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Journal of Computational and Applied Mathematics 50 (1994) 33–50

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

On the theory and application of a domain decomposition method for computing conformal modules

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Received 17 September 1992

Abstract

We consider the theory and application of a domain decomposition method for computing the conformal modules of long quadrilaterals. The method has been studied already by us and also by Gaier and Hayman. Our main purpose here is to extend its area of application and, in the same time, improve some of our earlier error estimates.

Key words: Conformal mapping; Quadrilaterals; Conformal modules; Domain decomposition

1. Introduction

Let $Q := \{\Omega;\, z_1, z_2, z_3, z_4\}$ denote a quadrilateral consisting of a Jordan domain Ω and four specified points z_1, z_2, z_3, z_4 on $\partial\Omega$. The conformal module $m(Q)$ of Q is defined as follows. Let R_h denote a rectangle of the form

$$R_h := \{(\xi, \eta) \colon 0 < \xi < 1,\ 0 < \eta < h\}.$$

Then $m(Q)$ is the unique value of h for which Q is conformally equivalent to the rectangular quadrilateral

$$\{R_h;\, 0, 1, 1 + \mathrm{i}h, \mathrm{i}h\}.$$

By this we mean that for $h = m(Q)$ and for this value only there exists a unique conformal map $F \colon \Omega \to R_h$, which takes the four points z_1, z_2, z_3, z_4 respectively onto the four vertices 0, 1, $1 + \mathrm{i}h$, $\mathrm{i}h$ of R_h.

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[1] The work of this author was supported by NATO Collaborative Research Grant GRG 910078.

SSDI 0377-0427(93)E0162-F

We note the following in connection with the above.

- The conformal map F has many applications in, for example, integrated-circuit design and steady-state diffusion, and in these the value of $m(Q)$ is often of special significance. In fact, in many of these applications only the value of $m(Q)$ (rather than the full conformal map) is of interest (see, e.g., the list of references given in [7, pp. 65, 66]).
- The conformal map $F : \Omega \to R_{m(Q)}$ can be expressed as

$$F = S \circ f, \tag{1.1}$$

where f is a conformal map of Ω onto the unit disc $D := \{\zeta: |\zeta| < 1\}$ and $S : D \to R_{m(Q)}$ is a simple Schwarz–Christoffel transformation that can be written down in terms of an inverse elliptic sine. In addition, the conformal module $m(Q)$ can be determined, quite simply, from the ratio of two complete elliptic integrals of the first kind whose moduli depend only on the images $\zeta_j := f(z_j)$, $j = 1, 2, 3, 4$, of the four boundary points z_j, $j = 1, 2, 3, 4$ (see, e.g., [8, Section 2]). The above approach, of using the unit disc (or, equivalently, the half plane) as an intermediate mapping domain, may be regarded as the conventional method for determining F and $m(Q)$. This is so because (a) D is the standard canonical domain for the mapping of simply-connected regions; (b) the problem of approximating $f : \Omega \to D$ is, by far, the most extensively studied numerical conformal mapping problem. Unfortunately, however, the use of (1.1) suffers from a well-known numerical drawback which is caused by the fact that if Q is "long" (and consequently $m(Q)$ is "large"), then either the two points ζ_1 and ζ_2, or the two points ζ_3 and ζ_4 (or indeed both pairs of points) are very close to each other. This crowding of points may be regarded as a form of ill-conditioning, in the sense that a numerical procedure based on the use of (1.1) may fail to provide a meaningful approximation to F or $m(Q)$, even if an accurate approximation to f is available. In particular, the process will break down completely if, due to the crowding, the computer fails to recognize the four points ζ_j, $j = 1, 2, 3, 4$, in the correct order on the unit circle. To be more precise, if ϕ is the length of the smaller of the two arcs that join ζ_1 with ζ_2 and ζ_3 with ζ_4, then serious difficulties will ensue (i.e., severe loss of accuracy or, even, complete breakdown of the procedure) when ϕ is "small" by comparison with the precision of the computed approximation to the conformal map f. The seriousness of this numerical drawback is highlighted by the fact that if $m(Q)$ is "large", then in the best possible situation (where the points ζ_j are arranged symmetrically on the unit circle so that $\zeta_1 = -\zeta_3$ and $\zeta_2 = -\zeta_4$)

$$\phi \approx 8e^{-\pi m(Q)/2}. \tag{1.2}$$

In fact, the right-hand side of (1.2) gives a good estimate of ϕ even for relatively small values of $m(Q)$, for example for $m(Q) = 2$ (see, e.g., [7, Section 3] and [8, Section 2]).

This paper contains a study of a domain decomposition method (DDM) for computing the conformal modules of long quadrilaterals. The method involves decomposing the original quadrilateral Q into two or more component quadrilaterals Q_j, $j = 1, \ldots,$ and then approximating $m(Q)$ by the sum $\Sigma m(Q_j)$ of the conformal modules of the component quadrilaterals. The objectives for doing this are as follows. (a) To overcome the difficulties associated with the crowding phenomenon described above. (b) To take advantage of the fact that many applications involve complicated quadrilaterals which, however, can be decomposed into very simple components.

The DDM was introduced by us [9,10] for the purpose of computing the conformal modules and associated conformal maps of a special class of quadrilaterals, viz. quadrilaterals $Q := \{\Omega; z_1, z_2, z_3, z_4\}$, where (a) the domain Ω is bounded by two parallel straight lines and two Jordan arcs; (b) the points z_1, z_2, z_3, z_4 are the four corners where the two boundary arcs meet the two boundary straight lines. In this connection, the method was also studied by Gaier and Hayman [2,3], who derived several important results that enhanced considerably the associated DDM theory. In particular, the results of [2,3] provided us with the means for extending the area of application of the DDM to a much wider class of quadrilaterals than that indicated above. This was done recently in [11]. The purpose of the present paper is to extend the application of the method still further and, in the same time, to improve some of our earlier error estimates. We shall do this by making use of two new corollaries of the two central theorems that we gave in [11].

2. Error estimates

We shall adopt throughout the following notations of our earlier paper [11].

