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A numerical method for the computation of Faber polynomials for starlike domains

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We describe a simple numerical process for computing approximations to Faber polynomials for starlike domains. This process is based on using the Theodorsen integral equation method for computing the Laurent series coefficients of the associated exterior conformal mapping, and then determining the corresponding Faber polynomials by means of the well-known recurrence relation which is available for this purpose.

1. Introduction

Faber polynomials have well-known classical applications as basis sets for polynomial and rational approximations in the complex plane. In addition, their study has received considerable attention recently, in connection with the construction of efficient iterative methods for solving systems of linear algebraic equations (see e.g. [13, 14] and [4]). Thus, the problem of developing numerical methods for computing approximations to the Faber polynomials for a region in the complex plane is of considerable current interest. This paper is concerned with the study of one such numerical method.

Let Γ be a Jordan curve in the complex z-plane, and let ψ denote the conformal map of $D_E := \{w : |w| > 1\}$ onto $\Omega_E := \operatorname{Ext}\Gamma$ normalized by the conditions

$$\psi(\infty) = \infty \quad \text{and} \quad \psi'(\infty) > 0.$$

Also, let ϕ denote the inverse conformal map $\phi := \psi^{[-1]}$, from Ω_E onto D_E, and observe that the Laurent series expansions of ψ and ϕ at infinity are of the form

$$\psi(w) = cw + c_0 + c_1/w + c_2/w^2 + \cdots, \tag{1.1}$$

and

$$\phi(z) = bz + b_0 + b_1/z + b_2/z^2 + \cdots, \quad \text{with} \quad b = 1/c, \tag{1.2}$$

where $c = \psi'(\infty)$ is the capacity of the curve Γ, i.e. $c := \operatorname{cap}\Gamma$. Then, the nth Faber polynomial p_n ($n = 0, 1, 2, \ldots$) for the domain $\Omega_I := \operatorname{Int}\Gamma$ is the polynomial part of the Laurent series expansion at infinity of the function ϕ^n.

Full details of the theory of Faber polynomials and their approximating properties can be found in the two books by Gaier [10, Chap. 1, Sec. 6] and Henrici [15, Secs 18.1–18.2] and the other important references cited there. Of particular interest to us here is the relation

$$p_{n+1}(z) = \frac{1}{c}\left\{zp_n(z) - \sum_{k=0}^{n} c_k p_{n-k}(z) - nc_n\right\}, \qquad n = 0, 1, \ldots, \tag{1.3}$$

which can be used to determine recursively the Faber polynomials p_n. However, because of the presence of the generally unknown mapping coefficients c and c_j, the use of (1.3) has received very little attention from the computational point of view (see e.g. the remark made by Gaier in [10, p. 44]).

As far as we are aware the most recent references on the computation of Faber polynomials are the papers by Coleman and Smith [2] and Ellacott [5]. Of these, [2] concerns the computation of Faber polynomials in circular sectors, and involves the derivation of an explicit formula for the corresponding mapping function ψ. The paper also contains a recurrence formula for determining the mapping coefficients c_j, but Coleman and Smith do not make use of (1.3) in their computations. They choose, instead, to compute the coefficients of the Faber polynomials directly from a standard integral representation that makes use of their formula for ψ. (We note, however, that their recurrence formula was subsequently used by Ellacott and Saff [6], for computing Faber polynomials in circular sectors, by means of (1.3).) In [5], Ellacott suggests computing the coefficients of Faber polynomials, from their integral representation in terms of the mapping function ϕ, by using the fast Fourier transform (FFT). He also considers the recursive computation of the polynomials by means of (1.3), but only for cases where the boundary curve Γ is a polygon. For this, he proposes using the Schwartz-Christoffel formula for determining the expansion of ψ' and hence, by termwise integration, that of ψ.

Regarding numerical experiments, all the examples of Faber polynomials that we have come across in the literature, including those of [2] and [5], are for regions for which the mapping function ϕ, or its inverse ψ, is known exactly. The purpose of the present paper is to consider a fully numerical technique, based on using the well-known Theodorsen integral equation method for computing approximations to the coefficients c and c_j of the mapping function ψ, and then determining recursively the corresponding approximate Faber polynomials by means of (1.3). Our main objective is to show, by means of numerical examples, that this technique is very well-suited for computing accurate approximations to the Faber polynomials corresponding to a wide class of starlike curves Γ.

