A domain decomposition method for approximating the conformal modules of long quadrilaterals

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Summary. This paper is concerned with the study of a domain decomposition method for approximating the conformal modules of long quadrilaterals. The method has been studied already by us and also by D. Gaier and W.K. Hayman, but only in connection with a special class of quadrilaterals, viz. quadrilaterals where: (a) the defining domain is bounded by two parallel straight lines and two Jordan arcs, and (b) the four specified boundary points are the four corners where the arcs meet the straight lines.

Our main purpose here is to explain how the method may be extended to a wider class of quadrilaterals than that indicated above.

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1 Introduction

Let Ω be a Jordan domain in the complex z-plane ($z = x + \mathrm{i}\,y$), and consider a system consisting of Ω and four distinct points z_1, z_2, z_3, z_4 in counterclockwise order on its boundary $\partial\Omega$. Such a system is said to be a quadrilateral Q and is denoted by

$$(1.1) \qquad Q := \{\Omega; z_1, z_2, z_3, z_4\}.$$

The *conformal module* $m(Q)$ of Q is defined as follows:

Let R_H be a rectangle of the form

$$(1.2) \qquad R_H := \{(\xi, \eta): 0 < \xi < 1, 0 < \eta < H\},$$

in the w-plane ($w = \xi + \mathrm{i}\,\eta$). Then, $m(Q)$ is the unique value of H for which Q is conformally equivalent to the rectangular quadrilateral

$$(1.3) \qquad Q' := \{R_H; 0, 1, 1 + \mathrm{i}H, \mathrm{i}H\}.$$

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By this we mean that for $H = m(Q)$ and for this value only there exists a unique conformal map $\Omega \to R_H$, which takes the four points z_1, z_2, z_3, z_4, respectively onto the four corners 0, 1, $1 + iH$, iH of R_H.

This paper is concerned with the study of a domain decomposition method (DDM) for computing approximations to the conformal modules of "long" quadrilaterals Q. This involves decomposing Q into two or more component quadrilaterals Q_j, $j = 1, 2, \ldots$, and approximating $m(Q)$ by the sum $\sum m(Q_j)$ of the conformal modules of the component quadrilaterals. The method has been studied already by us [9, 10] and also by Gaier and Hayman [2, 3], but only in connection with a special class of quadrilaterals, viz. quadrilaterals $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ where: (a) The domain Ω is bounded by two parallel straight lines l_1, l_2, and two Jordan arcs γ_1, γ_2. (b) The points z_1, z_2, z_3, z_4 are the four corners where the arcs γ_1, γ_2, meet the straight lines l_1, l_2.

Of the four references cited above, our two earlier papers [9, 10] are concerned with the problems of approximating both the conformal module $m(Q)$ and the conformal map $R_{m(Q)} \to \Omega$, and include the following:

(i) Computable estimates for the error in the DDM approximation to $m(Q)$, and also for various other errors connected with the DDM approximation to the conformal map $R_{m(Q)} \to \Omega$. These are derived in [9, Sect. 4], by assuming that the two boundary arcs γ_1, γ_2, satisfy certain smoothness conditions.
(ii) A number of conjectures concerning the assumptions under which the error estimates of [9] hold (cf. [10, Sect. 3]).
(iii) The results of several numerical examples that confirm the theory of [9] and support the conjectures made in [10] (cf. [9, Sect. 5] and [10, Sect. 3]).

The two more recent papers [2, 3] by Gaier and Hayman are concerned only with the problem of approximating $m(Q)$, and contain several important results that enhance considerably the associated theory. In particular, [2, 3] include the following:

(iv) An estimate for the error in the DDM approximation to $m(Q)$, which (unlike our estimate in [9]) is derived without imposing any conditions on the boundary segments γ_1, γ_2, other than requiring that they are Jordan arcs. (This proves one of the conjectures made in [10].)
(v) Certain other estimates that connect the conformal modules of the quadrilaterals involved in the decomposition with the conformal radii of certain curves associated with the arcs γ_1, γ_2.

In this paper we consider again the problem of computing approximations to $m(Q)$, and investigate the possibility of extending the application of the DDM to a wider class of quadrilaterals than that studied in [2, 3, 9, 10]. Our main objective is to show that the method does, indeed, have much wider applicability and, for each new application, to derive an estimate of the error in the resulting DDM approximation to $m(Q)$. We shall do this by making use of two new theorems, which are simple consequences of the results of Gaier and Hayman [2, 3] mentioned in (iv) and (v) above.

The paper is organized as follows: In Sect. 2 we set up various notations and state, without proof, the results of [2, 3] which are needed for our subsequent work. In Sect. 3 we prove two new theorems concerning the decomposition of general quadrilaterals. These theorems are then used in Sect. 4 to identify several new types of quadrilaterals for which the DDM is applicable, and to

derive estimates for the errors in the resulting approximations to $m(Q)$. Finally, in Sect. 5 we present three numerical examples that illustrate the much wider applicability of the method.

We end this introductory section by noting that the DDM is of considerable practical interest for the following two reasons: (a) The method can be used to overcome the "crowding" difficulties associated with the conventional approach of seeking to determine $m(Q)$ by going via the unit disc or the upper half plane (see [8, Sect. 3.1] and the other references cited there). (b) It is often possible to decompose a complicated quadrilateral into very simple component quadrilaterals (see the examples in Sect. 5).

2 Preliminary results and notations

We shall use throughout the following notations:

(i) Ω and $Q:=\{\Omega; z_1, z_2, z_3, z_4\}$ will denote respectively the original domain and the corresponding quadrilateral.
(ii) $\Omega_1, \Omega_2, \ldots,$ and $Q_1, Q_2, \ldots,$ will denote the principal subdomains and corresponding component quadrilaterals of the decomposition under consideration.

In addition, we shall use a multisubscript notation to denote the other subdomains and quadrilaterals that arise when the decomposition of Q involves more than one crosscut. For example, consider the situation illustrated in Fig. 1, where the decomposition of

$$Q:=\{\Omega; z_1, z_2, z_3, z_4\}$$

is defined by two non-intersecting arcs γ_1, γ_2, that join respectively two distinct points a and b on the boundary segment (z_2, z_3) to two distinct points d and c on the boundary segment (z_4, z_1). These two arcs subdivide Ω into three non-intersecting subdomains which we denote by Ω_1, Ω_2 and Ω_3. In addition the arc γ_1 subdivides Ω into Ω_1 and another subdomain which we denote by $\Omega_{2,3}$, i.e. we take

$$\bar{\Omega}_{2,3}:=\bar{\Omega}_2 \cup \bar{\Omega}_3.$$

Similarly, we say that γ_2 subdivides Ω into $\Omega_{1,2}$ and Ω_3, i.e. we take

$$\bar{\Omega}_{1,2}:=\bar{\Omega}_1 \cup \bar{\Omega}_2.$$

Finally, we use the notations $Q_1, Q_2, Q_3, Q_{1,2}$ and $Q_{2,3}$ to denote respectively the quadrilaterals corresponding to the subdomains $\Omega_1, \Omega_2, \Omega_3, \Omega_{1,2}$ and $\Omega_{2,3}$, i.e.