- Ω and $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ will denote respectively the original domain and corresponding quadrilateral.
- $\Omega_1, \Omega_2, \ldots$ and $Q_1, Q_2 \ldots$ will denote respectively the "principal" subdomains and corresponding quadrilaterals of the decomposition under consideration.
- The additional subdomains and associated quadrilaterals that arise when the decomposition of Q involves more than one crosscut will be denoted by using (in an obvious manner) a multisubscript notation.

For example, the five component quadrilaterals of the decomposition illustrated in Fig. 2.1 are

$$Q_1 := \{\Omega_1; z_1, z_2, a, d\}, \qquad Q_2 := \{\Omega_2; d, a, b, c\}, \qquad Q_3 := \{\Omega_3; c, b, z_3, z_4\}$$

and

$$Q_{1,2} := \{\Omega_{1,2}; z_1, z_2, b, c\}, \qquad Q_{2,3} := \{\Omega_{2,3}; d, a, z_3, z_4\},$$

where

$$\overline{\Omega}_{1,2} := \overline{\Omega}_1 \cup \overline{\Omega}_2, \qquad \overline{\Omega}_{2,3} := \overline{\Omega}_2 \cup \overline{\Omega}_3.$$

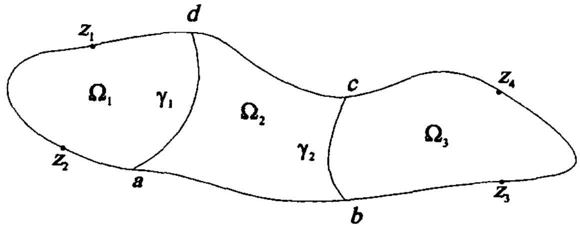


Fig. 2.1.

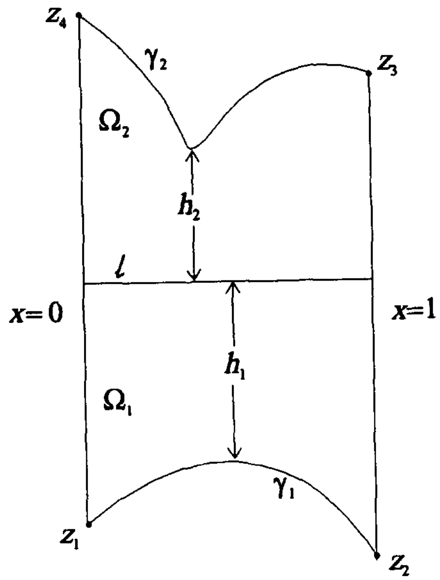


Fig. 2.2.

The following two theorems from our earlier paper [11] will also play a central role in our work here.

Theorem 2.1 (Papamichael and Stylianopoulos [11, Section 3]). *For the decomposition illustrated in Fig. 2.1 we have*

$$|m(Q) - \{m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2)\}| \leqslant 8.82\mathrm{e}^{-\pi m(Q_2)}, \tag{2.1}$$

provided that $m(Q_2) \geqslant 3$.

Theorem 2.2 (Papamichael and Stylianopoulos [11, Section 3]). *Consider the decomposition illustrated in Fig. 2.1 and suppose that the image of the crosscut* γ_1 *under the conformal map*

$$Q \to \{R_{m(Q)};\ 0,\ 1,\ 1 + \mathrm{i}m(Q),\ \mathrm{i}m(Q)\},$$

where

$$R_{m(Q)} := \{(\xi, \eta) \colon 0 < \xi < 1,\ 0 < \eta < m(Q)\},$$

is a straight line parallel to the real axis. Then,

$$-4.41\mathrm{e}^{-2\pi m(Q_2)} \leqslant m(Q) - \{m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2)\} \leqslant 0, \tag{2.2}$$

provided that $m(Q_2) \geqslant 1.5$.

We note the following in connection with the above.

Remark 2.3. As is indicated in [11], the two theorems are straightforward consequences of certain earlier results of Gaier and Hayman [2,3] (see also [9,10]), in connection with a special class of decompositions (like the one illustrated in Fig. 2.2), where (a) the defining domain Ω of the quadrilateral $Q := \{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$ is bounded by two parallel straight lines λ_1, λ_2 and two Jordan arcs γ_1, γ_2; (b) the four points z_1, z_2, z_3, z_4 are the four corners where the arcs γ_1, γ_2 meet the straight lines λ_1, λ_2; (c) the crosscuts of the decomposition are straight lines perpendicular to the boundary lines λ_1, λ_2 of Ω.

Of the results given in [2,3], the following two are of particular interest in connection with our work here (see also [11, Section 2]).

- Consider the decomposition of the quadrilateral illustrated in Fig. 2.2. Let h_1, h_2 be respectively the distances of the crosscut l from the two boundary arcs γ_1, γ_2 and let $h := \min(h_1, h_2)$. Then,

$$0 \leqslant m(Q) - \{m(Q_1) + m(Q_2)\} \leqslant 0.761\mathrm{e}^{-2\pi h}, \tag{2.3}$$

provided that $h \geqslant 1$.

- In the special case where the boundary arc γ_1 is a straight line parallel to the real axis (so that $m(Q_1) = h_1$), then

$$0 \leqslant m(Q) - \{h_1 + m(Q_2)\} \leqslant \tfrac{1}{2} \cdot 0.381\mathrm{e}^{-2\pi h_2}, \tag{2.4}$$

provided that $h_2 \geqslant 1$.

Remark 2.4. With reference to Fig. 2.1, it follows at once from the proofs given in [11] that (a) Theorem 2.1 remains valid when one or both of the endpoints d, a of the crosscut γ_1 (the endpoints b, c of the crosscut γ_2) coincide respectively with the vertices z_1, z_2 (the vertices z_3, z_4) of Q; (b) Theorem 2.2 remains valid when one or both of the endpoints b, c of the crosscut γ_2 coincide respectively with the vertices z_3, z_4 of Q.

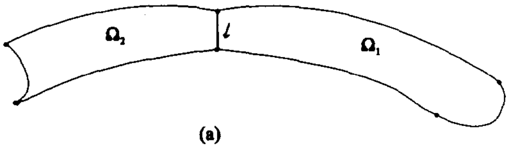
Remark 2.5. Theorems 2.1 and 2.2 were used in [11] for justifying the application of the DDM to several types of quadrilaterals and for deriving, in each case, an estimate of the error in the resulting approximation to $m(Q)$. In what follows we shall extend the application of the method to a wider class of quadrilaterals and, in the same time, shall improve some of our earlier estimates by making use of the following two corollaries.

