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On the zeros of asymptotically extremal polynomial sequences in the plane

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To Herbert Stahl, an exceptional mathematician, a delightful personality, and a dear friend

Abstract

Let E be a compact set of positive logarithmic capacity in the complex plane and let $\{P_n(z)\}_1^\infty$ be a sequence of asymptotically extremal monic polynomials for E in the sense that

$$\limsup_{n\to\infty} \|P_n\|_E^{1/n} \le \mathrm{cap}(E).$$

The purpose of this note is to provide sufficient geometric conditions on E under which the (full) sequence of normalized counting measures of the zeros of $\{P_n\}$ converges in the weak-star topology to the equilibrium measure on E, as $n \to \infty$. Utilizing an argument of Gardiner and Pommerenke dealing with the balayage of measures, we show that this is true, for example, if the interior of the polynomial convex hull of E has a single component and the boundary of this component has an "inward corner" (more generally, a "non-convex singularity"). This simple fact has thus far not been sufficiently emphasized in the literature. As applications we mention improvements of some known results on the distribution of zeros of some special polynomial sequences.

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1. Introduction

Let E be a compact set of positive logarithmic capacity ($\mathrm{cap}(E) > 0$) contained in the complex plane $\mathbb{C}$. We denote by Ω the unbounded component of $\overline{\mathbb{C}} \setminus E$ and by μ_E the *equilibrium measure* (energy minimizing Borel probability measure on E) for the logarithmic potential on E; see e.g. [12, Chapter 3] and [13, Section I.1]. As is well-known, the support $\mathrm{supp}(\mu_E)$ lies on the boundary $\partial\Omega$ of Ω.

For any polynomial $p_n(z)$, of degree n, we denote by ν_{p_n} the *normalized counting measure* for the zeros of $p_n(z)$; that is,

$$\nu_{p_n} := \frac{1}{n} \sum_{p_n(z)=0} \delta_z, \tag{1.1}$$

where δ_z is the unit point mass (Dirac delta) at the point z.

Let $\mathcal{N}$ denote an increasing sequence of positive integers. Then, following [13, p. 169] we say that a sequence of monic polynomials $\{P_n(z)\}_{n\in\mathcal{N}}$, of respective degrees n, is *asymptotically extremal on E* if

$$\limsup_{n\to\infty,\, n\in\mathcal{N}} \|P_n\|_E^{1/n} \le \mathrm{cap}(E), \tag{1.2}$$

where $\|\cdot\|_E$ denotes the uniform norm on E. (We remark that this inequality implies equality for the limit, since $\|P_n\|_E \ge \mathrm{cap}(E)^n$.) Such sequences arise, for example, in the study of polynomials orthogonal with respect to a measure μ belonging to the class **Reg**, see Definition 3.1.2 in [14].

Concerning the asymptotic behavior of the zeros of an asymptotically extremal sequence of polynomials, we recall the following result, see e.g. [10, Theorem 2.3] and [13, Theorem III.4.7].

Theorem 1.1. *Let $\{P_n\}_{n\in\mathcal{N}}$, denote an asymptotically extremal sequence of monic polynomials on E. If μ is any weak-star limit measure of the sequence $\{\nu_{P_n}\}_{n\in\mathcal{N}}$, then μ is a Borel probability measure supported on $\overline{\mathbb{C}} \setminus \Omega$ and $\mu^b = \mu_E$, where μ^b is the balayage of μ out of $\overline{\mathbb{C}} \setminus \Omega$ onto $\partial\Omega$. Similarly, the sequence of balayaged counting measures converges to μ_E:*

$$\nu_{P_n}^b \xrightarrow{*} \mu_E, \quad n \to \infty,\ n \in \mathcal{N}. \tag{1.3}$$

By the weak-star convergence of a sequence of measures λ_n to a measure λ we mean that, for any continuous f with compact support in $\mathbb{C}$, there holds

$$\int f d\lambda_n \to \int f d\lambda, \quad \text{as } n \to \infty.$$

For properties of balayage, see [13, Section II.4].