$$Q_1:=\{\Omega_1; z_1, z_2, a, d\}, \quad Q_2:=\{\Omega_2; d, a, b, c\}, \quad Q_3:=\{\Omega_3; c, b, z_3, z_4\},$$

and

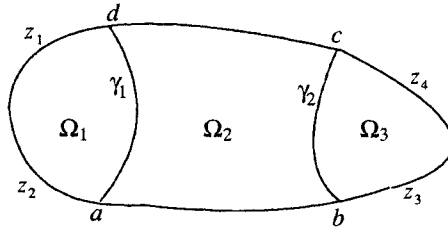$$Q_{1,2}:=\{\Omega_{1,2}; z_1, z_2, b, c\}, \quad Q_{2,3}:=\{\Omega_{2,3}; d, a, z_3, z_4\}.$$

Fig. 1

Let γ be a Jordan arc that joins the boundary lines $x=0$ and $x=1$ of the strip

$$S:=\{z=x+\mathrm{i}\,y:0<x<1\},$$

and lies entirely within S, except for its two end points. In our work we shall refer frequently to the so-called *exponential radius r* of such an arc γ. This is defined as follows (cf. [2, Sect. 1.2] and see Fig. 2):

Let γ^* be the arc obtained by translating γ along the imaginary axis until it lies on $\mathrm{Im}\,z\leq 0$, with at least one point on $\mathrm{Im}\,z=0$. Next, let Γ^* be the image of γ^* under the transformation $z\to\mathrm{e}^{\mathrm{i}\pi z}$, and let $\bar{\Gamma}^*$ denote the reflection of Γ in the real axis. Finally, let Γ denote the symmetric curve $\Gamma:=\Gamma^*\cup\bar{\Gamma}^*$ (Observe that Γ surrounds the unit circle and meets the circle in at least two points; see Fig. 2.) Then, r is defined to be the conformal radius of the interior of the curve Γ with respect to the origin 0. That is, the exponential radius of the arc γ is the radius of the disc $|w|<r$ onto which $\mathrm{Int}\,\Gamma$ can be mapped by a conformal map f such that $f(0)=0$ and $f'(0)=1$.
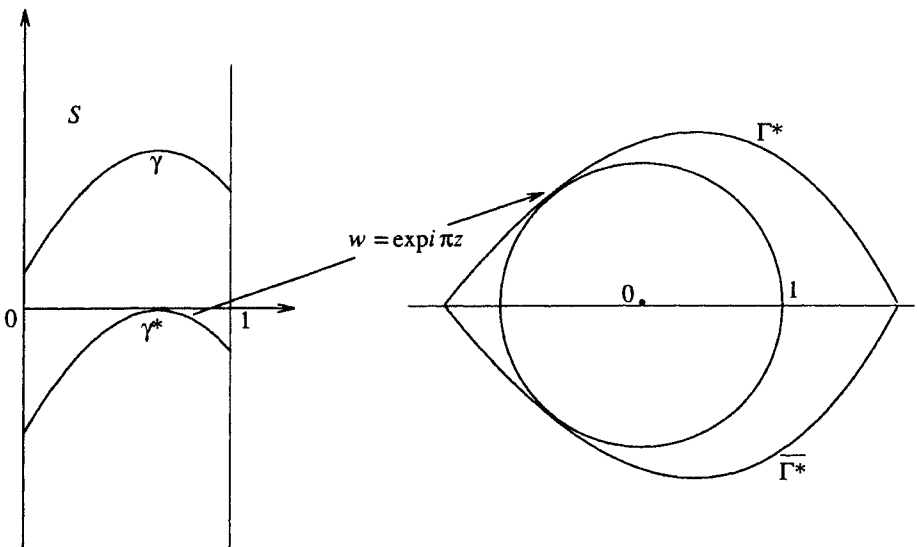

Fig. 2

In the remainder of this section we state without proof various results which are needed for our subsequent work. These include, in particular, three theorems due to Gaier and Hayman [2, 3] which play a crucial role in our work.

Consider a decomposition of the form illustrated in Fig. 3, where:

(i)   The domain Ω of the original quadrilateral $Q:=\{\Omega; z_1, z_2, z_3, z_4\}$ is bounded by two segments of the lines $x=0$ and $x=1$, and two Jordan arcs γ_1, γ_2.
(ii) The points z_1, z_2, z_3, z_4 are the four corners where the arcs γ_1, γ_2 meet the lines $x=0$ and $x=1$.
(iii) The crosscut l of subdivision is a straight line

$$l:=\{(x, y): 0\leqq x\leqq 1, y=c\},$$

parallel to the real axis, so that the component quadrilaterals are

$$Q_1:=\{\Omega_1; z_1, z_2, 1+ic, ic\} \quad \text{and} \quad Q_2:=\{\Omega_2; ic, 1+ic, z_3, z_4\}.$$

Let
$$h_1:=c-\max\{\operatorname{Im}z: z\in\gamma_1\} \quad \text{and} \quad h_2:=\min\{\operatorname{Im}z: z\in\gamma_2\}-c,$$

as illustrated in Fig. 3, and set

$$h_{12}:=h_1+h_2 \quad \text{and} \quad h:=\min(h_1, h_2).$$

Also, let r_1 and r_2 denote respectively the exponential radii of the arcs γ_1 and $\bar{\gamma}_2$. (Here, and throughout the paper, $\bar{\gamma}$ will denote the reflection of an arc γ in the real axis.)
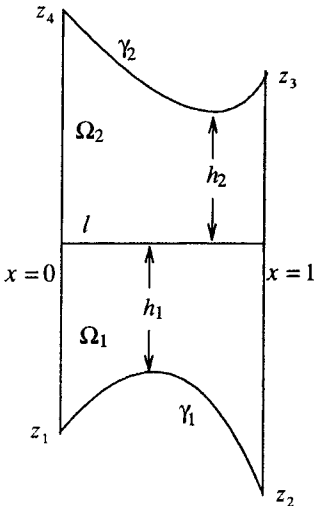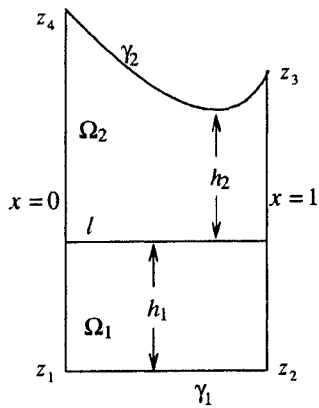

Fig. 3   Fig. 4

For the purposes of this papers, it is convenient to state Theorems 4 and 6 of [3] and Theorem 5 of [2] as follows:

Theorem 2.1 ([3, Theorem 4], [2, Theorem 1.4]). *With the notations of Fig. 3 and the above terminology we have:*
(i)

$$(2.1) \qquad \left| m(Q) - h_{12} - \frac{1}{\pi}(\log r_1 + \log r_2) \right| \leq 0.381\, e^{-\pi h_{12}},$$

provided that $h_{12} := h_1 + h_2 \geq 2$.
(ii)

$$(2.2) \qquad -\tfrac{1}{2} \times 0.381\, e^{-2\pi h_j} \leq m(Q_j) - h_j - \frac{1}{\pi}\log r_j \leq 0, \quad j = 1, 2,$$

provided that $h_j \geq 1$.

Theorem 2.2 ([3, Theorem 6], [2, Theorem 2]). *For the decomposition illustrated in Fig. 3 we have*

$$(2.3) \qquad 0 \leq m(Q) - \{m(Q_1) + m(Q_2)\} \leq 0.761\, e^{-2\pi h},$$

provided that $h := \min(h_1, h_2) \geq 1$.