Corollary 2.6. *Consider a quadrilateral* $Q := \{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$ *of the form illustrated in Fig.* 2.3(a). *The special feature of this is that the defining domain* Ω *can be decomposed by means of a straight line crosscut* l *into* Ω_1 *and* Ω_2, *so that* Ω_2 *is a reflection in* l *of some subdomain of* Ω_1. *Then, for the decomposition defined by* l,

$$0 \leqslant m(Q) - \{m(Q_1) + m(Q_2)\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_2)}, \tag{2.5}$$

provided that $m(Q_2) \geqslant 1.5$.

Proof. The proof is similar to that used for deriving estimate [11, Section 4.4, (4.11)]. That is, reflect Ω_1 in l and consider the decomposition of the resulting quadrilateral $\hat{Q}$ illustrated in



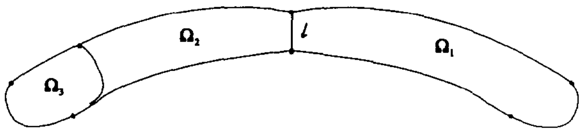


Fig. 2.3.

Fig. 2.3(b). Then, because l is a line of symmetry, the application of Theorem 2.2 to this decomposition gives

$$-4.41\mathrm{e}^{-2\pi m(Q_2)} \leqslant m(\hat{Q}) - \{m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2)\} \leqslant 0,$$

provided that $m(Q_2) \geqslant 1.5$. The desired result follows because

$$m(\hat{Q}) = 2m(Q_1), \qquad m(Q_{1,2}) = m(Q) \quad \text{and} \quad m(Q_{2,3}) = m(Q_1). \qquad \square$$

Corollary 2.7. *Consider a quadrilateral* $Q := \{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$ *of the form illustrated in Fig.* 2.4. *The special feature of this is that the defining domain* Ω *can be decomposed by means of a straight line crosscut* l *and two other (auxiliary) crosscuts* l_1, l_2 *into four subdomains* Ω_1, Ω_2, Ω_3, Ω_4, *so that* Ω_3 *is the reflection in* l *of* Ω_2. *Then, for the decomposition defined by* l,

$$0 \leqslant m(Q) - \{m(Q_{1,2}) + m(Q_{3,4})\} \leqslant 17.64\mathrm{e}^{-2\pi m(Q_2)}, \tag{2.6}$$

provided that $m(Q_2) \geqslant 1.5$.

Proof. The proof is similar to that used for deriving estimate [11, Section 4.5, (4.13)]. That is, the application of Theorem 2.1 to the decomposition of Q defined by the two crosscuts l_1 and l_2 gives

$$|\, m(Q) - \{m(Q_{1,2,3}) + m(Q_{2,3,4}) - m(Q_{2,3})\}\,| \leqslant 8.82\mathrm{e}^{-\pi m(Q_{2,3})}, \tag{2.7}$$

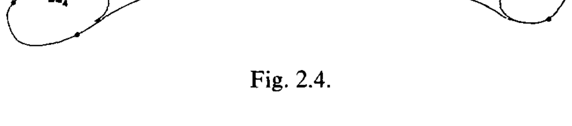


Fig. 2.4.

provided that $m(Q_{2,3}) \geqslant 3$. In addition, the application of Corollary 2.6 to each of the quadrilaterals $Q_{1,2,3}$ and $Q_{2,3,4}$ gives

$$0 \leqslant m(Q_{1,2,3}) - \{m(Q_{1,2}) + m(Q_3)\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_3)} \tag{2.8}$$

and

$$0 \leqslant m(Q_{2,3,4}) - \{m(Q_{3,4}) + m(Q_2)\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_2)}, \tag{2.9}$$

provided that $m(Q_3) \geqslant 1.5$ and $m(Q_2) \geqslant 1.5$. The desired result follows from (2.7)–(2.9), by observing that

$$m(Q_2) = m(Q_3), \qquad m(Q_{2,3}) = m(Q_2) + m(Q_3)$$

and

$$m(Q) \geqslant m(Q_{1,2}) + m(Q_{3,4}).$$

(The last inequality follows at once from the well-known composition law for conformal modules; see, e.g., [11, Remark 2.2].) □

The two corollaries given above contain as special cases the estimates of the three special decompositions that we studied in [11, Sections 4.1, 4.4 and 4.5]. Furthermore, the two corollaries can be used to improve and, in the same time, extend the applicability of [11, Sections 4.2 and 4.3, (4.4) and (4.8)]. These improved results can be states as follows.

Let Q_1, Q_2, Q_3 be the three principal component quadrilaterals in each of the two decompositions illustrated in Fig. 2.5. In each case, let $\hat{\Omega}_1$ and $\hat{\Omega}_3$ denote the reflections of Ω_1 and Ω_3 in the crosscuts l_1 and l_2 respectively. Then (depending on the "size" of Ω_2 relative to

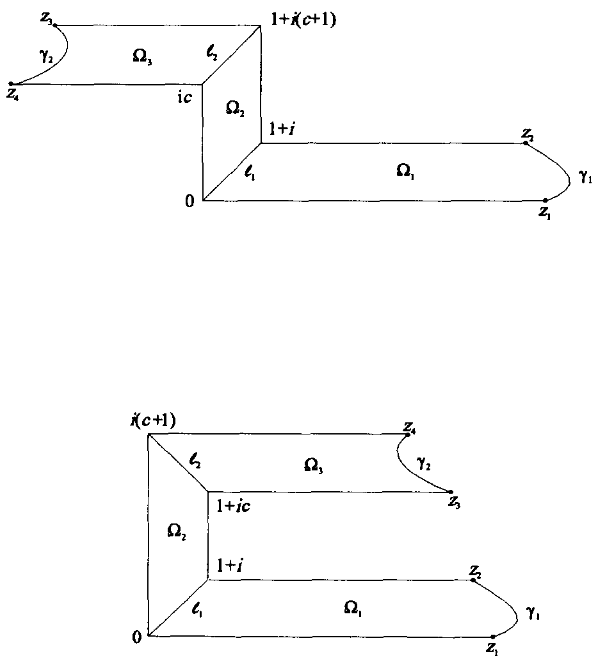


Fig. 2.5.