The goal of the present paper is to describe simple geometric conditions under which the normalized counting measures ν_{P_n} of an asymptotically extremal sequence $\{P_n\}$ on E *themselves* converge weak-star to the equilibrium measure. For example, this is the case whenever E is a closed non-convex polygonal region, a simple fact that has thus far not been sufficiently emphasized in the literature. Here we introduce more general sufficient conditions based on arguments of Gardiner and Pommerenke [3] dealing with the balayage of measures.

The outline of the paper is as follows: In Section 2 we describe a geometric condition, which we call the *non-convex singularity* (NCS) condition and state the main result regarding the counting measures ν_{P_n} of the zeros of polynomials that form an asymptotically extremal sequence. Its proof is given in Section 4.

In Section 3, we apply the main result to obtain improvements in several known results on the behavior of the zeros of orthogonal polynomials, whereby the NCS condition yields convergence conclusions for the full sequence $\mathcal{N}$ rather than for some subsequence.

2. A geometric property

Definition 2.1. Let G be a bounded simply connected domain in the complex plane. A point z_0 on the boundary of G is said to be a *non-convex type singularity* (NCS) if it satisfies the following two conditions:

(i) There exists a closed disk $\overline{D}$ with z_0 on its circumference, such that $\overline{D}$ is contained in G except for the point z_0.

(ii) There exists a line segment L connecting a point ζ_0 in the interior of $\overline{D}$ to z_0 such that

$$\lim_{\substack{z\to z_0\\ z\in L}} \frac{g_G(z,\zeta_0)}{|z-z_0|} = +\infty, \tag{2.1}$$

where $g_G(z,\zeta_0)$ denotes the Green function of G with pole at $\zeta_0 \in G$.

Recall that $g_G(\cdot,\zeta)$ is a positive harmonic function in $G\setminus\{\zeta\}$. Also note that the assumption that G is bounded and simply connected implies that G is regular with respect to the Dirichlet problem in G. This means that $\lim_{\substack{z\to t\\ z\in G}} g_G(z,\zeta) = 0$, for any $t \in \partial G$; see, e.g., [12, pp. 92 and 111].

Remark. With respect to condition (ii), we note that the existence of a straight line L and a point ζ_0 for which (2.1) holds, implies that the same is true for any other straight line connecting a point in the open disk D with z_0. This can be easily deduced from Harnack's Lemma (see e.g. [13, Lemma 4.9, p.17], [1, p. 14]) in conjunction with the symmetry property of Green functions, which imply that for a compact set $K \subset G$ containing ζ_0 and $\operatorname{dist}(z,K) > \delta > 0$, $z \in G$, there is a constant $C = C(K,\delta,\zeta_0) > 0$ such that the inequality

$$g_G(z,\zeta) \geq C g_G(z,\zeta_0) \tag{2.2}$$

holds for all $\zeta \in K$.

As we shall easily show, a point z_0 satisfying the following sector condition is an NCS point.

Definition 2.2. Let G be a bounded simply connected domain. A point z_0 on the boundary of G is said to be an *inward-corner* (IC) *point* if there exists a circular sector of the form $S := \{z : 0 < |z-z_0| < r,\ \alpha\pi < \arg(z-z_0) < \beta\pi\}$ with $\beta - \alpha > 1$ whose closure is contained in G except for z_0.

To see that an IC point satisfies Definition 2.1, let $g_S(z,\zeta)$ denote the Green function of the sector S. Then $g_S(z,\zeta) = -\log|\varphi_\zeta(z)|$, where φ_ζ is a conformal mapping of S onto the unit disk $\mathbb{D} := \{z : |z| < 1\}$, satisfying $\varphi_\zeta(\zeta) = 0$. From the theory of conformal mapping it is known [7] that the following expansion is valid for any $z \in \overline{S}$ near z_0:

$$\varphi_\zeta(z) = \varphi_\zeta(z_0) + a_1(z-z_0)^{1/(\beta-\alpha)} + O(|z-z_0|^{2/(\beta-\alpha)}),$$

with $a_1 \neq 0$. Since $|\varphi_\zeta(z_0)| = 1$, the above implies that the limit in (2.1) holds with $g_S(z,\zeta_0)$ in the place of $g_G(z,\zeta_0)$. The desired limit then follows from the comparison principle for Green functions:

$$g_S(z,\zeta) \le g_G(z,\zeta), \quad z,\zeta \in S;$$

see, e.g., [12, p. 108].