Theorem 2.3 ([2, Theorem 5]). *If in Fig. 3 the arc* γ_1 *is a straight line parallel to the real axis, i.e. if*

$$\gamma_1 := \{(x, y) : 0 \leq x \leq 1, y = c_1 < c\},$$

as illustrated in Fig. 4, then

$$(2.4) \qquad 0 \leq m(Q) - \{h_1 + m(Q_2)\} \leq \tfrac{1}{2} \times 0.381\, e^{-2\pi h_2},$$

provided that $h_2 \geq 1$.

We end this section by making the following four remarks:

Remark 2.1. The bounds in (2.1)–(2.4) were deduced from the precise estimates given by Gaier and Hayman in their second paper [3]. For this reason, they are sharper than those given in Theorems 1, 4, 2 and 5 of their earlier paper [2].

Remark 2.2. For a general decomposition of the form illustrated in Fig. 1 we always have

$$(2.5) \qquad m(Q) \geq m(Q_1) + m(Q_2) + m(Q_3),$$

and equality occurs only in the special case where the crosscuts γ_1 and γ_2 are equipotentials of the function u satisfying the Laplacian problem

$$\Delta u = 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } (z_1, z_2), \quad u = 1 \quad \text{on } (z_3, z_4),$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } (z_2, z_3) \cup (z_4, z_1),$$

(see e.g. [6, p. 437]). In particular, for the special decomposition of Fig. 3,

$$(2.6) \qquad\qquad m(Q) \geqq m(Q_1) + m(Q_2),$$

and equality occurs only in the two trivial cases where: (a) Ω is a rectangle, and (b) γ_2 is the reflection of γ_1 in the crosscut l. In addition, for the quadrilaterals Q, Q_1 and Q_2 of Fig. 3 we have

$$(2.7) \qquad\qquad m(Q) - 1 \leqq h_{12} := h_1 + h_2 \leqq m(Q)$$

and

$$(2.8) \qquad\qquad m(Q_j) - \tfrac{1}{2} \leqq h_j \leqq m(Q_j), \qquad j = 1, 2,$$

(cf. [5, p. 36]). Finally, for the quadrilaterals Q and Q_2 of Fig. 4 the composition law (2.6) implies that

$$(2.9) \qquad\qquad m(Q) - (h_1 + h_2) \geqq m(Q_2) - h_2.$$

Remark 2.3. With reference to the quadrilaterals of Figs. 3 and 4, estimates for the errors in the DDM approximations to $m(Q)$ (similar to those of Theorems 2.2 and 2.3), and also for various other errors associated with the DDM approximation to the full conformal map $R_{m(Q)} \to \Omega$, were first given by us in [9], without imposing any restrictions on the sizes of h and h_2. However, the estimates given in [9] were derived by imposing certain additional assumptions on the arcs γ_1, γ_2, and one of these is rather restrictive. (It requires that the slopes of the two arcs are numerically less than 1.)

Remark 2.4. Let T_l denote the quadrilateral illustrated in Fig. 5. That is, $T_l := \{\Omega; z_1, z_2, z_3, z_4\}$ where: (a) Ω is the trapezium bounded by the real and imaginary axes and the lines $x = 1$ and $y = x + l - 1$, with $l > 1$, and (b) the points z_1, z_2, z_3, z_4 are the corners of Ω.

We note the following, for future reference:

(i) The exact value of $m(T_l)$ is known exactly, for any $l > 1$, in terms of elliptic integrals (cf. [1, p. 104]). For example, the exact values of $m(T_2)$, $m(T_3)$, $m(T_{3.5})$ and $m(T_4)$ are given to twelve decimal places by

$$(2.10a) \qquad m(T_2) = 1.279\,261\,571\,171, \qquad m(T_3) = 2.279\,364\,207\,968,$$

$$(2.10b) \qquad m(T_{3.5}) = 2.779\,364\,391\,556, \qquad m(T_4) = 3.279\,364\,399\,489.$$

(ii) For any $c > 0$, Theorem 2.3 gives

$$0 \leqq m(T_{l+c}) - \{m(T_l) + c\} \leqq \tfrac{1}{2} \times 0.381\,e^{-2\pi(l-1)},$$

provided that $l \geqq 2$. This means, in particular, that

and

$$0 \leqq m(T_{3+c}) - \{m(T_3) + c\} \leqq 6.65 \times 10^{-7}$$

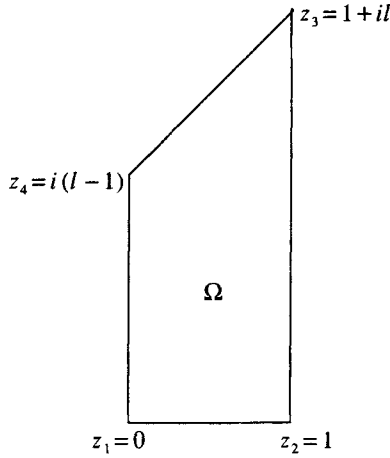$$0 \leqq m(T_{4+c}) - \{m(T_4) + c\} \leqq 1.25 \times 10^{-9}.$$

Fig. 5

It follows that, for any $c > 0$, $m(T_{3+c})$ can be computed correctly to at least five decimal places (from the value of $m(T_3)$ given in (2.10a)), by means of

(2.11) $$m(T_{3+c}) \approx m(T_3) + c.$$

Similarly, $m(T_{4+c})$ can be computed correctly to at least eight decimal places (from the value of $m(T_4)$ given in (2.10b)) by means of

(2.12) $$m(T_{4+c}) \approx m(T_4) + c$$

(see also [8, p. 78–81] and [10, Ex. 3.1]).

3 Two theorems for general domain decomposition

In this section we prove two theorems concerning general domain decomposition of the form illustrated in Fig. 6a. As will become apparent, these two theorems are simple consequences of the results of Gaier and Hayman stated in Sect. 2.
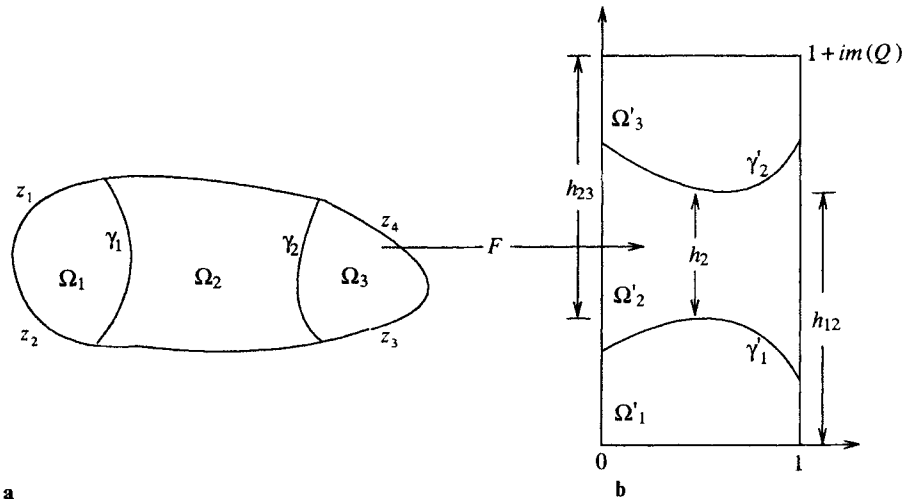

Fig. 6a, b

Theorem 3.1. *With the notations of Fig. 6a and the terminology introduced in Sect. 2, we have*

$$(3.1) \qquad |m(Q)-\{m(Q_{1,2})+m(Q_{2,3})-m(Q_2)\}| \leqq 8.82\,e^{-\pi m(Q_2)},$$

provided that $m(Q_2) \geqq 3$.