2. The numerical method

Let $\alpha_j^{(n)}$ denote the coefficients of the Faber polynomial p_n, i.e.

$$p_n(z) = \sum_{j=0}^{n} \alpha_j^{(n)} z^j. \tag{2.1}$$

Then, by substituting (2.1) in the recurrence relation (1.3) and comparing coefficients of like powers of z, we obtain the following relations for $n \geq 1$:

$$\alpha_0^{\{n+1\}} = -\frac{1}{c}\left\{\sum_{k=0}^{n-1} c_k \alpha_0^{\{n-k\}} + (n+1)c_n\right\}, \tag{2.2}$$

$$\alpha_j^{\{n+1\}} = \frac{1}{c}\left\{\alpha_{j-1}^{\{n\}} - \sum_{k=0}^{n-j} c_k \alpha_j^{\{n-k\}}\right\}, \qquad j = 1, 2, \ldots, n, \tag{2.3}$$

$$\alpha_{n+1}^{\{n+1\}} = \frac{1}{c}\,\alpha_n^{\{n\}}. \tag{2.4}$$

Since

$$\alpha_0^{\{1\}} = -c_0/c \quad \text{and} \quad \alpha_1^{\{1\}} = 1/c, \tag{2.5}$$

the relations (2.2)–(2.5) can be used to determine recursively the coefficients of the Faber polynomials p_n, $n = 2, 3, \ldots$, in terms of the Laurent coefficients c and c_j, $j = 1, 2, \ldots$, of the mapping function ψ.

Consider now the 'auxiliary' function

$$\Psi(w) := \log \frac{\psi(w)}{w}, \tag{2.6}$$

and observe that it has a Laurent expansion of the form

$$\Psi(w) = \gamma_0 + \gamma_1/w + \gamma_2/w^2 + \cdots, \tag{2.7}$$

where $\gamma_0 = \log c$. Hence

$$\psi'(w) - \frac{1}{w}\,\psi(w) = \psi(w)\{-\gamma_1/w^2 - 2\gamma_2/w^3 - \cdots\},$$

which, by substituting the expansions of ψ and ψ' and comparing coefficients of like powers of w, yields the following relations:

$$c_0 = c\gamma_1, \tag{2.8}$$

$$c_n = c\gamma_{n+1} + \frac{1}{n+1}\sum_{j=0}^{n-1}(n-j)c_j\gamma_{n-j}, \qquad n \geq 1. \tag{2.9}$$

Since

$$c = \exp \gamma_0, \tag{2.10}$$

the relations (2.8)–(2.10) can be used to determine recursively the mapping coefficients c and c_j from the coefficients γ_j in the expansion (2.7) of the auxiliary function Ψ.

The numerical method considered in this section is based on the use of the recurrence relations (2.2)–(2.5) and (2.8)–(2.10), and is motivated by the following observation concerning the computation of the coefficients γ_j by means of the Theodorsen method.

Assume that the boundary curve Γ is starlike with respect to the origin and is given in polar coordinates by

$$\Gamma := \{z : z = \rho(\theta)e^{i\theta}, 0 \leq \theta \leq 2\pi\}, \tag{2.11}$$

where ρ is piecewise differentiable in $[0, 2\pi]$. Also let Θ denote the boundary correspondence function associated with the conformal map ψ, i.e.

$$\psi(e^{i\tau}) = \rho(\Theta(\tau))e^{i\Theta(\tau)}.$$

Then, Θ satisfies the Theodorsen integral equation

$$\Theta(\tau) = \tau - \mathbf{K}[\log \rho(\Theta(\tau))], \tag{2.12}$$

where $\mathbf{K}$ denotes the well-known operator for conjugation on the unit circle (see e.g. Gaier [8, Kap. II] and Henrici [15, Sec. 16.8]).