Remark. It is interesting to note that if the boundary ∂G is a piecewise analytic Jordan curve, then at any IC point z_0 of ∂G the density of the equilibrium measure is zero. This can be easily deduced from the relation $d\mu_{\partial G}(z) = |\Phi'(z)|ds$ connecting the equilibrium measure to the arclength measure ds on ∂G, where Φ is a conformal mapping of Ω onto $\{w : |w| > 1\}$, taking ∞ to ∞. Then, if $\lambda\pi$ $(1 < \lambda < 2)$ is the interior opening angle at z_0, $\Phi(z)$ has near z_0 an expansion of the form

$$\Phi(z) = \Phi(z_0) + b_1(z-z_0)^{1/(2-\lambda)} + o(|z-z_0|^{1/(2-\lambda)}), \tag{2.3}$$

with $b_1 \neq 0$, which leads to $\Phi'(z_0) = 0$.

We can now state our main result.

Theorem 2.1. *Let $E \subset \mathbb{C}$ be a compact set of positive capacity, Ω the unbounded component of $\overline{\mathbb{C}} \setminus E$, and $\mathcal{E} := \overline{\mathbb{C}} \setminus \Omega$ denote the polynomial convex hull of E. Assume there is closed set $E_0 \subset \mathcal{E}$ with the following three properties:*

(i) $\operatorname{cap}(E_0) > 0$;
(ii) *either* $E_0 = \mathcal{E}$ *or* $\operatorname{dist}(E_0, \mathcal{E} \setminus E_0) > 0$;
(iii) *either the interior* $\operatorname{int}(E_0)$ *of* E_0 *is empty or the boundary of each open component of* $\operatorname{int}(E_0)$ *contains an NCS point.*

Let V be an open set containing E_0 such that $\operatorname{dist}(V, \mathcal{E} \setminus E_0) > 0$ *if* $E_0 \neq \mathcal{E}$. *Then for any asymptotically extremal sequence of monic polynomials $\{P_n\}_{n\in\mathcal{N}}$ for E,*

$$\nu_{P_n}|_V \xrightarrow{*} \mu_E|_{E_0}, \quad n \to \infty,\ n \in \mathcal{N}, \tag{2.4}$$

where $\mu|_K$ denotes the restriction of a measure μ to the set K.

We remark that, for the case of a Jordan region, the hypothesis of Theorem 3 of [3] implies the existence of an NCS point. We also note that the assumption $\operatorname{dist}(E_0, \mathcal{E} \setminus E_0) > 0$ implies that any (open) component of $\operatorname{int}(E_0)$ is simply connected.

As a consequence of Theorem 2.1 and [13, Theorem III.4.1] we have the following.

Corollary 2.1. *With the hypotheses of Theorem 2.1, if G denotes a component of* $\operatorname{int}(E_0)$, *then for any asymptotically extremal sequence $\{P_n\}_{n\in\mathcal{N}}$ of monic polynomials for E, there exists a point ζ in G such that*

$$\limsup_{n\to\infty,\, n\in\mathcal{N}} |P_n(\zeta)|^{1/n} = \operatorname{cap}(E). \tag{2.5}$$

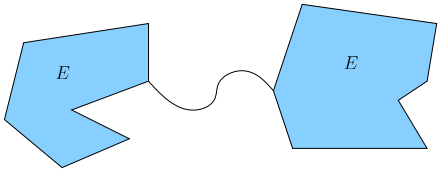

Fig. 1. Form (iii).

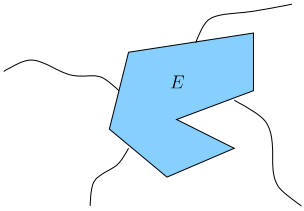

Fig. 2. Form (iv).