Proof. The quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ is conformally equivalent to the rectangular quadrilateral

$$Q' := \{R_{m(Q)}; 0, 1, 1+i\,m(Q), i\,m(Q)\}.$$

Let F be the corresponding conformal map of Q onto Q', and let γ_1', γ_2' be respectively the images of the crosscuts γ_1, γ_2 under this map. Then, γ_1' and γ_2' define the decomposition of Q' illustrated in Fig. 6b, where

$$(3.2) \qquad m(Q_j')=m(Q_j),\quad j=1,2,3,\quad m(Q_{1,2}')=m(Q_{1,2})\quad\text{and}\quad m(Q_{2,3}')=m(Q_{2,3}),$$

because the modules are conformally invariant.

Let

$$h_2 := \min\{\operatorname{Im} z': z'\in\gamma_2'\} - \max\{\operatorname{Im} z': z'\in\gamma_1'\},$$

$$h_{12} := \min\{\operatorname{Im} z': z'\in\gamma_2'\},\qquad h_{23} := m(Q)-\max\{\operatorname{Im} z': z'\in\gamma_1'\},$$

and observe that

$$(3.3) \qquad h_{12}+h_{23}-h_2=m(Q).$$

Observe also that the assumption $m(Q_2)\geqq 3$ implies the following:

$$(3.4) \qquad h_2 \geqq m(Q_2)-1 \geqq 2,$$

$$(3.5) \qquad h_{12} \geqq m(Q_{1,2})-\tfrac{1}{2} \geqq m(Q_2)-\tfrac{1}{2} \geqq \tfrac{5}{2},$$

$$(3.6) \qquad h_{23} \geqq m(Q_{2,3})-\tfrac{1}{2} \geqq m(Q_2)-\tfrac{1}{2} \geqq \tfrac{5}{2}$$

(see Remark 2.2).

Next, apply the results of Theorem 2.1 to each of the quadrilaterals Q_2', $Q_{1,2}'$ and $Q_{2,3}'$. This, in conjunction with (3.2), gives

$$\left| m(Q_2)-h_2-\frac{1}{\pi}(\log r_1 + \log r_2)\right| \leqq 0.381\,e^{-\pi h_2},$$

$$-\tfrac{1}{2}\times 0.381\,e^{-2\pi h_{12}} \leqq m(Q_{1,2})-h_{12}-\frac{1}{\pi}\log r_2 \leqq 0,$$

$$-\tfrac{1}{2}\times 0.381\,e^{-2\pi h_{23}} \leqq m(Q_{2,3})-h_{23}-\frac{1}{\pi}\log r_1 \leqq 0,$$

where r_1 and r_2 are respectively the exponential radii of the arcs $\overline{\gamma_1'}$ and $\overline{\gamma_2'}$. It follows that

$$-E_1 \leqq m(Q)-\{m(Q_{1,2})+m(Q_{2,3})-m(Q_2)\} \leqq E_2,$$

where, by using (3.4)–(3.6),

$$E_1 := 0.381\, e^{-\pi h_2} \leqq 0.381\, e^{\pi}\, e^{-\pi m(Q_2)} \leqq 8.82\, e^{-\pi m(Q_2)},$$

and

$$E_2 := 0.381\, (e^{-\pi h_2} + \tfrac{1}{2}\, e^{-2\pi h_{12}} + \tfrac{1}{2}\, e^{-2\pi h_{23}}) \leqq 8.82\, e^{-\pi m(Q_2)}.$$

This completes the proof of the theorem.   □

Theorem 3.2. *Consider the decomposition illustrated in Fig. 6a and, with the terminology used in the proof of Theorem 3.1, assume that the image of the crosscut γ_1 under the conformal map $F: Q \to Q'$ is a straight line parallel to the real axis (see Fig. 7). Then,*

$$(3.7) \qquad -4.41\, e^{-2\pi m(Q_2)} \leqq m(Q) - \{m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2)\} \leqq 0,$$

provided that $m(Q_2) \geqq 1.5$.

Proof. The assumption $m(Q_2) \geqq 1.5$ implies that

$$(3.8) \qquad\qquad\qquad h_2 \geqq m(Q_2) - \tfrac{1}{2} \geqq 1$$

(see (2.8)). Therefore, for the quadrilateral $Q'_{1,2}$ of Fig. 7, Theorem 2.3 gives

$$(3.9\,a) \qquad\qquad 0 \leqq m(Q_{1,2}) - \{h_1 + m(Q_2)\} \leqq E,$$

where, by using (3.8),

$$(3.9\,b) \qquad E := \tfrac{1}{2} \times 0.381\, e^{-2\pi h_2} \leqq \tfrac{1}{2} \times 0.381\, e^{\pi}\, e^{-2\pi m(Q_2)} \leqq 4.41\, e^{-2\pi m(Q_2)}.$$

The desired result follows at once from (3.9), because

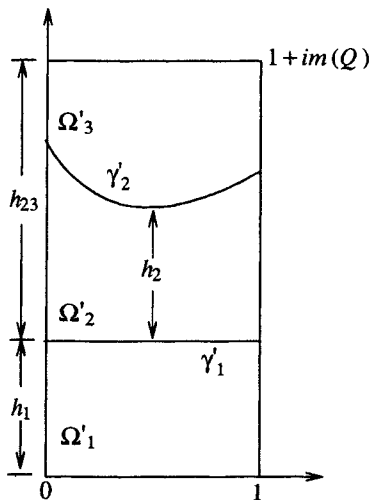$$h_1 = m(Q) - h_{23} = m(Q) - m(Q_{23}).   \square$$

Fig. 7

4 Applications

4.1. Let Q_1 and Q_2 be the two component quadrilaterals of the decomposition illustrated in Fig. 8a, and assume that the reflection $\hat{\gamma}_1$ of the arc γ_1 in the crosscut

$$l := \{(x, y): 0 \leq x \leq 1, y = x\}$$

lies above the arc γ_2. Then,

(4.1) $$0 \leq m(Q) - \{m(Q_1) + m(Q_2)\} \leq 4.41 \, e^{-2\pi m(Q_2)},$$

provided that $m(Q_2) \geq 1.5$. This can be shown as follows:

Reflect Ω_1 in l, and consider the decomposition of the resulting quadrilateral $\hat{Q}$ illustrated in Fig. 8b. (That is the decomposition of $\hat{Q}$ is defined by the two crosscuts l and γ_2.) Then, because of the symmetry,

(4.2) $$m(Q_{2,3}) = m(Q_1), \qquad m(\hat{Q}) = 2 \, m(Q_1),$$

and the image of the crosscut l under the associated conformal map

$$F: \hat{Q} \to \{R_{m(\hat{Q})}; 0, 1, 1 + i\,m(\hat{Q}), i\,m(\hat{Q})\}$$

will be the straight line

$$\{(\xi, \eta): 0 \leq \xi \leq 1, \eta = m(Q_1)\}.$$

Therefore, Theorem 3.2 is applicable to the decomposition of Fig. 8b, and its application gives