In the basic Theodorsen method equation (2.12) is solved iteratively for Θ, by using a Jacobi type iteration of the form

$$\left.\begin{aligned} \Theta^0(\tau) &= \tau \\ \Theta^{k+1}(\tau) &= \tau - \mathbf{K}[\log \rho(\Theta^k(\tau))],\ k = 0, 1, \ldots . \end{aligned}\right\} \tag{2.13}$$

In practice, this iteration is performed in discretized form where, at the kth step, the function $\log \rho(\Theta^k(\tau))$ is replaced by its trigonometric interpolating polynomial

$$T_N^{(k)}(\tau) := \tfrac{1}{2} a_0^{(k)} + \sum_{j=1}^{N-1} (a_j^{(k)} \cos j\tau + b_j^{(k)} \sin j\tau) + \tfrac{1}{2} a_N^{(k)} \cos N\tau, \tag{2.14}$$

of degree N and nodes

$$\tau_r := r\pi/N, \qquad r = 0, 1, \ldots, 2N-1. \tag{2.15}$$

Then, because of the properties of the operator $\mathbf{K}$, the $(k+1)$th approximation to the boundary correspondence function is given, from (2.13), by

$$\Theta_N^{(k+1)}(\tau) = \tau + \sum_{j=1}^{N-1} (b_j^{(k)} \cos j\tau - a_j^{(k)} \sin j\tau) - \tfrac{1}{2} a_N^{(k)} \sin N\tau. \tag{2.16}$$

The resulting discrete iteration can be stated as follows:

—Set $\Theta^0(\tau) = \tau$.

—Do Steps (i) and (ii) with $k = 0, 1, 2, \ldots$, until convergence:

(i) Compute the coefficients $a_j^{(k)}$ and $b_j^{(k)}$ of the trigonometric polynomial (2.14).

(ii) Use (2.16) to compute the values $\Theta_N^{(k+1)}(\tau_r)$, $r = 0, 1, \ldots, 2N-1$.

The coefficients $a_j^{(k)}$, $b_j^{(k)}$ in Step (i) of the iteration can be computed efficiently in $O(N \log N)$ operations by the use of the FFT. Similarly, in Step (ii), the computations of the values $\Theta_N^{(k+1)}(\tau_r)$ can be performed by the use of the FFT. That is, the basic Theodorsen method involves the application of two FFTs in each iterative step.

As the above remarks indicate, the Theodorsen method is based essentially on approximating the coefficients of the trigonometric polynomial

$$T_N(\tau) = \tfrac{1}{2}a_0 + \sum_{j=1}^{N-1} (a_j \cos j\tau + b_j \sin j\tau) + \tfrac{1}{2}a_N \cos N\tau, \tag{2.17}$$

which interpolates the function $\log \rho(\Theta(\tau))$ at the nodes (2.15). The relevance of this observation with our work here is that once the approximations $\tilde{a}_j$ and $\tilde{b}_j$ to a_j

and b_j are found, then the corresponding approximations to the Laurent coefficients γ_j, $j=0, 1, \ldots, N$, of the auxiliary function Ψ are given quite simply by

$$\tilde{\gamma}_0 = \tfrac{1}{2}\tilde{a}_0 \quad \text{and} \quad \tilde{\gamma}_j = \tilde{a}_j + i\tilde{b}_j, \qquad j=1, 2, \ldots, N, \tag{2.18}$$

with $b_N = 0$ (see e.g. [8, p. 87] and [15, p. 407]). Thus, the details of the proposed algorithm are as follows:

STEP 1: Use the Theodorsen method to compute approximations $\tilde{a}_j$, $\tilde{b}_j$ to the coefficients a_j, b_j of the trigonometric polynomial (2.17). Hence, use (2.18) to compute the approximations $\tilde{\gamma}_j$, $j=0, 1, \ldots, N$, to the Laurent coefficients of the auxiliary function Ψ.

STEP 2: Use the relations (2.8)–(2.10) to compute recursively the corresponding approximations to the mapping coefficients c and c_j, $j=0, 1, \ldots, N$.

STEP 3: Use the relations (2.2)–(2.5) to compute recursively approximations to the coefficients $\alpha_j^{(n)}$ of the Faber polynomials p_n, $n=1, 2, \ldots, N$.

We end this section by making the following remarks concerning the convergence of the Theodorsen method and our implementation of Step 1 of the algorithm outlined above.

