1,b_2,\dots\}$ be the set of zeros of w lying on $\mathbb{C}\setminus G$. Write w as

$$w(z):=h(z)\prod_{j=1}^{\ell}(z-a_i)^{v_i}\,,$$

where $v_i=\operatorname{mult}(a_i)$, $1\le i\le\ell$, is the multiplicity of the zero a_i. Then, by (21) and Lemma 3.5(b), we have the following representation for $K_w(z,\zeta)$ in terms of any f, g, t, and s satisfying (26):

$$(62)\quad K_w(z,\zeta) = \frac{f(z)\overline{g(\zeta)}}{\pi\left[1-t(z)\overline{s(\zeta)}\right]^2} \times \frac{\prod_{i=1}^{\ell}[t(z)-t(a_i)]^{v_i}\left[\overline{s(\zeta)}-\overline{s(a_i)}\right]^{v_i}}{h(z)\overline{h(\zeta)}\prod_{i=1}^{\ell}(z-a_i)^{v_i}(\overline{\zeta}-\overline{a}_i)^{v_i}}$$
$$\times\frac{Q_w\left(t(z),\overline{s(\zeta)}\right)}{\prod_{i=1}^{\ell}\left[1-t(a_i)\overline{s(\zeta)}\right]^{v_i+1}\left[1-t(z)\overline{s(a_i)}\right]^{v_i+1}},$$

where $Q_w(\tau,\xi)$ is a polynomial in two variables (that depends on the choice of t and s) with the properties stated in Lemma 3.5(b)(i)–(iv).

We first prove that

(I) *$K_w(\cdot,\zeta)$ is a meromorphic function in $\mathbb{C}$ such that $h(\cdot)K_w(\cdot,\zeta)$ is analytic in $\overline{G}$ and, for all but finitely many $\zeta\in G$, i and $-i$ are zeros of $h(\cdot)K_w(\cdot,\zeta)$ of multiplicity $N-1$.*

Let φ be a conformal map of G onto $\mathbb{D}$. By (20), we can set $f(z)=\varphi'(z)$, $g(\zeta)=\varphi'(\zeta)$, $t(z)=\varphi(z)$, and $s(\zeta)=\varphi(\zeta)$. Then, since φ is a rational function that has an analytic continuation (also denoted by φ) across ∂G, we see from (62) with the above choice of f, g, t, and s, that for all $\zeta\in G$, $K_w(\cdot,\zeta)$ is a meromorphic function in $\mathbb{C}$. On the other hand, since $|\varphi(\pm i)|=1$, we have

$$\varphi(\pm i)^{-1}\notin\left\{1/\varphi(a_1),\dots,1/\varphi(a_\ell),\overline{\varphi(a_1)},\dots,\overline{\varphi(a_\ell)}\right\},$$

so that by Lemma 3.5(b)(ii), $Q_w(\varphi(\pm i),\cdot)\not\equiv 0$. Thus, it follows from (62) that for all but finitely many $\zeta\in G$, $\pm i$ is a zero of $K_w(\cdot,\zeta)$ if and only if $\pm i$ is a zero of $K(\cdot,\zeta)$, so that the rest of (I) follows from (61).

Now, we see from (61) that for all $z\neq i$, $K(z,\zeta)$ can be expressed in the form of (26), this time with the choice of functions

$$f(z)=\frac{(z+i)^{N-1}}{(z-i)^{N+1}},\qquad \overline{g(\zeta)}=-\frac{4N^2e^{-2N\alpha i}(\overline{\zeta}+i)^{N-1}}{(\overline{\zeta}-i)^{N+1}},$$

$$(63)\qquad t(z)=\left(\frac{z+i}{z-i}\right)^N,\qquad \overline{s(\zeta)}=\left(\frac{\overline{\zeta}+i}{\overline{\zeta}-i}\cdot e^{-2\alpha i}\right)^N.$$

Then, looking at the denominator of (62), we see that the possible poles of $K_w(\cdot,\zeta)$ are contained in the set

$$\mathcal{S}:=\{b_1,b_2,\dots\}\cup\mathcal{S}(\zeta)\cup\mathcal{S}(a_1)\cup\cdots\cup\mathcal{S}(a_\ell),$$

where $\mathcal{S}(\zeta)$, $\mathcal{S}(a_i)$ denote, respectively, the solution sets of the equations in the variable z

$$(64)\qquad 1-t(z)\overline{s(\zeta)}=0,\qquad 1-t(z)\overline{s(a_i)}=0,\quad i=1,\dots,\ell;$$

Next, we show that

(II) *for all but countably many $\zeta \in G$, $z \in \mathcal{S}$ is* **not** *a pole of $K_w(\cdot,\zeta)$ if and only if there exists $1 \leq k < \infty$ such that $z = b_k$ and either one of the following statements holds:*
(II') $b_k \in \{-i, i\}$ *and* $\operatorname{mult}(b_k) \leq N-1$*;*
(II") $b_k = c$ *for some c that is a zero of the rational function*

$$\prod_{i=1}^{\ell}[t(z) - t(a_i)]^{v_i}$$

with multiplicity at least $\operatorname{mult}(b_k)$.