(4.3) $$-4.41 \, e^{-2\pi m(Q_2)} \leq m(\hat{Q}) - \{m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2)\} \leq 0.$$

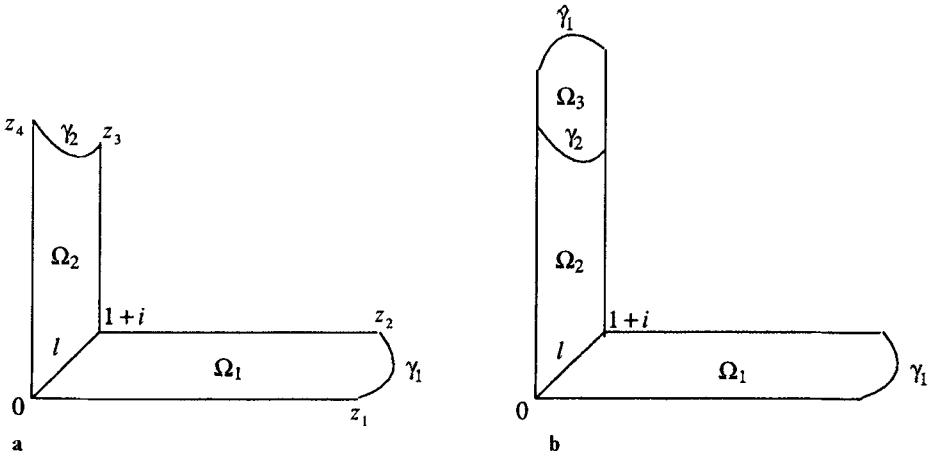Since $m(Q_{1,2}) = m(Q)$, the desired result follows at once from (4.2) and (4.3).

Fig. 8a, b

4.2. Let Q_1, Q_2 and Q_3 be the three component quadrilaterals of the decomposition illustrated in Fig. 9, let $\hat{\Omega}_1$ and $\hat{\Omega}_3$ denote the reflections of Ω_1 and Ω_3 in the crosscuts l_1 and l_2, and assume that $\Omega_2 \subset \hat{\Omega}_1$ and $\Omega_2 \subset \hat{\Omega}_3$. Then,

$$(4.4) \qquad 0 \leqq m(Q) - \{m(Q_1) + m(Q_2) + m(Q_3)\} \leqq 8.821\, e^{-\pi m(Q_2)},$$

provided that $m(Q_2) \geqq 3$. This can be shown as follows:

The application of Theorem 3.1 gives

$$(4.5) \qquad |m(Q) - \{m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2)\}| \leqq 8.82\, e^{-\pi m(Q_2)}.$$

Further, the application of estimate (4.1) to each of the quadrilaterals $Q_{1,2}$ and $Q_{2,3}$ gives

$$(4.6) \qquad 0 \leqq m(Q_{1,2}) - \{m(Q_1) + m(Q_2)\} \leqq 4.41\, e^{-2\pi m(Q_2)}$$

and

$$(4.7) \qquad 0 \leqq m(Q_{2,3}) - \{m(Q_2) + m(Q_3)\} \leqq 4.41\, e^{-2\pi m(Q_2)}.$$

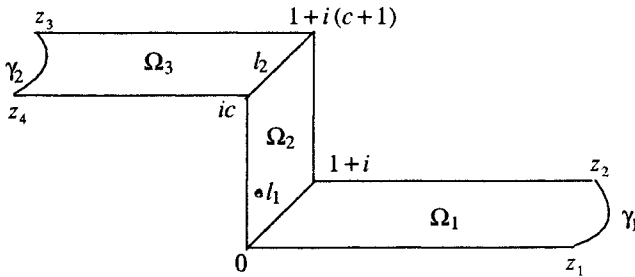The desired result then follows at once from (4.5)–(4.7) and the composition law (2.5).

Fig. 9

4.3. Consider the decomposition illustrated in Fig. 10, let $\hat{\Omega}_1$ and $\hat{\Omega}_3$ denote respectively the reflections of Ω_1 and Ω_3 in the crosscuts l_1 and l_2, and assume that $\Omega_2 \subset \hat{\Omega}_1$ and $\Omega_2 \subset \hat{\Omega}_3$. Then, an argument similar to that used in Sect. 4.2 above gives

$$(4.8) \qquad 0 \leqq m(Q) - \{m(Q_1) + m(Q_2) + m(Q_3)\} \leqq 8.821\, e^{-\pi m(Q_2)},$$
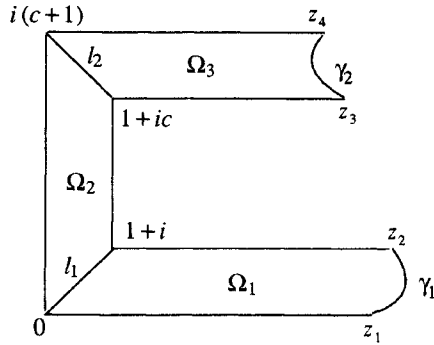
provided that $m(Q_2) \geqq 3$.

Fig. 10

4.4. Consider the decomposition of the quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ illustrated in Fig. 11 a, where:

(i) The domain Ω is the union of a rectangle

(4.9) $\{(x, y): 0 < x < 2c, 0 < y < 1\}$

and some other simply-connected domain.
(ii) The decomposition is defined by the straight line crosscut

(4.10) $l := \{(x, y): x = c, 0 \leq y \leq 1\},$

i.e. $m(Q_2) = c$.
   Then,

(4.11) $0 \leq m(Q) - \{c + m(Q_1)\} \leq 4.41 \, e^{-2\pi c},$
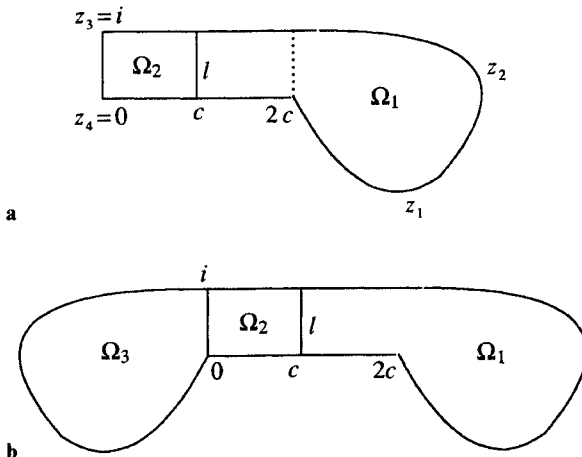
provided that $c \geq 1.5$.

a

b

Fig. 11 a, b

The estimate (4.11) can be derived, in a manner similar to that used in Sect. 4.1, by reflecting the subdomain Ω_1 in the crosscut l and considering the decomposition of the resulting quadrilateral $\hat{Q}$ illustrated in Fig. 11b. Then, because l is a line of symmetry, the application of Theorem 3.2 to this decomposition of $\hat{Q}$ gives

$$(4.12) \qquad -4.41\,e^{-2\pi m(Q_2)} \leq m(\hat{Q}) - \{m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2)\} \leq 0.$$

The desired result follows at once from (4.12), because

$$m(\hat{Q}) = 2\,m(Q_1), \quad m(Q_2) = c, \quad m(Q_{1,2}) = m(Q) \quad \text{and} \quad m(Q_{2,3}) = m(Q_1).$$