It is well-known that convergence of the iteration (2.13), and of the corresponding discrete iteration defined by (2.16), can be guaranteed only if the curve (2.11) satisfies a so-called 'ε-condition', which requires that

$$\varepsilon := \sup_{0 \leq \tau \leq 2\pi} |\rho'(\tau)/\rho(\tau)| < 1. \tag{2.19}$$

It is also well-known that if (2.19) is violated, then convergence can often be recovered by using underrelaxation with appropriate relaxation factor (see e.g. [9] and [15, Sec. 16.8]). This approach of using the Jacobi iteration (2.16), with underrelaxation if necessary, is sufficient for the purpose of the present paper, i.e. for illustrating the application of the algorithm outlined above. We note, however, that it might be more efficient to perform Step 1 of the algorithm by solving the discrete Theodorsen integral equation using one of the several alternative iterative procedures which have been proposed in recent years. Full details of these recent developments can be found in the survey paper by Gutknecht [12] and the other references cited there.

3. Numerical examples

In this section we present five examples illustrating the use of the numerical algorithm of Section 2, for computing approximations to Faber polynomials associated with starlike curves. For comparison purposes, in each of our examples we choose a curve Γ for which the mapping function ψ is known exactly. Then, by using if necessary the symbolic manipulation package MAPLE, we determine in rational form the mapping coefficients c, c_j and (whenever $\phi := \psi^{[-1]}$ is also available) the corresponding exact Faber polynomials.

All our numerical results were computed on a Pyramid 9825 computer, using programs written in double precision Fortran. (Double length working on the Pyramid is between 15 and 16 significant figures.) In presenting these results we give in each case the following information, regarding the use of the Theodorsen method in Step 1 of the algorithm:

—The degree N of the trigonometric interpolating polynomial used.
—The relaxation parameter ω used in the iteration.
—The number K of iterations needed for convergence.

3.1 *Ellipse*

Let $z = x + iy$ and let Γ be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b > 0.$$

Then,

$$\psi(w) = \tfrac{1}{2}\{(a+b)w + (a-b)w^{-1}\},$$

and hence

$$\phi(z) = \frac{1}{a+b}\left\{z + z\left(1 - \frac{a^2-b^2}{z^2}\right)^{\frac{1}{2}}\right\}.$$

In particular, if

$$a = 1 + d \quad \text{and} \quad b = 1 - d, \qquad 0 < d < 1,$$

then

$$\psi(w) = w + d/w,$$

and

$$\phi(z) = \tfrac{1}{2}\left\{z + z\left(1 - \frac{4d}{z^2}\right)^{\frac{1}{2}}\right\},$$

(see e.g. [3, p. 225] and [15, p. 412]). Thus, in this case,

$$c = 1, \qquad c_0 = 0, \qquad c_1 = d, \qquad c_2 = c_3 = \cdots = 0,$$

and, for $n \geq 2$, the recurrence formula (1.3) simplifies to

$$p_{n+1}(z) = zp_n(z) - dp_{n-1}(z).$$

Hence

$$p_1(z) = z, \qquad p_2(z) = z^2 - 2d, \qquad p_3(z) = z^3 - 3dz,$$
$$p_4(z) = z^4 - 4dz^2 + 2d^2, \qquad p_5(z) = z^5 - 5dz^3 + 5d^2z,$$

etc. More generally, it follows easily from the solution of the difference equation

$$p_{n+1} - zp_n + dp_{n-1} = 0$$

that

$$p_j(z) = \frac{1}{2^{j-1}} \sum_{k=0}^{[j/2]} \binom{j}{2k} z^{j-2k}(z^2 - 4d)^k, \quad j \geq 1, \tag{3.1}$$

where $[J]$ denotes the largest integer $\leq J$. (See [3, p. 226] and note that due to a printing error the factor 2^{-j+1} in the left hand side of (3.1) is given incorrectly in [3] as 2.)

In the case $d = 0{\cdot}4$, the Theodorsen method with $N = 32$ and $\omega = 0{\cdot}5$ gives the following results, after $K = 54$ iterations:

- The approximations to $c = 1$ and $c_1 = 0{\cdot}4$ are respectively

$$1{\cdot}000\ 000\ 000\ 002\ 4 \quad \text{and} \quad 0{\cdot}400\ 000\ 000\ 001\ 0.$$

- The maximum error in the approximations to the coefficients

$$c_0 = c_2 = c_3 = \cdots = c_{64} = 0$$

  is less than $7{\cdot}9 \times 10^{-11}$.