Corollary 2.2. *Let E consist of the union of a finite number of closed Jordan regions $E := \cup_{j=1}^{N} \overline{G_j}$, where $\overline{G_i} \cap \overline{G_j} = \emptyset, i \neq j$, and assume that for each $k = 1, \ldots, m$ the boundary of G_k contains an NCS point. Then for any asymptotically extremal sequence of monic polynomials $\{P_n\}_{n\in\mathcal{N}}$ for E,*

$$\nu_{P_n}|_{\mathcal{V}} \xrightarrow{*} \mu_E|_{\mathcal{V}}, \quad n \to \infty,\ n \in \mathcal{N}, \tag{2.6}$$

where $\mathcal{V}$ is an open set containing $\bigcup_{k=1}^{m} \overline{G}_k$, such that if $m < N$ the distance of $\overline{\mathcal{V}}$ from $\bigcup_{j=m+1}^{N} \overline{G}_j$ is positive.

We now give some examples that follow from Theorem 2.1 and Corollary 2.2. If E has one of the following forms, then for any asymptotically extremal sequence $\{P_n\}_{n\in\mathbb{N}}$ of monic polynomials on E, we have

$$\nu_{P_n} \xrightarrow{*} \mu_E, \quad n \to \infty,\ n \in \mathbb{N}. \tag{2.7}$$

(i) E is a closed non-convex polygon or a finite union of mutually exterior closed non-convex polygons.
(ii) E is the union of two mutually exterior closed non-convex polygons, except for a single common boundary point.
(iii) E is the union of two mutually exterior closed non-convex polygons joined by a Jordan arc in their exterior, such that the complement of E is connected and does not separate the plane; see Fig. 1.
(iv) E is a closed non-convex polygon Π together with a finite number of closed Jordan arcs lying exterior to Π except for the initial point on the boundary of Π, and such that the complement of E does not separate the plane; see Fig. 2.
(v) E is any of the preceding forms with the polygons replaced by closed bounded Jordan regions, each one having an NCS point.
(vi) E is any of the preceding forms union with a compact set K in the complement of E such that K has empty interior and $E \cup K$ does not separate the plane.

We remark that if E is a convex region, so that the hypotheses of Theorem 2.1 are not fulfilled, then the zero behavior of an asymptotically extremal sequence of monic polynomials P_n can be quite different. For example, if E is the closed unit disk centered at the origin for which μ_E is the uniform measure on the circumference $|z| = 1$, the polynomials $P_n(z) = z^n$ form an extremal sequence for which $\nu_{P_n} = \delta_0$, the unit point mass at zero. A less trivial example is illustrated in Fig. 3, where the zeros of orthonormal polynomials with respect to area measure on a circular sector E with opening angle $\pi/2$ are plotted for degrees $n = 50, 100$, and 150. These so-called Bergman polynomials $B_n(z)$ form an asymptotically extremal sequence of polynomials for the sector, yet their normalized zero counting measures converge weak-star to a measure ν that is supported on the union of three curves lying in the interior of E except for their three endpoints, the vertices of the sector; see [11]. On the other hand, for any compact set E of positive capacity, whether convex or not, if $\{q_n(z)\}_{n\in\mathbb{N}}$ denotes a sequence of Fekete polynomials for E, then this sequence is asymptotically extremal on E, all their zeros stay on the outer boundary $\partial\Omega$, and $\nu_{q_n} \xrightarrow{*} \mu_E$, as $n \to \infty$, $n \in \mathbb{N}$; see e.g., [13, p. 176].

In every case, according to Theorem 1.1, a limit measure of a sequence of asymptotically extremal monic polynomials must have a balayage to the outer boundary of E that equals the equilibrium measure μ_E. The question then of what types of point sets can support a measure with such a balayage is a relevant inverse problem. In this connection, there is a conjecture of the first author on the existence of *electrostatic skeletons* for every closed convex polygon (more generally, for any closed convex region with boundary consisting of line segments or circular arcs). By an electrostatic skeleton on E we mean a positive measure with closed support in E, such that its logarithmic potential matches the equilibrium potential in Ω and its support has empty interior and does not separate the plane. For example, a square region has a skeleton whose support is the union of its diagonals; the circular sector in Fig. 3 has a skeleton supported on the illustrated curve joining the three vertices. See the discussion in [9, p. 55] and in [2].