4.5. Let Q_1 and Q_2 be the component quadrilaterals of the decomposition illustrated in Fig. 12a, where again the domain Ω consists in part by a rectangle of the form (4.9), and the crosscut is the straight line (4.10). Then,

$$(4.13) \qquad 0 \leq m(Q) - \{m(Q_1) + m(Q_2)\} \leq 17.64\,e^{-2\pi c},$$
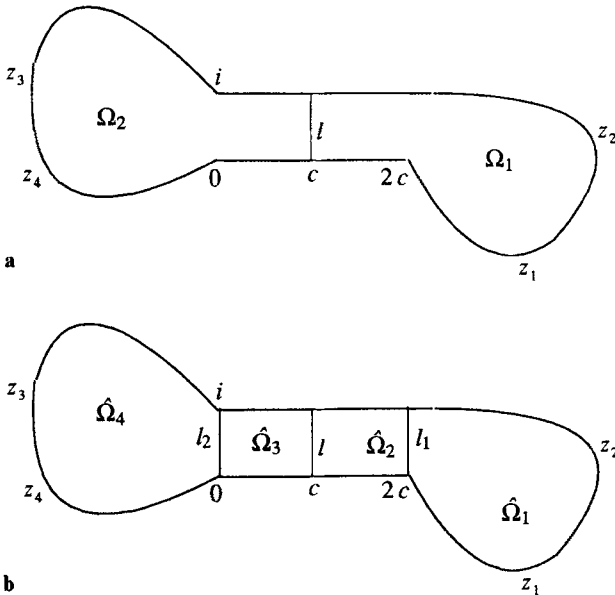
provided that $c \geq 1.5$.

Fig. 12a, b

For the proof of (4.13), we introduce two additional crosscuts by means of the straight lines

$$l_1 := \{(x, y): x = 2c, 0 \leq y \leq 1\} \quad \text{and} \quad l_2 := \{(x, y): x = 0, 0 \leq y \leq 1\},$$

and label the subdomains of the resulting decomposition $\hat{\Omega}_1$, $\hat{\Omega}_2$, $\hat{\Omega}_3$, $\hat{\Omega}_4$, as illustrated in Fig. 12b. Then, the application of Theorem 3.1 to the decomposition of Q defined by the two crosscuts l_1 and l_2 gives

$$(4.14) \qquad |m(Q) - \{m(\hat{Q}_{1,2,3}) + m(\hat{Q}_{2,3,4}) - 2c\}| \leq 8.82 \, e^{-2\pi c},$$

because $m(\hat{Q}_{2,3}) = 2c$. In addition, the application of (4.11) to each of the quadrilaterals $\hat{Q}_{1,2,3}$ and $\hat{Q}_{2,3,4}$ gives

$$(4.15) \qquad 0 \leq m(\hat{Q}_{1,2,3}) - \{c + m(\hat{Q}_{1,2})\} \leq 4.41 \, e^{-2\pi c}$$

and

$$(4.16) \qquad 0 \leq m(\hat{Q}_{2,3,4}) - \{c + m(\hat{Q}_{3,4})\} \leq 4.41 \, e^{-2\pi c}.$$

The desired result follows from (4.14)–(4.16), by observing that $m(\hat{Q}_{1,2}) = m(Q_1)$ and $m(\hat{Q}_{3,4}) = m(Q_2)$.

5 Numerical examples

Example 5.1. Let $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ be the quadrilateral illustrated in Fig. 13, where the two boundary arcs γ_1, γ_2 have cartesian equations

$$x = 0.25 y^4 - 0.5 y^2 + 7, \qquad 0 \leq y \leq 1,$$

$$x = 7.5 + 0.25 \cos 2\pi(6 - y), \qquad 6 \leq y \leq 7,$$
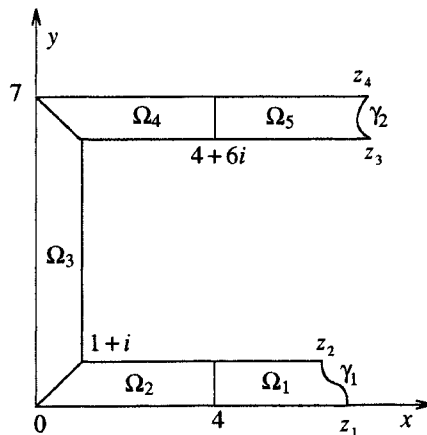
respectively.

Fig. 13

We consider approximating $m(Q)$ by

$$\tilde{m}(Q) := \sum_{j=1}^{5} m(Q_j),$$

and note the following in connection with the determination of the modules of the component quadrilaterals $Q_j, j = 1, \ldots, 5$:

(i) The modules of quadrilaterals of the form illustrated in Fig. 3 can be determined easily by means of the Garrick iterative algorithm given in [4]. In addition, such quadrilaterals admit further decompositions to which the estimates (2.3) and (2.4) of Gaier and Hayman are applicable. For the component quadrilaterals Q_1 and Q_5 under consideration the algorithm of [4] gives the values

$$m(Q_1) = 2.859\,569\,035 \quad \text{and} \quad m(Q_5) = 3.364\,089\,632,$$

and these are expected to be correct to all the figures quoted (cf. [10, Ex. 3.2]).
(ii) As was indicated in Sect. 2, the values of $m(Q_2)$, $m(Q_3)$ and $m(Q_4)$ are known exactly in terms of elliptic integrals. More specifically, if T_i denotes the quadrilateral of Fig. 5, then these three modules are given correct to nine decimal places by

$$m(Q_2) = m(Q_4) = m(T_4) = 3.279\,364\,399$$

and

$$m(Q_3) = 2\,m(T_{3,5}) = 5.558\,728\,783$$

(cf. Remark 2.4).
   Thus, $\tilde{m}(Q)$ is given by

$$\tilde{m}(Q) = 18.341\,116\,249.$$

For the error in the above approximation, we first note that estimate (4.8) implies

$$0 \leqq m(Q) - \{m(Q_{1,2}) + m(Q_3) + m(Q_{4,5})\} \leqq 8.821\,\mathrm{e}^{-\pi m(Q_3)},$$

(cf. Sect. 4.3). In addition, Theorem 2.2 implies

$$0 \leqq m(Q_{1,2}) - \{m(Q_1) + m(Q_2)\} \leqq 0.761\,\mathrm{e}^{-2\pi h_1},$$

and

$$0 \leqq m(Q_{4,5}) - \{m(Q_4) + m(Q_5)\} \leqq 0.761\,\mathrm{e}^{-2\pi h_4},$$

where $h_1 = 2.75$ and $h_4 = 3$. Therefore,

$$0 \leqq m(Q) - \sum_{j=1}^{5} m(Q_j) \leqq E,$$

where

$$E := 8.821\,\mathrm{e}^{-\pi m(Q_3)} + 0.761\,(\mathrm{e}^{-2\pi h_1} + \mathrm{e}^{-2\pi h_4}) \leqq 2.6 \times 10^{-7},$$

i.e.