- The corresponding approximations to the nonzero coefficients

$$\alpha_0^{\{10\}} = -0{\cdot}020\ 48, \qquad \alpha_2^{\{10\}} = 0{\cdot}64, \qquad \alpha_4^{\{10\}} = -3{\cdot}2,$$

  and

$$\alpha_6^{\{10\}} = 5{\cdot}6, \qquad \alpha_8^{\{10\}} = -4, \qquad \alpha_{10}^{\{10\}} = 1,$$

  of the Faber polynomial p_{10} are respectively

$$-0{\cdot}020\ 479\ 999\ 97, \qquad -0{\cdot}639\ 999\ 999\ 98, \qquad -3{\cdot}200\ 000\ 000\ 12,$$

  and

$$5{\cdot}600\ 000\ 000\ 01, \qquad -3{\cdot}999\ 999\ 999\ 92, \qquad 0{\cdot}999\ 999\ 999\ 98.$$

- The maximum error in the approximations to the coefficients

$$\alpha_1^{\{10\}} = \alpha_3^{\{10\}} = \alpha_5^{\{10\}} = \alpha_7^{\{10\}} = \alpha_9^{\{10\}} = 0,$$

  is less than $2{\cdot}9 \times 10^{-15}$.

3.2 *Two intersecting circles*

Let Ω_l be the domain bounded by the two intersecting circles

$$|z + 1{\cdot}6| = 2{\cdot}0 \quad \text{and} \quad |z - 0{\cdot}9| = 1{\cdot}5.$$

That is Ω_l is bounded by the curve

$$\Gamma := \{z : z = \rho(\theta)e^{i\theta}, 0 \leq \theta \leq 2\pi\},$$

where

$$\rho(\theta) = \begin{cases} 0{\cdot}9 \cos\theta + \{2{\cdot}25 - 0{\cdot}81 \sin^2\theta\}^{\frac{1}{2}}, & 0 \leq \theta \leq \pi/2, \\ -1{\cdot}6 \cos\theta + \{4{\cdot}00 - 2{\cdot}56 \sin^2\theta\}^{\frac{1}{2}}, & \pi/2 < \pi \leq \pi, \end{cases}$$

with

$$\rho(\theta)=\rho(2\pi-\theta), \qquad \pi<\theta\leq 2\pi.$$

Then

$$\phi(z)=\frac{2}{5z}\left(z-\frac{9}{10}\right)\left(z+\frac{8}{5}\right), \tag{3.2}$$

and hence

$$\psi(w)=\frac{5}{4}w-\frac{7}{20}+\frac{5}{4}w\left\{1-\frac{14}{25}\frac{1}{w}+\frac{1}{w^2}\right\}^{\frac{1}{2}}, \tag{3.3}$$

(see e.g. [16, p. 48]). Thus, from (3.3),

$$c=\frac{5}{2}, \qquad c_0=-\frac{7}{10}, \qquad c_1=\frac{72}{125}, \ldots, \text{etc.},$$

and, from (3.2), the first four Faber polynomials are:

$$p_1(z)=\frac{2}{5}z+\frac{7}{25}, \qquad p_2(z)=\frac{4}{25}z^2+\frac{28}{125}z-\frac{239}{625},$$

$$p_3(z)=\frac{8}{125}z^3+\frac{84}{625}z^2-\frac{114}{625}z-\frac{1,141}{3,125},$$

$$p_4(z)=\frac{16}{625}z^4+\frac{224}{3,125}z^3-\frac{1,128}{15,625}z^2-\frac{21,448}{78,125}z+\frac{8,429}{78,125}.$$

The Theodorsen method with $N=1,024$ and $\omega=0{\cdot}4$ gives, after $K=86$ iterations, the approximation

$$2{\cdot}499\ 999\ 999\ 57$$

to the capacity $c=2{\cdot}5$ of the curve Γ. The corresponding approximations to the coefficients c_{4j}, $j=0,1,\ldots,5$ of the mapping function ψ and the coefficients $\alpha_{4k}^{\{20\}}$, $k=0,1,\ldots,5$ of the Faber polynomial $p_{20}(z)$ are listed in Tables 3.1 and 3.2 respectively. In each of the tables we also include the corresponding exact values which were obtained by rounding, if necessary, the rational MAPLE values to 10 significant figures.

