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CONSTRUCTIVE APPROXIMATION

A Domain Decomposition Method for Conformal Mapping onto a Rectangle

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Abstract. Let g be the function which maps conformally a rectangle R onto a simply connected domain G so that the four vertices of R are mapped respectively onto four specified points z_1, z_2, z_3, z_4 on ∂G. This paper is concerned with the study of a domain decomposition method for computing approximations to g and to an associated domain functional in cases where: (i) G is bounded by two parallel straight lines and two Jordan arcs. (ii) The four points z_1, z_2, z_3, z_4, are the corners where the two straight lines meet the two arcs.

1. Introduction

Let G be a simply connected Jordan domain in the complex z-plane ($z = x + iy$), and consider a system consisting of G and four distinct points z_1, z_2, z_3, z_4, in counterclockwise order on its boundary ∂G. Such a system is said to be a quadrilateral Q and is denoted by

$$Q := \{G; z_1, z_2, z_3, z_4\}.$$

The conformal module $m(Q)$ of Q is defined as follows:

Let R be a rectangle of the form

$$R := \{(\xi, \eta): a < \xi < b, c < \eta < d\}, \tag{1.1}$$

in the w-plane ($w = \xi + i\eta$), and let h denote its aspect ratio, i.e.,

$$h := (d - c)/(b - a).$$

Then $m(Q)$ is the unique value of h for which Q is conformally equivalent to a rectangle of the form (1.1), in the sense that for $h = m(Q)$ and for this value only there exists a unique conformal map $R \to G$ which takes the four corners $a + ic, b + ic, b + id$, and $a + id$ of R respectively onto the four points z_1, z_2, z_3, and z_4. In particular, with $h = m(Q)$, Q is conformally equivalent to a rectangle of the form

$$R_h\{\alpha\} := \{(\xi, \eta): 0 < \xi < 1, \alpha < \eta < \alpha + h\}. \tag{1.2}$$

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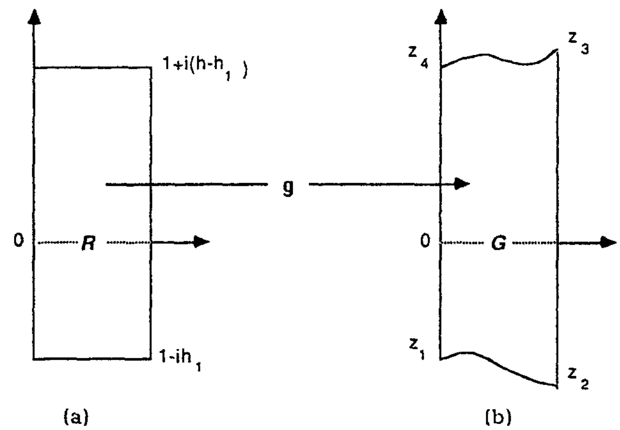

Fig. 1.1

Consider now the case where Q is of the form illustrated in Fig. 1.1(b) and let the arcs (z_1, z_2) and (z_3, z_4) have cartesian equations $y = \tau_1(x)$ and $y = \tau_2(x)$, where $\tau_j, j = 1, 2$, are positive in $[0, 1]$. That is, let

(1.3a) $$Q := \{G; z_1, z_2, z_3, z_4\},$$

where

(1.3b) $$G := \{(x, y): 0 < x < 1, -\tau_1(x) < y < \tau_2(x)\},$$

with

$$\tau_j(x) > 0, \qquad j = 1, 2 \quad \text{for } x \in [0, 1],$$

and

(1.3c) $$z_1 = -i\tau_1(0), \qquad z_2 = 1 - i\tau_1(1),$$
$$z_3 = 1 + i\tau_2(1), \qquad z_4 = i\tau_2(0).$$

Also let

(1.4a) $$G_1 := \{(x, y): 0 < x < 1, -\tau_1(x) < y < 0\}$$

and

(1.5a) $$G_2 := \{(x, y): 0 < x < 1, 0 < y < \tau_2(x)\},$$

so that $\bar{G} = \bar{G}_1 \cup \bar{G}_2$, and let Q_1 and Q_2 denote the quadrilaterals

(1.4b) $$Q_1 := \{G_1; z_1, z_2, 1, 0\}$$

and

(1.5b) $$Q_2 := \{G_2; 0, 1, z_3, z_4\}.$$

Finally, let

(1.6) $$h := m(Q) \quad \text{and} \quad h_j := m(Q_j), \qquad j = 1, 2,$$

and let g and g_j, $j = 1, 2$, be the associated conformal maps

$$g: R \to G, \tag{1.7}$$

$$g_1: R_1 \to G_1, \tag{1.8}$$

and

$$g_2: R_2 \to G_2, \tag{1.9}$$

where, with the notation (1.2),

$$R := R_h\{-h_1\} = \{(\xi, \eta): 0 < \xi < 1, -h_1 < \eta < h - h_1\}, \tag{1.10}$$

$$R_1 := R_{h_1}\{-h_1\} = \{(\xi, \eta): 0 < \xi < 1, -h_1 < \eta < 0\}, \tag{1.11}$$

and

$$R_2 := R_{h_2}\{0\} = \{(\xi, \eta): 0 < \xi < 1, 0 < \eta < h_2\}. \tag{1.12}$$

(The conformal map g is illustrated in Fig. 1.1.)