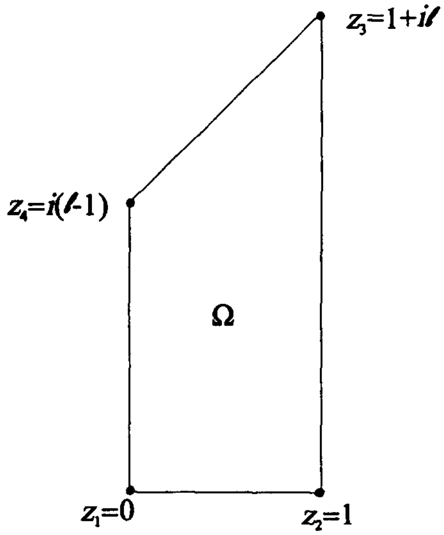


Fig. 2.6.

those of Ω_1 and Ω_3) we have the following estimates for the error in the DDM approximation to $m(Q)$.

- If $\Omega_2 \subset \hat{\Omega}_1$ and $\Omega_2 \subset \hat{\Omega}_3$, then

$$0 \leqslant m(Q) - \{m(Q_1) + m(Q_2) + m(Q_3)\} \leqslant 22.05\mathrm{e}^{-2\pi m(Q_2)}, \tag{2.10}$$

provided that $m(Q_2) \geqslant 1.5$.

- If $\Omega_2 \subset \hat{\Omega}_1$ and $\Omega_2 \supset \hat{\Omega}_3$, then

$$0 \leqslant m(Q) - \{m(Q_1) + m(Q_2) + m(Q_3)\} \leqslant 4.41\{\mathrm{e}^{-2\pi m(Q_2)} + \mathrm{e}^{-2\pi m(Q_3)}\}, \tag{2.11}$$

provided that $\min\{m(Q_2), m(Q_3)\} \geqslant 1.5$.

- If $\Omega_2 \supset \hat{\Omega}_1$ and $\Omega_2 \subset \hat{\Omega}_3$, then

$$0 \leqslant m(Q) - \{m(Q_1) + m(Q_2) + m(Q_3)\} \leqslant 4.41\{\mathrm{e}^{-2\pi m(Q_1)} + \mathrm{e}^{-2\pi m(Q_2)}\}, \tag{2.12}$$

provided that $\min\{m(Q_1), m(Q_2)\} \geqslant 1.5$.

- If $\Omega_2 \supset \hat{\Omega}_1$ and $\Omega_2 \supset \hat{\Omega}_3$, then

$$0 \leqslant m(Q) - \{m(Q_1) + m(Q_2) + m(Q_3)\} \leqslant 4.41\{\mathrm{e}^{-2\pi m(Q_1)} + \mathrm{e}^{-2\pi m(Q_3)}\}, \tag{2.13}$$

provided that $\min\{m(Q_1), m(Q_3)\} \geqslant 1.5$.

The first estimate (2.10) is obtained by combining the two inequalities

$$0 \leqslant m(Q) - \{m(Q_1) + m(Q_{2,3})\} \leqslant 17.64\mathrm{e}^{-2\pi m(Q_2)}$$

and

$$0 \leqslant m(Q_{2,3}) - \{m(Q_2) + m(Q_3)\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_2)}.$$

These are derived by applying respectively (a) Corollary 2.7 to the decomposition of Q defined by the crosscut l_1; (b) Corollary 2.6 to the decomposition of $Q_{2,3}$ defined by the crosscut l_2.

The other estimates (2.11)–(2.13) are obtained in a similar manner, but by using only Corollary 2.6. For example, (2.11) is obtained by combining the inequalities

$$0 \leqslant m(Q) - \{m(Q_{1,2}) + m(Q_3)\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_3)}$$

and

$$0 \leqslant m(Q_{1,2}) - \{m(Q_1) + m(Q_2)\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_2)}$$

that result from the application of Corollary 2.6 to (a) the decomposition of Q defined by the crosscut l_2; (b) the decomposition of $Q_{1,2}$ defined by the crosscut l_1.

Naturally, considerable simplifications and improvements occur when $\Omega_2 = \hat{\Omega}_1$ or $\Omega_2 = \hat{\Omega}_3$. For example, if $\Omega_2 = \hat{\Omega}_1$, then

$$m(Q_1) = m(Q_2), \qquad m(Q_{1,2}) = m(Q_1) + m(Q_2) = 2m(Q_2)$$

and, as a consequence, (2.11) can be replaced by

$$0 \leqslant m(Q) - \{m(Q_1) + m(Q_2) + m(Q_3)\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_3)}.$$

We end this section by repeating a remark that we made in our earlier paper [11, Section 2], concerning the conformal modules of quadrilaterals of the form illustrated in Fig. 2.6. Such quadrilaterals are of special interest in connection with the use of the DDM, because in many applications (for example in integrated-circuit design) the boundary of the original quadrilateral Q consists only of straight lines inclined at angles of 90° and 45°.

Remark 2.8. Let $T_l := \{\Omega;\, z_1, z_2, z_3, z_4\}$ denote the quadrilateral illustrated in Fig. 2.6, where (a) the domain Ω is the trapezium bounded by the real and imaginary axes and the lines $x = 1$ and $y = x + l - 1$, with $l > 1$; (b) the points z_1, z_2, z_3, z_4 are the corners of Ω. Then we have the following.