3. Applications to special polynomial sequences

We begin with some results for Bergman polynomials $\{B_n\}_{n\in\mathbb{N}}$ that are orthogonal with respect to the area measure dA over a bounded Jordan domain G; i.e.,

$$\int_G B_m(z)\overline{B_n(z)}\,dA(z) = 0, \quad m \neq n. \tag{3.1}$$

The following theorem was established in [8].

Theorem 3.1. *Let G be a bounded Jordan domain and φ a conformal map of G onto the unit disk $\mathbb{D}$. Then, there is a subsequence $\mathcal{N}$ of $\mathbb{N}$ such that*

$$\nu_{B_n} \xrightarrow{*} \mu_{\partial G}, \quad n \to \infty,\ n \in \mathcal{N} \tag{3.2}$$

if and only if φ cannot be analytically continued to some open set containing $\overline{G}$.

It is not difficult to see that this property of G is independent of the choice of the conformal map φ.

As a consequence of Corollary 2.2, we obtain a sufficient condition for (3.2) to hold for the full sequence.

Corollary 3.1. *If the Jordan domain G has a point on its boundary that satisfies the NCS condition, then (3.2) holds for $\mathcal{N} = \mathbb{N}$.*

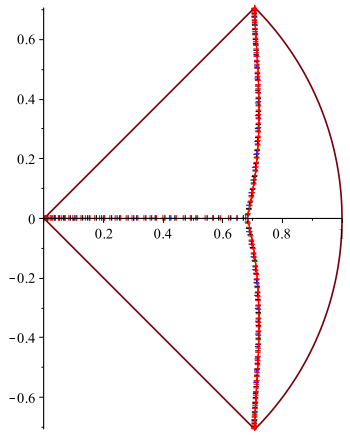

Fig. 3. Zeros of the Bergman polynomials B_n, $n = 50, 100, 150$, for the circular sector with opening angle $\pi/2$.

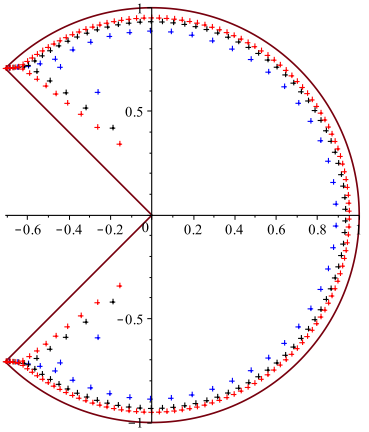

Fig. 4. Zeros of the Bergman polynomials B_n, $n = 50, 100, 150$, for the circular sector with opening angle $3\pi/2$.

In Fig. 4, we depict zeros of the Bergman polynomials for the circular sector $G := \{z : z = e^{i\theta},\ -3\pi/4 < \theta < 3\pi/4\}$. The computations of the Bergman polynomials for this sector as well as for the sector in Fig. 3 were carried out in Maple 16 with 300 significant figures, using the Arnoldi Gram–Schmidt algorithm; see [15, Section 7.4] for a discussion regarding the stability of the algorithm.

For Fig. 4 the origin is an NCS point and, therefore, Corollary 3.1 implies that the only limit distribution of the zeros is the equilibrium measure, a fact reflected in Fig. 4. It is interesting to note that this figure also depicts the facts that the density of the equilibrium measure is zero at corner points with opening angle greater than π, and it is infinite at corners with opening angle less that π.

In [4, p. 1427] the following question has been raised: If G is the union of two mutually exterior Jordan domains G_1 and G_2 whose boundaries are singular, does there exist a common sequence of integers n for which $\nu_{B_n}|_{\mathcal{V}_j}$ converges to $\mu_{\partial G}|_{\partial G_j}$, where $\mathcal{V}_j$ is an open set containing G_j, $j = 1, 2$. Thanks to Theorem 2.1, the answer is affirmative for the full sequence $\mathbb{N}$ if both G_1 and G_2 have the NCS property.