$$0 \leqq m(Q) - \tilde{m}(Q) \leqq 2.6 \times 10^{-7}.$$

Thus,

$$18.341\,116\,24 \leqq m(Q) \leqq 18.341\,116\,51.$$

Example 5.2. Let $Q:=\{\Omega; z_1, z_2, z_3, z_4\}$ be the quadrilateral illustrated in Fig. 14, where the boundary arcs γ_1, γ_2, have cartesian equations

$$x = y(y-1)+6.5, \qquad\qquad 0\leq y\leq 1,$$

$$x = -7.5-0.2\,\mathrm{sech}^2\,2.5\,y, \qquad 0\leq y\leq 1,$$
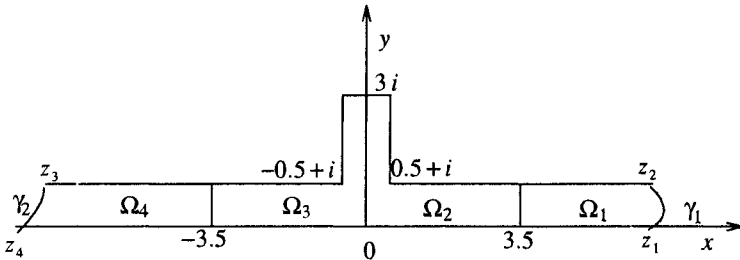
respectively.

Fig. 14

Here we approximate $m(Q)$ by

$$\tilde{m}(Q):=\sum_{j=1}^{4} m(Q_j),$$

where:

(i) The modules of Q_1 and Q_4 are determined by using the Garrick algorithm of [4]. The computed values are

$$m(Q_1) = 3.12181333 \quad \text{and} \quad m(Q_4) = 4.06551571,$$

and these are expected to be correct to all the figures quoted (cf. [10, Ex. 3.3]).
(ii) The module of Q_2 (which, by symmetry, is equal to that of Q_3) is determined by using the subroutine RESIST of the Schwarz-Christoffel package SCPACK of Terefethen [11]. This gives the value

$$m(Q_2) = 3.42344546 = m(Q_3),$$

which is also expected to be correct to all the figures quoted.
    Thus, $\tilde{m}(Q)$ is given by

$$\tilde{m}(Q):=m(Q_1)+2\,m(Q_2)+m(Q_4) = 14.03421996.$$

For the error in the above DDM approximation, we first apply estimate (4.13), of Sect. 4.5, to the decomposition of Q defined by the single crosscut at $x=3.5$. This gives

(5.1) $0\leq m(Q)-\{m(Q_1)+m(Q_{2,3,4})\}\leq 17.64\mathrm{e}^{-6\pi}.$

The same estimate, when applied to the decomposition of $Q_{2,3,4}$ defined by the crosscut at $x = -3.5$, gives

$$(5.2) \qquad 0 \leqq m(Q_{2,3,4}) - \{m(Q_{2,3}) + m(Q_4)\} \leqq 17.64\,e^{-6\pi},$$

where, by symmetry,

$$(5.3) \qquad m(Q_{2,3}) = m(Q_2) + m(Q_3) = 2\,m(Q_2).$$

Therefore, from (5.1)–(5.3), we have that

$$0 \leqq m(Q) - \tilde{m}(Q) \leqq 35.28\,e^{-6\pi} \leqq 2.3 \times 10^{-7},$$

i.e.

$$14.03421996 \leqq m(Q) \leqq 14.03422019.$$

Example 5.3. Let $Q := \{\Omega;\, z_1,\, z_2,\, z_3,\, z_4\}$ be the quadrilateral illustrated in Fig. 15, where the width of each strip of the spiral Ω is 1, and the lengths of the "outer" segments of $\partial\Omega$ (in clockwise order, starting from the right hand side) are 18, 19, 18, 16, 15, 13, 12, 10, 9, 7, 6, 4 and 3.
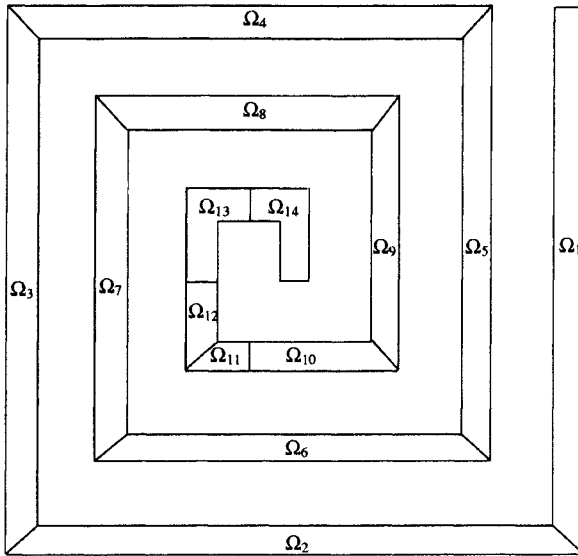

Fig. 15

This is one of the examples considered by Howell and Trefethen [7], in connection with the use of a modified Schwarz-Christoffel technique for the conformal mapping of elongated polygonal regions. Here we consider approximating $m(Q)$ by

$$(5.4) \qquad \tilde{m}(Q) := \sum_{j=1}^{9} m(Q_j) + m(Q_{10,11}) + \sum_{j=12}^{14} m(Q_j),$$

and note the following concerning the decomposition used and the determination of the modules of the corresponding component quadrilaterals:

(i)   The crosscut that subdivides $Q_{10,11}$ into Q_{10} and Q_{11} is needed only for estimating the error in the DDM approximation (5.4).

(ii) Let T_i be the quadrilateral of Fig. 5. Then,

(5.5a) $m(Q_1)=m(T_{18})$, $m(Q_2)=2m(T_{9.5})$, $m(Q_3)=2m(T_9)$,

(5.5b) $m(Q_4)=2m(T_8)$, $m(Q_5)=2m(T_{7.5})$, $m(Q_6)=2m(T_{6.5})$,

(5.5c) $m(Q_7)=2m(T_6)$, $m(Q_8)=2m(T_5)$, $m(Q_9)=2m(T_{4.5})$,

(5.5d) $m(Q_{10,11})=2m(T_{3.5})$ and $m(Q_{12})=m(T_3)$.

This means that all the above modules can be written down, correct to at least eight decimal places, by using the values of $m(T_3)$, $m(T_{3.5})$ and $m(T_4)$, given in (2.10), in conjunction with the formula

$$m(T_{4+c}) \approx m(T_4)+c, \qquad c>0,$$

(cf. Remark 2.4).

(iii) By symmetry

(5.6a) $m(Q_{11,12})=m(Q_{13})=m(Q_{14})$

and

(5.6b) $m(Q_{11,\ldots,14})=3m(Q_{13})$.

(iv) For the computation of $m(Q_{13})$ we use (as in Example 5.2) the subroutine RESIST of [11]. This gives the value

$$m(Q_{13})=3.558\,625\,812,$$

which is expected to be correct to all the figures quoted.

    Thus, by determining the modules involved in (5.4), as indicated in (ii) and (iv) above, we find that

(5.7) $\tilde{m}(Q)=132.704\,539\,41$.

This should be compared with the approximation

(5.8) $m(Q) \approx 132.704\,54$,

obtained by Howell and Trefethen [7, p. 943].