- The value of $m(T_l)$ is known exactly, for any $l > 1$, in terms of elliptic integrals (cf. [1, p.104]). For example, the exact values of $m(T_2)$, $m(T_3)$, $m(T_{3.5})$ and $m(T_4)$ are given to twelve decimal places by

$$m(T_2) = 1.279\,261\,571\,171, \qquad m(T_3) = 2.279\,364\,207\,968 \tag{2.14}$$

and

$$m(T_{3.5}) = 2.779\,364\,391\,556, \qquad m(T_4) = 3.279\,364\,399\,489. \tag{2.15}$$

- For any $c > 0$, estimate (2.4) of Gaier and Hayman gives

$$0 \leqslant m(T_{l+c}) - \{m(T_l) + c\} \leqslant \tfrac{1}{2} \cdot 0.381\mathrm{e}^{-2\pi(l-1)}, \tag{2.16}$$

provided that $l \geqslant 2$. In particular, for any $c > 0$, $m(T_{4+c})$ can be computed correct to at least eight decimal places (from the value of $m(T_4)$ given in (2.15)) by means of

$$m(T_{4+c}) \approx m(T_4) + c \tag{2.17}$$

(see also [7, pp. 78–81] and [9, Example 3.1]).

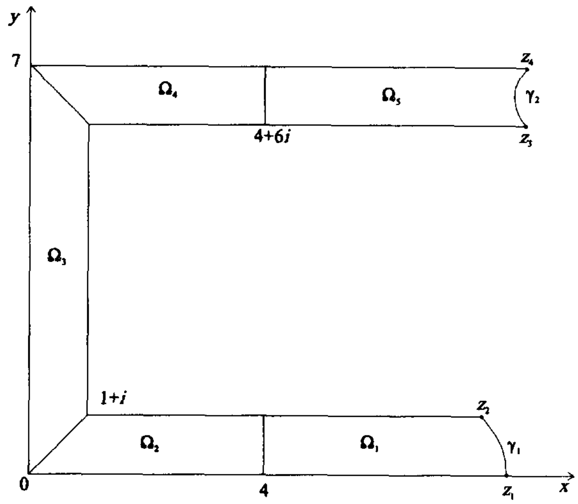


Fig. 3.1.

3. Numerical examples

In this section we present five numerical examples, illustrating the usefulness of the two corollaries given in Section 2. The first two of these (Examples 3.1 and 3.2) are, in fact, taken from our earlier paper [11]. They are reconsidered here for the purpose of showing how the two corollaries can be used to improve some of our earlier DDM error estimates. The other three examples (Examples 3.3–3.5) are new. Their purpose is to illustrate how the two corollaries can be used to extend the applicability of the DDM error estimation analysis to a wider class of quadrilaterals than that studied in [11].

Example 3.1 (Papamichael and Stylianopoulos [11, Example 5.1]). Consider the decomposition illustrated in Fig. 3.1, where the two boundary arcs γ_1, γ_2 are given respectively by

$$\gamma_1 := \{(x,\ y):\ x = 0.25y^4 - 0.5y^2 + 7,\ 0 \leqslant y \leqslant 1\}$$

and

$$\gamma_2 := \{(x,\ y):\ x = 7.5 + 0.25\cos 2\pi(6 - y),\ 6 \leqslant y \leqslant 7\}.$$

As in [11, Example 5.1], we let

$$\tilde{m}(Q) := \sum_{j=1}^{5} m(Q_j),$$

where

- the modules $m(Q_1)$ and $m(Q_5)$ are determined by using the Garrick iterative algorithm of [4]; the computed values are

$$m(Q_1) = 2.859\,569\,035 \quad \text{and} \quad m(Q_5) = 3.364\,089\,632,$$

and these are expected to be correct to all the figures quoted (cf. [9, Example 3.2]);

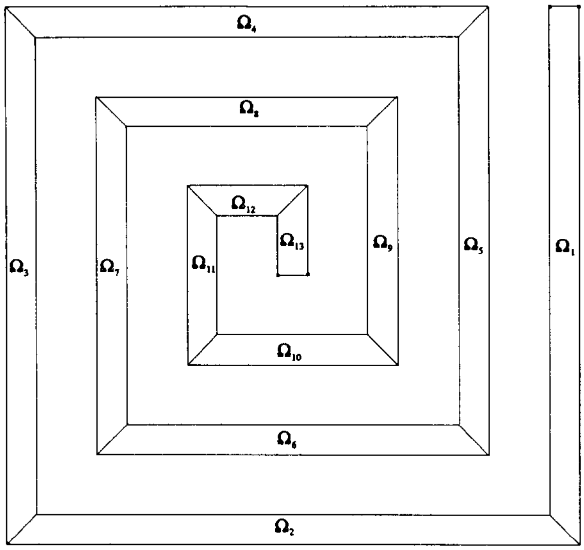


Fig. 3.2.

• the modules of the remaining component quadrilaterals can be written down immediately from the values listed in (2.14), (2.15). More precisely, if T_l denotes the quadrilateral of Fig. 2.6, then $m(Q_2) = m(Q_4) = m(T_4)$ and $m(Q_3) = 2m(T_{3.5})$. Therefore, to nine decimal places,

$$m(Q_2) = m(Q_4) = 3.279\,364\,399 \quad \text{and} \quad m(Q_3) = 5.558\,728\,783.$$

Thus, $\tilde{m}(Q)$ is given to nine decimal places by

$$\tilde{m}(Q) = 18.341\,116\,249. \tag{3.1}$$

The error in the above DDM approximation can be estimated as follows. The application of estimate (2.10) to the decomposition of Q defined by the single crosscut that separates Ω_2 from Ω_3 gives

$$0 \leqslant m(Q) - \{m(Q_{1,2}) + m(Q_3) + m(Q_{4,5})\} \leqslant 22.05\mathrm{e}^{-2\pi m(Q_3)}.$$

Also, the application of estimate (2.3) to each of the quadrilaterals $Q_{1,2}$ and $Q_{4,5}$ gives

$$0 \leqslant m(Q_{1,2}) - \{m(Q_1) + m(Q_2)\} \leqslant 0.761\mathrm{e}^{-2\pi h_1}$$

and

$$0 \leqslant m(Q_{4,5}) - \{m(Q_4) + m(Q_5)\} \leqslant 0.761\mathrm{e}^{-2\pi h_4},$$

where $h_1 = 2.75$ and $h_4 = 3$ (see Remark 2.3). Thus,

$$0 \leqslant m(Q) - \sum_{j=1}^{5} m(Q_j) \leqslant E,$$

where

$$E := 22.05\mathrm{e}^{-2\pi m(Q_3)} + 0.761\{\mathrm{e}^{-2\pi h_1} + \mathrm{e}^{-2\pi h_4}\},$$

that is,

$$0 \leqslant m(Q) - \tilde{m}(Q) \leqslant 2.9 \cdot 10^{-8}.$$

Hence, from (3.1),

$$18.341\,116\,24 < m(Q) < 18.341\,116\,28.$$

This improves our earlier estimate

$$18.341\,116\,24 < m(Q) < 18.341\,116\,51$$

that we derived in [11].