Suppose that z is a solution of any of the equations (64) that is not a pole of $K_w(\cdot,\zeta)$. Since obviously $z \neq \pm i$, we have that at least one of the equations

$$t(z) - t(a_i) = 0, \qquad 1 \leq i \leq \ell\,, \tag{65}$$

$$Q_w\left(1/\overline{s(\zeta)}, \overline{s(\zeta)}\right) = 0, \qquad Q_w\left(1/\overline{s(a_i)}, \overline{s(\zeta)}\right) = 0, \quad 1 \leq i \leq \ell\,, \tag{66}$$

must be satisfied. But for all $\eta, \lambda \in G$, $1 - t(\eta)\overline{s(\lambda)} \neq 0$, so that z cannot satisfy any of the equations (65). Also, since Q_w is a polynomial and $s(\zeta)$ is given by (63), Lemma 3.5(b)(ii)(iii) implies that only a finite number of $\zeta \in G$ can be a solution to one of the equations (66). Hence, for all but finitely many $\zeta \in G$, every element of $\mathcal{S}(\zeta) \cup \mathcal{S}(a_1) \cup \cdots \cup \mathcal{S}(a_\ell)$ is a pole of $K_w(\cdot,\zeta)$.

Now, it follows from (I) that for all but finitely many $\zeta \in G$, $b_k \in \{-i, i\}$ is a pole of $K_w(\cdot,\zeta)$ if and only if $\operatorname{mult}(b_k) > N-1$. On the other hand, for any $b \in \mathbb{C} \setminus (G \cup \{i, -i\})$, the polynomial $Q_w(t(b), \cdot)$ is not identically zero (this is guaranteed by Lemma 3.5(b)(i) if $t(b) = t(a_i)$ for some $1 \leq i \leq \ell$, by (iii) if $t(b) = 1/\overline{s(a_i)}$ for some $1 \leq i \leq \ell$, and by (ii) if $t(b)$ is otherwise). Thus, for all but finitely many $\zeta \in G$, the zero $b_k \in \mathbb{C} \setminus (G \cup \{i, -i\})$ of $h(z)$ is a pole of $K_w(\cdot,\zeta)$ unless $b_k = c$ for some c that is a solution to

$$\prod_{i=1}^{\ell}[t(z) - t(a_i)]^{v_i} = 0\,,$$

of multiplicity at least $\operatorname{mult}(b_k)$. This completes the proof of (II).

Notice that according to the definition in (39) and (40),

$$1/\rho(K_w(\cdot,\zeta)) = \max\left\{|1/\Phi(z)| : z \text{ is a pole of } K_w(\cdot,\zeta)\right\}. \tag{67}$$

Now, it easily follows from (11) and (10) that for any $\zeta \in G$,

$$\max\{|1/\Phi(\zeta_\alpha)|, |1/\Phi(\zeta_\beta)|\} = \begin{cases} |1/\Phi(\zeta_\alpha)| & \text{if } \zeta \in G_\alpha \cup \Gamma \\ |1/\Phi(\zeta_\beta)| & \text{if } \zeta \in G_\beta \end{cases} = |\widehat{\Phi}(\zeta)| \tag{68}$$

and it is not difficult to verify by using the explicit expressions of t and s in (63) that

$$\mathcal{S}(\zeta) = \left\{ \frac{\overline{\zeta}\cot(\alpha - k\pi/N) + 1}{\overline{\zeta} - \cot(\alpha - k\pi/N)}\,,\, 1 \le k \le N \right\}.$$

In particular, $\zeta_\alpha, \zeta_\beta \in \mathcal{S}(\zeta)$ (cases $k = N$ and $k = 1$, respectively).

Suppose we have proven that

$$\max\{|1/\Phi(\eta)| : \eta \in \mathcal{S}(\zeta)\} = \max\{|1/\Phi(\zeta_\alpha)|, |1/\Phi(\zeta_\beta)|\}\,. \tag{69}$$

Then by (67), (II), (69) and (68), we get that for all but finitely many $\zeta \in G$,

$$1/\rho(K_w(\cdot,\zeta)) = \max\left\{|\widehat{\Phi}(\zeta)|, |\widehat{\Phi}(a_1)|, \ldots, |\widehat{\Phi}(a_\ell)|, |\Phi(b_1)|^{-1}, |\Phi(b_2)|^{-1}, \cdots\right\} \setminus \mathcal{B},$$

where $\mathcal{B} := \{|\Phi(b_k)|^{-1} : b_k$ satisfies either (II') or (II")$\}$. We will show, however, that

$$t(c) - t(a_i) = 0 \Rightarrow |\Phi(c)|^{-1} \le |\Phi(a_i)|^{-1}\,, \tag{70}$$

and therefore (recall (12)), $1/\rho(K_w(\cdot,\zeta)) = \max\{|\widehat{\Phi}(\zeta)|, r\}$, where r is the largest number of the set

$$\left\{R_\Gamma, |\widehat{\Phi}(a_1)|, \ldots, |\widehat{\Phi}(a_\ell)|, |\Phi(b_1)|^{-1}, |\Phi(b_2)|^{-1}, \cdots\right\} \setminus \left\{|\Phi(b_k)|^{-1} : b_k \in \{-i, i\} \text{ and } \operatorname{mult}(b_k) \le N-1\right\}.$$