    The error in $\tilde{m}(Q)$ can be estimated as follows:

    Theorem 3.1 implies that

$$|m(Q)-\{m(Q_{1,2})+m(Q_{2,\ldots,14})-m(Q_2)\}| \leqq 8.82\,e^{-\pi m(Q_2)}.$$

Also, the application of estimate (4.1) to $Q_{1,2}$ gives

$$0 \leqq m(Q_{1,2})-\{m(Q_1)+m(Q_2)\} \leqq 4.41\,e^{-2\pi m(Q_1)}.$$

Hence, by recalling (2.5),

(5.9) $0 \leqq m(Q) - \{m(Q_1) + m(Q_{2,\ldots,14})\} \leqq 8.82\,e^{-\pi m(Q_2)} + 4.41\,e^{-2\pi m(Q_1)}.$

Similarly, by combining (5.9) with the results obtained by applying respectively Theorem 3.1 and estimate (4.1) to $Q_{2,\ldots,14}$ and $Q_{2,3}$; $Q_{3,\ldots,14}$ and $Q_{3,4}$; $\ldots$; $\ldots$; $Q_{8,\ldots,14}$ and $Q_{8,9}$; $Q_{9,\ldots,14}$ and $Q_{9,10,11}$, we find that

(5.10a) $0 \leqq m(Q) - \left\{ \sum_{j=1}^{9} m(Q_j) + m(Q_{10,\ldots,14}) \right\} \leqq E_1,$

where

(5.10b) $E_1 := 8.82 \left\{ \sum_{j=2}^{9} e^{-\pi m(Q_j)} + e^{-\pi m(Q_{10,11})} \right\}$

$+ 4.41 \left\{ e^{-2\pi m(Q_1)} + \sum_{j=3}^{9} e^{-2\pi m(Q_j)} + e^{-2\pi m(Q_{10,11})} \right\}.$

Next, by applying Theorem 3.1 to $Q_{10,\ldots,14}$ and estimate (4.1) to $Q_{10,11,12}$, we obtain

$0 \leqq m(Q_{10,\ldots,14}) - \{m(Q_{10,11,12}) + m(Q_{11,\ldots,14}) - m(Q_{11,12})\} \leqq 8.82\,e^{-\pi m(Q_{11,12})},$

and

$0 \leqq m(Q_{10,11,12}) - \{m(Q_{10,11}) + m(Q_{12})\} \leqq 4.41\,e^{-2\pi m(Q_{12})}.$

These two estimates, in conjunction with the symmetry relations (5.6), then yield

(5.11a) $0 \leqq m(Q_{10,\ldots,14}) - \left\{ m(Q_{10,11}) + \sum_{j=12}^{14} m(Q_j) \right\} \leqq E_2,$

where

(5.11b) $E_2 := 8.82\,e^{-\pi m(Q_{13})} + 4.41\,e^{-2\pi m(Q_{12})}.$

Finally, by combining (5.10) and (5.11), we obtain

(5.12a) $0 \leqq m(Q) - \tilde{m}(Q) \leqq E,$

where

(5.12b) $E := E_1 + E_2 \leqq 1.26 \times 10^{-4}.$

Therefore, from (5.7) and (5.12),

$$132.704\,539 \leqq m(Q) \leqq 132.704\,666.$$

We end by considering the alternative decomposition of Q illustrated in Fig. 16. In this, the component quadrilaterals Q_j, $j = 1, 2, \ldots, 9$, are the same as those in Fig. 15, i.e. $m(Q_j)$, $j = 1, 2, \ldots, 9$, are given by (5.5a)–(5.5c). In addition,

$$m(Q_{10}) = 2 m(T_{3.5}), \quad m(Q_{11}) = 2 m(T_3), \quad m(Q_{12}) = 2 m(T_2) \text{ and } m(Q_{13}) = m(T_3).$$

Thus, the corresponding DDM approximation $\hat{m}(Q)$ can be calculated trivially (as indicated in (ii) above) from the four values of $m(T_2)$, $m(T_3)$, $m(T_{3.5})$ and $m(T_4)$ given in (2.10). The resulting value

$$\hat{m}(Q) := 132.704\,539\,35$$

agrees to ten significant figures with the DDM approximation (5.7), and to eight significant figures with the approximation (5.8) of Howell and Trefethen [7]. Unfortunately, however, our method of estimating the DDM errors cannot be applied to the decomposition of Fig. 16. More specifically, this method cannot be used to estimate the error in $\hat{m}(Q)$, because $m(Q_{12}) < 3$ and, as a result, Theorem 3.1 cannot be used to estimate

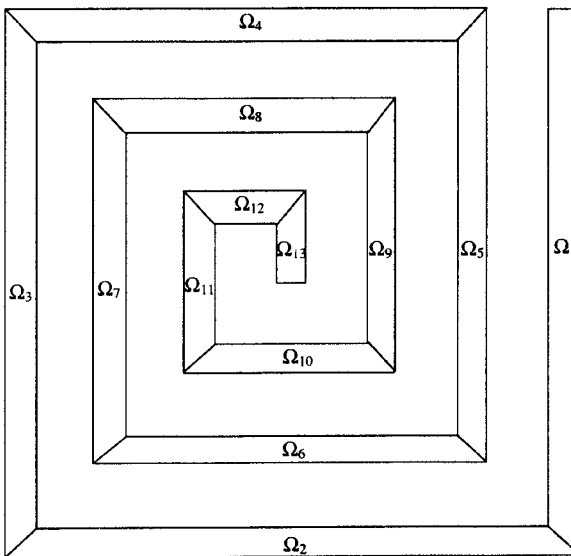$$m(Q_{11,12,13}) - \{m(Q_{11,12}) + m(Q_{12,13}) - m(Q_{12})\}.$$

Fig. 16

References

1. Bowman, F. (1953): Introduction to Elliptic Functions. English University Press, London
2. Gaier, D., Hayman, W.K. (1991): On the computation of modules of long quadrilaterals. Constr. Approx. 7, 453–467
3. Gaier, D., Hayman, W.K. (1990): Modules of long quadrilaterals and thick ring domains. Rendiconti di Matematica Roma 10, 809–834

4. Gaier, D., Papamichael, N. (1987): On the comparison of two numerical methods for conformal mapping. IMA J. Numer. Anal. **7**, 261–282
5. Hayman, W.K. (1948): Remarks on Ahlfors' distortion theorem. Quart. Appl. Math. Oxford Ser. **19**, 33–53
6. Henrici, P. (1986): Applied and Computational Complex Analysis, Vol. III. Wiley, New York
7. Howell, I.H., Trefethen, L.N. (1990): A modified Schwarz-Christoffel transformation for elongated regions. SIAM J. Sci. Statist. Comput. **11**, 928–949
8. Papamichael, N. (1989): Numerical conformal mapping onto a rectangle with applications to the solution of Laplacian problems. J. Comput. Appl. Math. **28**, 63–83
9. Papamichael, N., Stylianopoulos, N.S. (1991): A domain decomposition method for conformal mapping onto a rectangle. Constr. Approx. **7**, 349–379
10. Papamichael, N., Stylianopoulos, N.S. (1990): On the numerical performance of a domain decomposition method for conformal mapping, pp. 155–169. In: S. Ruscheweyh, E.B. Saff, L.C. Salinas, R.S. Varga, eds., Computational Methods and Function Theory. Lecture Notes in Maths. **1435**. Springer, Berlin Heidelberg New York
11. Trefethen, L.N. (1989): SCPACK User's Guide. Numerical Analysis Report 89-2, Dept. of Math., Massachusetts Institute of Technology, Cambridge, MA