Example 3.2 (Papamichael and Stylianopoulos [11, Example 5.3]). Let $Q := \{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$ be the quadrilateral illustrated in Fig. 3.2, where the width of each strip of the spiral Ω is 1, and the lengths of the "outer" segments of $\partial\Omega$ (in clockwise order, starting from the right-hand side) are 18, 19, 18, 16, 15, 13, 12, 10, 9, 7, 6, 4 and 3.

We consider approximating $m(Q)$ by

$$\tilde{m}(Q) := \sum_{j=1}^{13} m(Q_j),$$

and note that

$$\begin{aligned}
&m(Q_1) = m(T_{18}), && m(Q_2) = 2m(T_{9.5}), && m(Q_3) = 2m(T_9), && m(Q_4) = 2m(T_8),\\
&m(Q_5) = 2m(T_{7.5}), && m(Q_6) = 2m(T_{6.5}), && m(Q_7) = 2m(T_6), && m(Q_8) = 2m(T_5),\\
&m(Q_9) = 2m(T_{4.5}), && m(Q_{10}) = 2m(T_{3.5}), && m(Q_{11}) = 2m(T_3), && m(Q_{12}) = 2m(T_2)
\end{aligned}$$

and

$$m(Q_{13}) = m(T_3),$$

where T_l is the quadrilateral illustrated in Fig. 2.6. This means that the modules of all the component quadrilaterals can be written down, correct to at least eight decimal places, by using the values of $m(T_2)$, $m(T_3)$, $m(T_{3.5})$ and $m(T_4)$ listed in (2.14), (2.15) together with formula (2.17), i.e.,

$$m(T_{4+c}) \approx m(T_4) + c, \quad c > 0.$$

Thus, by trivial calculation,

$$\tilde{m}(Q) := 132.704\,539\,35. \tag{3.2}$$

The error in $\tilde{m}(Q)$ can be estimated, by using only Corollary 2.6, as follows. The application of this corollary to the decomposition of Q defined by the single crosscut that separates Ω_{12} from Ω_{13} gives

$$0 \leqslant m(Q) - \{m(Q_{1,\dots,12}) + m(Q_{13})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_{13})}.$$

Similarly, its application to each of $Q_{1,\dots,12}$, $Q_{1,\dots,11,\dots}$, $Q_{1,2,3}$ gives

$$0 \leqslant m(Q_{1,\dots,12}) - \{m(Q_{1,\dots,11}) + m(Q_{12})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_{12})},$$

$$0 \leqslant m(Q_{1,\dots,11}) - \{m(Q_{1,\dots,10}) + m(Q_{11})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_{11})},$$

etc., until

$$0 \leqslant m(Q_{1,2,3}) - \{m(Q_{1,2}) + m(Q_3)\} \leqslant 4.41 e^{-2\pi m(Q_3)}.$$

Finally, the application of the corollary to $Q_{1,2}$ gives

$$0 \leqslant m(Q_{1,2}) - \{m(Q_1) + m(Q_2)\} \leqslant 4.41 e^{-2\pi m(Q_1)}.$$

(The above are valid because $m(Q_{12}) = 2m(T_2) > 1.5$ and $m(Q_{13}) = m(T_3) > 1.5$.) Therefore, by combining the above estimates,

$$0 \leqslant m(Q) - \sum_{j=1}^{13} m(Q_j) \leqslant E,$$

where

$$E := 4.41 \left\{ e^{-2\pi m(Q_1)} + \sum_{j=3}^{13} e^{-2\pi m(Q_j)} \right\} \leqslant 3.2 \cdot 10^{-6},$$

that is,

$$0 \leqslant m(Q) - \tilde{m}(Q) \leqslant 3.2 \cdot 10^{-6}.$$

Hence, from (3.2),

$$132.704539 < m(Q) < 132.704543. \tag{3.3}$$

Estimate (3.3) should be compared with the result

$$132.704539 < m(Q) < 132.704666$$

that we derived in [11] by considering a much more complicated decomposition than that of Fig. 3.2. We note, in particular, that the decomposition used in [11] involved computing by means of the Schwarz–Christoffel package SCPACK of [12] the conformal module of an L-shaped quadrilateral. We also note the approximation

$$m(Q) \approx 132.70454,$$

which was obtained in [6], using a modified Schwarz–Christoffel technique.

Example 3.3. Let $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ be the quadrilateral illustrated in Fig. 3.3, where

$$\Omega_2 := \left\{(x, y): (x-2)^2 + y^2 < 4,\ x < 2,\ y < 0\right\} \cup \{(x, y): 2 \leqslant x < 4,\ -2 < y < -1\},$$

and Ω_3 is the mirror image of Ω_2 in the real axis.

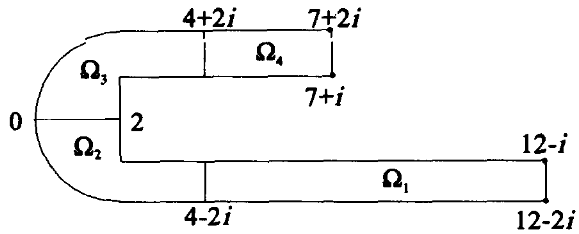


Fig. 3.3.

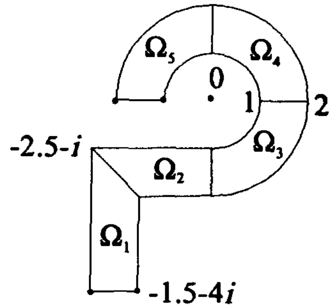


Fig. 3.4.