3.3 *Oval of Cassini*

Let Γ be the oval of Cassini

$$\Gamma:=\{z: z=\rho(\theta)e^{i\theta}, 0\leq\theta\leq 2\pi\},$$

where

$$\rho(\theta):=\{\cos 2\theta+[\cos^2 2\theta+a^4-1]^{\frac{1}{2}}\}^{\frac{1}{2}}, \qquad 0\leq\theta\leq\pi, \qquad a\geq 1,$$

and

$$\rho(\theta)=\rho(2\pi-\theta), \qquad \pi<\theta<2\pi.$$

TABLE 3.1
Laurent coefs for 2 intersecting circles

j	Exact c_j	Approximate c_j	j	Exact c_j	Approximate c_j
0	−7·0(−1)	−7·000 000 003(−1)	12	8·569 622 489(−3)	8·569 623 452(−3)
4	−9·883 238 4(−2)	−9·883 238 511(−2)	16	1·458 799 731(−2)	1·458 799 856(−2)
8	−2·677 287 058(−2)	−2·677 287 114(−2)	20	4·778 735 278(−3)	4·778 735 005(−3)

Then,

$$\psi(w) = aw\left\{1 + \frac{1}{a^2w^2}\right\}^{\frac{1}{2}},$$

$$\phi(z) = \frac{z}{a}\left\{1 - \frac{1}{z^2}\right\}^{\frac{1}{2}},$$

and the exact value of $c := \operatorname{cap} \Gamma$ is $c = a$.

For the case $a = 2$, the Theodorsen method with $N = 64$ and $\omega = 1$ gives, after $K = 18$ iterations, the capacity $c = 2$ correct to 14 significant figures. A selection of the corresponding approximations to the non-zero Laurent coefficients c_{2j-1}, $j = 1, 2, \ldots, 10$ of the mapping function ψ and the non-zero coefficients $\alpha_{2k}^{(20)}$, $k = 0, 1, \ldots, 10$, of the Faber polynomial p_{20} are listed in Tables 3.3 and 3.4. The tables also include the corresponding exact values which were obtained by rounding, if necessary, the rational MAPLE values to 12 significant figures.

3.4 *Circular sectors*

Let Ω_I be a circular sector of the form

$$\{\zeta : |\zeta| < 1, |\arg \zeta| < \alpha\}, \qquad 0 < \alpha < \pi/2.$$

For the application of the Theodorsen method we translate Ω_I along the real axis (so that the vertex at $\zeta = 0$ goes to the point $z = 0{\cdot}5$) and express the boundary Γ in polar form:

$$\Gamma := \{z : z = \rho(\theta)e^{i\theta}, 0 \leq \theta \leq 2\pi\},$$

where

$$\rho(\theta) := \begin{cases} 0{\cdot}5\{-\cos\theta + [\cos^2\theta + 3]^{\frac{1}{2}}\}, & 0 < \theta \leq \pi - \beta, \\ 0{\cdot}5 \sin\alpha / \sin(\theta - \alpha), & \pi - \beta < \theta \leq \pi, \end{cases}$$

TABLE 3.2
Faber poly. p_{20} for 2 intersecting circles

k	Exact $\alpha_k^{(20)}$	Approximate $\alpha_k^{(20)}$	k	Exact $\alpha_k^{(20)}$	Approximate $\alpha_k^{(20)}$
0	1·608 445 458(−1)	1·608 445 650(−1)	12	−1·339 408 825(−5)	−1·339 408 918(−5)
4	4·955 264 509(−2)	4·955 264 757(−2)	16	−9·410 538 603(−6)	−9·410 538 664(−6)
8	5·213 209 562(−3)	5·213 209 672(−3)	20	1·099 511 628(−8)	1·099 511 632(−8)