Then, the validity of relation (14) follows as a consequence of Corollary 4.2 and the definition of γ_r in (13).

The above argument assumes that (69) and (70) were true. Let us verify that this is the case.

With G the lens-shaped domain described in the paragraph preceding Theorem 2.9, the normalized exterior mapping $w = \Phi(z)$ is given by the composition of the following three transformations:

$$\xi(z) := e^{(\pi-\beta)i}\left(\frac{z-i}{z+i}\right), \tag{71}$$

$$t(\xi) = \xi^{N/(2N-1)}, \quad \arg\xi \in \left(-\tfrac{\pi}{N}, \tfrac{(2N-1)\pi}{N}\right), \tag{72}$$

$$w(t) := \frac{1-\lambda_\beta t}{t-\lambda_\beta}, \quad \lambda_\beta := e^{\frac{N(\pi-\beta)i}{2N-1}}\,. \tag{73}$$

Let us prove (69). If $\eta \in \mathcal{S}(\zeta)$, then by definition, $1 - t(\eta)\overline{s(\zeta)} = 0$ where t and s are given by (63). Hence, for some $1 \le k \le N$,

$$\frac{\eta - i}{\eta + i} = \frac{\overline{\zeta} + i}{\overline{\zeta} - i} \cdot e^{(2\pi k/N - 2\alpha)i},$$

so that $(\eta - i)/(\eta + i)$ lies on the circular arc

$$\mathcal{C}_\zeta := \left\{ \left| \frac{\overline{\zeta} + i}{\overline{\zeta} - i} \right| e^{i\theta} \,:\, \arg\left(\frac{\overline{\zeta} + i}{\overline{\zeta} - i}\right) + 2\beta \le \theta \le \arg\left(\frac{\overline{\zeta} + i}{\overline{\zeta} - i}\right) + 2\pi - 2\alpha \right\} .$$

Notice that the endpoints of $\mathcal{C}_\zeta$ correspond to the values $\eta = \zeta_\beta$, $\eta = \zeta_\alpha$. By (71), (72) and (73), $\{\Phi(\eta) : \eta \in \mathcal{S}(\zeta)\}$ is contained in the set

$$\mathcal{C}^*_\zeta := \left\{ (w \circ t)(\xi) : |\xi| = |(\overline{\zeta} + i)/(\overline{\zeta} - i)| \right\} ,$$

which is obviously a circle intersecting the unit circle at two points. Indeed, $\{\Phi(\eta) : \eta \in \mathcal{S}(\zeta)\}$ is contained in the subarc $(w \circ t)\left(e^{(\pi-\beta)i}\mathcal{C}_\zeta\right)$ of $\mathcal{C}^*_\zeta$, which lies on $\{|w| > 1\}$ and connects the points $\Phi(\zeta_\alpha)$, $\Phi(\zeta_\beta)$. Consequently, $\Phi(\zeta_\alpha)$ and $\Phi(\zeta_\beta)$ are the nearest points of $(w \circ t)\left(e^{(\pi-\beta)i}\mathcal{C}_\zeta\right)$ to the origin, whence (69) follows.

Now, to prove (70), assume that $t(c) - t(a_j) = 0$. Then for some $1 \le k \le N - 1$,

$$\frac{c - i}{c + i} = \overline{\left(\frac{\overline{a}_j + i}{\overline{a}_j - i}\right)} \cdot e^{2k\pi i/N} ,$$

so that $(c - i)/(c + i)$ lies on the circular arc

$$\left\{ \left| \frac{\overline{a}_j + i}{\overline{a}_j - i} \right| e^{i\theta} \,:\, 2\pi - \arg\left(\frac{\overline{a}_j + i}{\overline{a}_j - i}\right) + 2\pi/N \le \theta \le 4\pi - \arg\left(\frac{\overline{a}_j + i}{\overline{a}_j - i}\right) - 2\pi/N \right\} .$$

By the argument given above to prove (69), it suffices to show that this arc is a subset of $\mathcal{C}_{a_j}$. But this is a trivial fact since $\alpha + \beta = \pi/N$ and

$$\pi - \beta < \arg\left(\frac{\overline{a}_j + i}{\overline{a}_j - i}\right) < \pi + \alpha .$$

The proof is complete. ∎

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