We consider approximating $m(Q)$ by

$$\tilde{m}(Q) := \sum_{j=1}^{4} m(Q_j),$$

and note that

$$m(Q_1) = 8, \qquad m(Q_2) = m(Q_3), \qquad m(Q_4) = 3.$$

Here we determine $m(Q_2)$ by using the integral equation conformal mapping package CONFPACK of [5]. This gives the approximation

$$m(Q_2) \approx 3.205\,804, \tag{3.4}$$

which we expect to be correct to at least four decimal places, because

- the CONFPACK estimate of the error in the corresponding approximation to the conformal map onto the unit disc is $3 \cdot 10^{-6}$;
- the measure of crowding is greater than $5 \cdot 10^{-2}$, i.e., the crowding is not serious relative to the accuracy of the numerical conformal map (recall estimate (1.2) and see [7, Section 2]).

Therefore, we expect $\tilde{m}(Q)$ to be given to at least four decimal places by

$$\tilde{m}(Q) = 17.411\,608. \tag{3.5}$$

The error in (3.5) can be estimated as follows. The application of Corollary 2.7 to the decomposition of Q defined by the single crosscut that separates Ω_1 from Ω_2 gives

$$0 \leqslant m(Q) - \{8 + m(Q_{2,3,4})\} \leqslant 17.64\mathrm{e}^{-4\pi}.$$

Similarly, the application of the same corollary to the decomposition of $Q_{2,3,4}$ defined by the crosscut that separates Ω_3 and Ω_4 gives

$$0 \leqslant m(Q_{2,3,4}) - \{m(Q_{2,3}) + 3\} \leqslant 17.64\mathrm{e}^{-4\pi}.$$

Therefore, since $m(Q_{2,3}) = 2m(Q_2)$,

$$0 \leqslant m(Q) - \tilde{m}(Q) \leqslant E,$$

where

$$E := 35.28\mathrm{e}^{-4\pi} \leqslant 1.24 \cdot 10^{-4}.$$

Thus, from (3.5),

$$17.4116 < m(Q) < 17.4118,$$

provided that our expectation concerning the accuracy of the CONFPACK approximation (3.4) is valid.

Example 3.4. Consider the decomposition illustrated in Fig. 3.4, where the subdomain $\Omega_{3,4,5}$ is the three quarters annulus

$$\Omega_{3,4,5} := \{z = re^{i\theta}\colon 1 < r < 2, \ -\tfrac{1}{2}\pi < \theta < \pi\}.$$

(As will become apparent the crosscut that separates Ω_2 from Ω_3 is needed only for the DDM error estimation analysis.)

We let

$$\tilde{m}(Q) := m(Q_1) + m(Q_{2,3}) + m(Q_4) + m(Q_5), \tag{3.6}$$

and note that $m(Q_1)$ and $m(Q_2)$ are given correct to nine decimal places by

$$m(Q_1) = m(T_3) = 2.279\,364\,208 \quad \text{and} \quad m(Q_2) = m(T_{2.5}) = 1.779\,359\,959$$

(cf. Remark 2.8). We also note that

$$m(Q_{4,5}) = -\frac{\pi}{\log 0.5} = 4.532\,360\,142$$

(see, e.g., [7, Section 3.2]). Hence,

$$m(Q_3) = m(Q_4) = m(Q_5) = \tfrac{1}{2}m(Q_{4,5}) = 2.266\,180\,071.$$

Therefore, in the domain decomposition approximation (3.6) only the value of $m(Q_{2,3})$ is not known exactly. For the computation of this unknown module we use (as in Example 3.3) the conformal mapping package CONFPACK [5]. The resulting approximation

$$m(Q_{2,3}) \approx 4.051\,610\,5 \tag{3.7}$$

is, again, expected to be correct to at least four decimal places because

- the CONFPACK estimate of the error in the corresponding approximation to the conformal map onto the unit disc is $6 \cdot 10^{-6}$;
- the measure of crowding is greater than $1 \cdot 10^{-2}$.

Therefore, we expect $\tilde{m}$(Q) to be given to at least four decimal places by

$$\tilde{m}(Q) = 10.863\,335. \tag{3.8}$$

The error in $\tilde{m}(Q)$ can be estimated as follows. The application of Corollary 2.6 to the decomposition of Q defined by the single crosscut that separates Ω_4 from Ω_5 gives

$$0 \leqslant m(Q) - \{m(Q_{1,\dots,4}) + m(Q_5)\} \leqslant 4.41e^{-2\pi m(Q_5)}.$$

Similarly, the application of the same corollary to $Q_{1,\dots,4}$ gives

$$0 \leqslant m(Q_{1,\dots,4}) - \{m(Q_{1,2,3}) + m(Q_4)\} \leqslant 4.41e^{-2\pi m(Q_4)}.$$

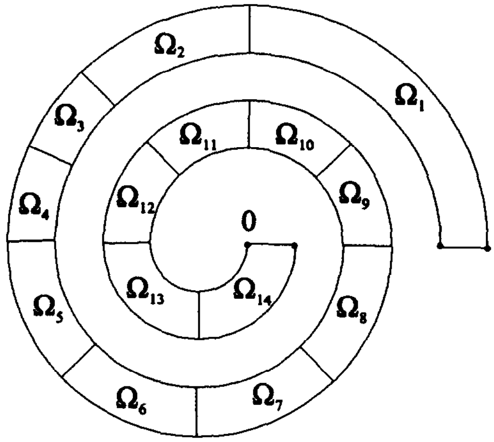


Fig. 3.5.

Finally, the application of Corollary 2.7 to the decomposition of $Q_{1,2,3}$ defined by the crosscut that separates Ω_1 from Ω_2 gives

$$0 \leqslant m(Q_{1,2,3}) - \{m(Q_1) + m(Q_{2,3})\} \leqslant 17.64\mathrm{e}^{-2\pi m(Q_2)}.$$

Thus, by combining the above estimates,

$$0 \leqslant m(Q) - \tilde{m}(Q) \leqslant E,$$

where

$$E := 8.82\mathrm{e}^{-2\pi m(Q_4)} + 17.64\mathrm{e}^{-2\pi m(Q_2)} < 2.52 \cdot 10^{-4}.$$

Therefore, from (3.8),

$$10.8633 < m(Q) < 10.8636,$$

provided that our expectation concerning the CONFPACK approximation (3.7) is valid.