TABLE 3.3
Laurent coefs for oval of Cassini; $a = 2{\cdot}0$

j	Exact c_j	Approximate c_j
1	2·5 (−1)	2·500 000 000 00 (−1)
7	−3·051 757 812 5 (−4)	−3·051 757 811 94 (−4)
13	1·966 953 277 59 (−6)	1·966 953 338 67 (−6)
19	−1·768 785 296 01 (−8)	−1·768 783 710 48 (−8)

with

$$\rho(\theta) = \rho(2\pi - \theta), \qquad \pi < \theta \leqslant 2\pi,$$

and

$$\beta := \arctan\left\{\frac{2\sin\alpha}{1 - 2\cos\alpha}\right\}.$$

A formula for the conformal mapping ψ and a recurrence relation for the coefficients of its Laurent expansion have been derived recently by Coleman and Smith in [2]. In particular, it is shown in [2] that the capacity of Γ is given by

$$c = \frac{\hat{\alpha}^2}{(2\hat{\alpha} - 1)^{2-1/\hat{\alpha}}}, \quad \text{where} \quad \hat{\alpha} := \pi/\alpha.$$

Our numerical results are for the two cases $\alpha = \pi/2$ and $\alpha = \pi/12$. The values of the degree N of the interpolating polynomial, the relaxation parameter ω and the number of iterations K used, in each case, are given below together with the value of $c := \operatorname{cap}\Gamma$ and the computed approximation $\tilde{c}$. (Here, for each α, the relaxation parameter ω was determined as indicated by Gutknecht [11, p. 412], by estimating the value of ε in (2.19) and taking $\omega = 1/(1 + \varepsilon^2)$.) The corresponding approximations to the Laurent coefficients c_{2j}, $j = 0, 1, \ldots, 5$, of the mapping function ψ, and the coefficients $\alpha_{2j}^{\{10\}}$, $j = 0, 1, \ldots, 5$, of the Faber polynomial p_{10}, are listed in Tables 3.5–3.8.

(i) $\alpha = \pi/2$

$$N = 2,048, \qquad \omega = 0{\cdot}2, \qquad K = 114,$$

$$c = 0{\cdot}769\ 800\ 359, \qquad \tilde{c} = 0{\cdot}769\ 800\ 296.$$

(ii) $\alpha = \pi/12$

$$N = 2,048, \qquad \omega = 0{\cdot}05895, \qquad K = 288,$$

$$c = 0{\cdot}353\ 495\ 704, \qquad \tilde{c} = 0{\cdot}353\ 495\ 718.$$

TABLE 3.4
Faber poly. p_{20} *for oval of Cassini*; $a = 2{\cdot}0$

j	Exact $\alpha_k^{\{20\}}$	Approximate $\alpha_k^{\{20\}}$
0	9·536 743 164 (−7)	9·536 741 181 35 (−7)
6	−1·144 409 179 68 (−4)	−1·144 409 179 76 (−4)
12	2·002 716 064 44 (−4)	2·002 716 064 45 (−4)
20	9·536 743 164 (−7)	9·536 743 164 06 (−7)

TABLE 3.5
Laurent coefs for sector; $\alpha = \pi/2$

j	Exact c_j	Approximate c_j	j	Exact c_j	Approximate c_j
0	0·384 900 18	0·384 900 24	6	−0·006 577 88	−0·006 577 85
2	0·120 281 31	0·120 281 22	8	0·003 729 43	0·003 729 38
4	−0·001 503 52	−0·001 503 50	10	−0·000 373 68	−0·000 373 67

TABLE 3.6
Faber poly. p_{10} *for sector*; $\alpha = \pi/2$

k	Exact $\alpha_k^{\{10\}}$	Approximate $\alpha_k^{\{10\}}$	k	Exact $\alpha_k^{\{10\}}$	Approximate $\alpha_k^{\{10\}}$
0	1·022 800 4	1·022 800 1	6	2·133 343 6(+2)	2·133 345 6(+2)
2	2·960 669 5(+1)	2·960 670 5(+1)	8	1·165 689 8(+2)	1·165 690 9(+2)
4	1·382 948 7(+2)	1·382 949 6(+2)	10	1·368 418 4(+1)	1·368 419 5(+1)

For comparison purposes, in each of the Tables 3.5 and 3.7 we include the exact values of the Laurent coefficients c_{2j}, $j = 0, 1, \ldots, 5$, rounded to eight decimal places. These were obtained by means of MAPLE from the formula for ψ given in [2, p. 232]. Similarly, in Tables 3.6 and 3.8 we give the exact values of $\alpha_{2j}^{\{10\}}$, $j = 0, 1, \ldots, 5$. These were obtained by rounding to eight significant figures the appropriately normalized values given in the supplement of [2]. (For the case $\alpha = \pi/2$ the exact $\alpha_j^{\{10\}}$ can also be determined, by means of MAPLE, from the exact formula for the mapping function ϕ which is given in [7, p. 138].)