We conclude by applying the results of the main theorem to the case of Faber polynomials. For this purpose, we assume that $\Omega = \overline{\mathbb{C}} \setminus E$ is simply connected and let Φ denote the conformal map $\Omega \to \Delta := \{w : |w| > 1\}$, normalized so that near infinity

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots, \quad \gamma > 0. \tag{3.3}$$

The Faber polynomials $\{F_n(z)\}_{n=0}^{\infty}$ of E are defined as the polynomial part of the expansion of $\Phi^n(z)$ near infinity.

The following theorem was established in [5].

Theorem 3.2. *Suppose that* $\mathrm{int}(E)$ *is connected and* ∂E *is a piecewise analytic curve that has a singularity other than an outward cusp. Then, there is a subsequence* $\mathcal{N}$ *of* $\mathbb{N}$ *such that*

$$\nu_{F_n} \xrightarrow{*} \mu_E, \quad n \to \infty,\ n \in \mathcal{N}. \tag{3.4}$$

Using Theorem 2.1 we can refine the above as follows; see also the question raised in Remark 6.1(c) in [5].

Corollary 3.2. *If* $\mathrm{int}(E)$ *has a point on its boundary that satisfies the NCS condition, then* (3.4) *holds for* $\mathcal{N} = \mathbb{N}$.

4. Proof of Theorem 2.1

Let μ be any weak-star limit measure of the sequence $\{\nu_{P_n}\}_{n\in\mathcal{N}}$ and recall from Theorem 1.1 that $\mathrm{supp}(\mu) \subset \mathcal{E}$ and

$$U^{\mu}(z) = U^{\mu_E}(z), \quad z \in \Omega, \tag{4.1}$$

where

$$U^{\nu}(z) := \int \log \frac{1}{|z-t|} d\nu(t)$$

denotes the logarithmic potential on a measure ν.

We consider first the case when $E_0 = \mathcal{E}$. It suffices to show that

$$\mathrm{supp}(\mu) \subset \partial\Omega, \tag{4.2}$$

because, this in view of (4.1) and Carleson's unicity theorem [13, Theorem II.4.13] will imply the relation $\mu = \mu_E$, which yields (2.4) with $V = \mathbb{C}$. Clearly, (4.2) is satisfied automatically in the case $\mathrm{int}(E_0) = \emptyset$, so we turn our attention now to the case $\mathrm{int}(E_0) \neq \emptyset$ and assume to the contrary that $\mathrm{supp}(\mu)$ is not contained in $\partial\Omega$. Then there exists a small closed disk K belonging to some open component of $\mathrm{int}(E_0)$, such that $\mu(K) > 0$. We call this particular component G, and note that it is simply connected. Since ∂G is regular with respect to the interior and exterior Dirichlet problem,

$$\lim_{\substack{z\to t\in\partial G \\ z\in G}} g_G(z,\zeta) = 0 \quad \text{and} \quad \lim_{\substack{z\to t\in\partial G \\ z\in\overline{\mathbb{C}}\setminus\overline{G}}} g_{\overline{\mathbb{C}}\setminus\overline{G}}(z,\infty) = 0,$$

where $g_{\overline{\mathbb{C}}\setminus\overline{G}}(z,\infty)$ denotes the Green function of $\overline{\mathbb{C}} \setminus \overline{G}$ with pole at infinity.

Following Gardiner and Pommerenke (see [3, Section 5]), we set $\mu_0 := \mu|_K$ and consider the function

$$S(z) := \begin{cases} \int g_G(z,\zeta)d\mu_0(\zeta), & z \in G, \\ U^{\mu_E}(z) - \log \dfrac{1}{\mathrm{cap}(E)}, & z \in \overline{\mathbb{C}} \setminus G. \end{cases} \tag{4.3}$$

From the properties of Green functions and equilibrium potentials it follows that $S(z)$ is harmonic in $G \setminus K$ and in $\Omega \setminus \{\infty\}$, positive in G, negative or zero in $\overline{\mathbb{C}} \setminus \overline{G}$ and vanishes *quasi-everywhere* (that is, apart from a set of capacity zero) on ∂G.