Example 3.5. Let $Q := \{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$ be the quadrilateral illustrated in Fig. 3.5 where
- $\Omega_{1,\ldots,4}$ is the upper half of the annulus $\{z\colon 4 < |z| < 5\}$;
- $\Omega_{5,\ldots,8}$ is the lower half of $\{z\colon 3 < |z+1| < 4\}$;
- $\Omega_{9,\ldots,12}$ is the upper half of $\{z\colon 2 < |z| < 3\}$;
- $\Omega_{13,14}$ is the lower half of $\{z\colon 1 < |z+1| < 2\}$.

In the figure the three crosscuts that separate respectively the subdomains Ω_4 from Ω_5, Ω_8 from Ω_9 and Ω_{12} from Ω_{13} are, in fact, needed only for the DDM error estimation analysis. That is, the DDM approximation under consideration is

$$\tilde{m}(Q) := \sum_{j=1}^{3} m(Q_j) + m(Q_{4,5}) + \sum_{j=6}^{7} m(Q_j) + m(Q_{8,9}) + \sum_{j=10}^{11} m(Q_j) + m(Q_{12,13}) + m(Q_{14}),$$

where

$$m(Q_1) = 2m(Q_2) = 4m(Q_3) = 4m(Q_4) = \frac{0.5\pi}{\log 5 - \log 4} = 7.039\,398\,260,$$

$$m(Q_5) = m(Q_6) = m(Q_7) = m(Q_8) = \frac{0.25\pi}{\log 4 - \log 3} = 2.730\,090\,745,$$

$$m(Q_9) = m(Q_{10}) = m(Q_{11}) = m(Q_{12}) = \frac{0.25\pi}{\log 3 - \log 2} = 1.937\,030\,210,$$

$$m(Q_{13}) = m(Q_{14}) = \frac{0.5\pi}{\log 2} = 2.266\,180\,071$$

(see, e.g., [7, Section 3.2]). Thus,

$$\tilde{m}(Q) = 23.919\,368\,935 + m(Q_{4,5}) + m(Q_{8,9}) + m(Q_{12,13}),$$

and for the unknown modules we use again the conformal mapping package CONFPACK [5]. The resulting approximations are

$$m(Q_{4,5}) \approx 4.491\,289\,1, \qquad m(Q_{8,9}) \approx 4.664\,681\,4, \qquad m(Q_{12,13}) \approx 4.204\,757\,2, \tag{3.9}$$

with respectively

- error estimates for the conformal maps onto the unit disc:

$$8 \cdot 10^{-6}, \qquad 4 \cdot 10^{-6}, \qquad 7 \cdot 10^{-6};$$

- measures of crowding:

$$6 \cdot 10^{-3}, \qquad 5 \cdot 10^{-3}, \qquad 1 \cdot 10^{-2}.$$

Although, in this case, the crowding is more serious than in the previous examples, it is again reasonable to expect that the approximations (3.9) are correct to four decimal places. Therefore, we expect $\tilde{m}(Q)$ to be given correct to four decimal places by

$$\tilde{m}(Q) = 37.280\,097. \tag{3.10}$$

The error in $\tilde{m}(Q)$ can be estimated by applying Corollaries 2.6 and 2.7 to the various quadrilaterals as follows.

- Corollary 2.6 to Q: $0 \leqslant m(Q) - \{m(Q_1) - m(Q_{2,\ldots,14})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_1)}$;
- Corollary 2.6 to $Q_{2,\ldots,14}$: $0 \leqslant m(Q_{2,\ldots,14}) - \{m(Q_2) + m(Q_{3,\ldots,14})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_2)}$;
- Corollary 2.6 to $Q_{3,\ldots,14}$: $0 \leqslant m(Q_{3,\ldots,14}) - \{m(Q_3) + m(Q_{4,\ldots,14})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_3)}$;
- Corollary 2.7 to $Q_{4,\ldots,14}$: $0 \leqslant m(Q_{4,\ldots,14}) - \{m(Q_{4,5}) + m(Q_{6,\ldots,14})\} \leqslant 17.64\mathrm{e}^{-2\pi m(Q_5)}$;
- Corollary 2.6 to $Q_{6,\ldots,14}$: $0 \leqslant m(Q_{6,\ldots,14}) - \{m(Q_6) + m(Q_{7,\ldots,14})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_6)}$;
- Corollary 2.6 to $Q_{7,\ldots,14}$: $0 \leqslant m(Q_{7,\ldots,14}) - \{m(Q_7) + m(Q_{8,\ldots,14})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_7)}$;
- Corollary 2.7 to $Q_{8,\ldots,14}$: $0 \leqslant m(Q_{8,\ldots,14}) - \{m(Q_{8,9}) + m(Q_{10,\ldots,14})\} \leqslant 17.64\mathrm{e}^{-2\pi m(Q_9)}$;
- Corollary 2.6 to $Q_{10,\ldots,14}$: $0 \leqslant m(Q_{10,\ldots,14}) - \{m(Q_{10}) + m(Q_{11,\ldots,14})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_{10})}$;
- Corollary 2.6 to $Q_{11,\ldots,14}$: $0 \leqslant m(Q_{11,\ldots,14}) - \{m(Q_{11}) + m(Q_{12,13,14})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_{11})}$;
- Corollary 2.6 to $Q_{12,13,14}$: $0 \leqslant m(Q_{12,13,14}) - \{m(Q_{12,13}) + m(Q_{14})\} \leqslant 4.41\mathrm{e}^{-2\pi m(Q_{14})}$.

Hence, by combining the above,

$$0 \leqslant m(Q) - \tilde{m}(Q) \leqslant E,$$

where

$$E := 4.41 \left\{ \sum_{j=1}^{3} e^{-2\pi m(Q_j)} + \sum_{j=6}^{7} e^{-2\pi m(Q_j)} + \sum_{j=10}^{11} e^{-2\pi m(Q_j)} + e^{-2\pi m(Q_{14})} \right\}$$
$$+ 17.64 \{ e^{-2\pi m(Q_5)} + e^{-2\pi m(Q_9)} \} < 2.11 \cdot 10^{-4}.$$

Therefore, from (3.10),

$$37.2800 < m(Q) < 37.2804,$$

provided our expectation regarding the CONFPACK approximations (3.9) is valid.

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