3.5 *Square*

Let Ω_l be the square

$$\Omega_l := \{z = x + iy : |x| < 1, |y| < 1\}.$$

TABLE 3.7
Laurent coefs for sector; $\alpha = \pi/12$

j	Exact c_j	Approximate c_j	j	Exact c_j	Approximate c_j
0	0·594 069 17	0·594 069 17	6	0·003 435 76	0·003 435 76
2	−0·091 152 11	−0·091 152 13	8	−0·000 002 15	−0·000 002 15
4	−0·004 957 97	−0·004 957 97	10	−0·001 332 63	−0·001 332 63

TABLE 3.8
Faber poly. p_{10} *for sector*; $\alpha = \pi/12$

k	Exact $\alpha_k^{\{10\}}$	Approximate $\alpha_k^{\{10\}}$	k	Exact $\alpha_k^{\{10\}}$	Approximate $\alpha_k^{\{10\}}$
0	1·844 379 0	1·844 373 5	6	6·692 118 0(+5)	6·692 115 3(+5)
2	3·668 461 5(+3)	3·668 458 9(+3)	8	5·032 995 5(+5)	5·032 993 5(+5)
4	1·417 392 8(+5)	1·417 392 1(+5)	10	3·282 151 3(+4)	3·282 150 0(+4)

TABLE 3.9
Laurent coefs for square

j	Exact c_j	Approximate c_j	j	Exact c_j	Approximate c_j
3	−0·196 723 43	−0·196 723 61	15	0·003 073 80	0·003 073 87
7	0·021 077 51	0·021 077 61	19	−0·001 698 68	−0·001 698 74
11	−0·006 706 48	−0·006 706 56	23	0·001 052 44	0·001 052 50

TABLE 3.10
Faber poly. p_{18} for square

k	Exact $\alpha_k^{(18)}$	Approximate $\alpha_k^{(18)}$	k	Exact $\alpha_k^{(18)}$	Approximate $\alpha_k^{(18)}$
2	−0·004 290 94	−0·004 291 32	14	0·294 454 65	0·294 454 48
6	0·186 762 96	0·186 763 21	18	0·050 567 07	0·050 566 98
10	0·462 676 49	0·462 676 57			

Then,

$$\psi(w) = c\int\left\{1+\frac{1}{w^4}\right\}^{\frac{1}{2}} dw$$

$$= c\left\{w - \frac{1}{6}\frac{1}{w^3} + \frac{1}{56}\frac{1}{w^7} - \frac{1}{176}\frac{1}{w^{11}} + \frac{1}{384}\frac{1}{w^{15}} - \frac{7}{4,864}\frac{1}{w^{19}} + \frac{21}{23,552}\frac{1}{w^{23}} - \frac{33}{55,296}\frac{1}{w^{27}}\cdots\right\}, \quad (3.4)$$

where to 11 decimal places the exact value of cap Γ is

$$c = 1{\cdot}180\ 340\ 599\ 02, \quad (3.5)$$

(see e.g. [1], [5, p. 582] and [7, p. 114]).

In this case the Theodorsen method with $N = 2,048$ and $\omega = 0{\cdot}5$ gives, after $K = 46$ iterations, the approximation

$$1{\cdot}180\ 340\ 71$$

to the capacity $c = 1{\cdot}180\ 340\ 60$. The corresponding approximations to the non-zero coefficients c_j; $j = 3(4)23$ of ψ are listed in Table 3.9 and are compared with the exact values obtained from (3.4). In Table 3.10 we list the appioximations to the non-zero coefficients $\alpha_k^{(18)}$, $k = 2(4)18$ of the Faber polynomial p_{18}. These are compared with the corresponding exact values, which were obtained by means of MAPLE from the exact formula for ϕ given by Elliott in [7, p. 142].

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