Let now $\widehat{\mu}_0$ denote the balayage of μ_0 out of K onto ∂G. Then, the relation $\widehat{\mu}_0 \le \mu^b$ follows from the discussion regarding balayage onto arbitrary compact sets in [6, pp. 222–228]. Since, by Theorem 1.1, $\mu^b = \mu_E$, the difference $\mu_E - \widehat{\mu}_0$ is a positive measure, a fact leading to the following useful representation:

$$S(z) = U^{\mu_0}(z) + U^{\mu_E - \widehat{\mu}_0}(z) - \log \frac{1}{\mathrm{cap}(E)}, \quad z \in \mathbb{C}, \tag{4.4}$$

which shows that $S(z)$ is superharmonic in $\mathbb{C}$.

By assumption, the boundary of G contains an NCS point z_0. Without loss of generality, we make the following simplifications regarding the two conditions in Definition 2.1: By performing a translation and scaling we take z_0 to be the origin and by rotation we take $\zeta_0 = i\gamma$, for some $\gamma \in (0,1)$. Finally, in view of the Remark following Definition 2.1, we take $\overline{D} := \{z : |z - i\gamma| \le \gamma\}$ and choose γ so that $\overline{D}$ is a subset of the open unit disk $\mathbb{D}$ and $\overline{D} \cap K = \emptyset$.

The contradiction we seek will be a consequence of the following two claims:

Claim (a) $$\int_{\pi}^{2\pi} \frac{S(re^{i\theta})}{r} d\theta \ge \tau, \quad \text{as } r \to 0+,$$

where τ is a negative constant and

Claim (b) $$\int_{0}^{\pi} \frac{S(re^{i\theta})}{r} d\theta \to +\infty, \quad \text{as } r \to 0+.$$

These claims follow as in [3], utilizing in the justification of Claim (b) the essential condition that the origin is an NCS point so that

$$\lim_{y \to 0+} \frac{g_G(iy, i\gamma)}{y} = +\infty. \tag{4.5}$$

Note that for small positive y the definition of $S(z)$ gives

$$\frac{S(iy)}{y} = \frac{1}{y} \int g_G(iy,\zeta) d\mu_0(\zeta). \tag{4.6}$$

Using (2.2) we have $g_G(iy,\zeta) \ge C g_G(iy, i\gamma)$ for any $\zeta \in K$ and small y, which in view of (4.5) leads to the limit

$$\lim_{y \to 0+} \frac{S(iy)}{y} = +\infty. \tag{4.7}$$

Using Claims (a) and (b) and the fact that $S(0) = 0$ (since the origin is a regular point of Ω), it is easy to arrive at a relation that contradicts the mean value inequality for superharmonic

functions (see also [3, p. 425]):

$$\begin{aligned}\frac{1}{r}\left(\frac{1}{2\pi}\int_0^{2\pi} S(re^{i\theta})d\theta - S(0)\right) &= \frac{1}{2\pi r}\int_0^{2\pi} S(re^{i\theta})d\theta \\ &\geq \frac{1}{2\pi r}\int_0^{\pi} S(re^{i\theta})d\theta + \tau \\ &\to \infty, \quad r \to +0.\end{aligned}$$

This establishes the theorem for the case $E_0 = \mathcal{E}$.

To conclude the proof, we observe that when $\operatorname{dist}(E_0,\ \mathcal{E} \setminus E_0) > 0$ our arguments above show that μ cannot have any point of its support in the interior of any open component of $E_0 \cap V$; hence it is supported on the outer boundary of E_0 inside V. Therefore, by following the proof of Theorem II.4.13 in [13], we see that the logarithmic potentials of μ and μ_E coincide in V and the required relation $\mu|_V = \mu_E|_{E_0}$ follows from the unicity theorem for logarithmic potentials. □

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