This paper is concerned with the study of a domain decomposition method for computing approximations to $h := m(Q)$ and to the associated conformal map g, defined by (1.7), in cases where the quadrilateral Q is of the form (1.3). More specifically, the method under consideration is based on decomposing Q into the two smaller quadrilaterals Q_1 and Q_2, given by (1.4) and (1.5), and then approximating h, R, and g respectively by

$$\tilde{h} := h_1 + h_2, \tag{1.13a}$$

$$\tilde{R} := R_{\tilde{h}}\{-h_1\} = \{(\xi, \eta): 0 < \xi < 1, -h_1 < \eta < h_2\}, \tag{1.13b}$$

and

$$\tilde{g}(w) := \begin{cases} g_2(w): R_2 \to G_2 & \text{for} \quad w \in R_2, \\ g_1(w): R_1 \to G_1 & \text{for} \quad w \in R_1. \end{cases} \tag{1.14}$$

The motivation for considering this method emerges from the intuitive observation that if $h^* := \min(h_1, h_2)$ is "large," then the segment $0 < x < 1$ of the real axis is "nearly" an equipotential of the function u satisfying the following Laplacian problem:

$$\Delta u = 0 \quad \text{in } G,$$
$$u = 0 \quad \text{on} \quad (z_1, z_2), \qquad u = 1 \quad \text{on} \quad (z_3, z_4),$$
$$\frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad (z_2, z_3) \cup (z_4, z_1).$$

This in turn indicates that if h^* is large, then

$$h \simeq h_1 + h_2,$$

and function (1.14) "approximates" the true conformal map g. (We note that

$$h \geq h_1 + h_2, \tag{1.15}$$

and equality occurs only in the two trivial cases where: (a) G is a rectangle, and (b) $\tau_1(x) = \tau_2(x)$, $x \in [0, 1]$; see, e.g. p. 437 of [8].)

The purpose of this paper is to provide a theoretical justification for the above decomposition method, and to show that (1.13), (1.14) are capable of producing close approximations to h and to g, even when $h^* := \min(h_1, h_2)$ is only moderately large. We do this by a method of analysis that makes extensive use of the theory given in Chapter 5, Section 3, of [2], in connection with the integral equation method of Garrick [6] for the conformal mapping of doubly connected domains. In particular, we derive estimates of

$$E_h := |h - \tilde{h}| = h - (h_1 + h_2), \tag{1.16a}$$

$$E_g^{(1)} := \max\{|g(w) - g_1(w)| : w \in \bar{R}_1\}, \tag{1.16b}$$

$$E_g^{(2)} := \max\{|g(w + iE_h) - g_2(w)| : w \in \bar{R}_2\}, \tag{1.16c}$$

and show that

$$E_h = O\{e^{-2\pi h^*}\},$$

and

$$E_g^{(1)} = 0\{e^{-\pi h^*}\}, \qquad E_g^{(2)} = 0\{e^{-\pi h^*}\},$$

provided that the functions τ_j, $j = 1, 2$, in (1.3b), satisfy certain smoothness conditions.

Although the main results of this paper are derived by considering quadrilaterals of the form (1.3), the domain decomposition method and the associated theory have a somewhat wider application. More specifically, it will become apparent from our work that both the method and the theory also apply to the mapping of quadrilaterals $Q := \{G; z_1, z_2, z_3, z_4\}$, in cases where the domain G and the crosscut c that decomposes Q into Q_1 and Q_2 are as described below:

- G is of the form illustrated in Fig. 1.2. That is, G is bounded by a segment $l_1 := (z_4, z_1)$ of the real axis, a straight line $l_2 := (z_2, z_3)$ inclined at an angle $\alpha\pi$, $0 < \alpha \leq 1$, to l_1, and two Jordan arcs $\gamma_1 := (z_1, z_2)$ and $\gamma_2 := (z_3, z_4)$ which are given in polar coordinates by

$$\gamma_j := \{z : z = \rho_j(\theta)e^{i\theta}, 0 \leq \theta \leq \alpha\pi\}, \qquad j = 1, 2. \tag{1.17}$$

- The functions $\rho_j, j = 1, 2$, in (1.17), are such that $\rho_1(\theta) > 1$ and $0 < \rho_2(\theta) < 1$, for $\theta \in [0, \alpha\pi]$, and the crosscut c is the arc $z = e^{i\theta}, 0 < \theta < \alpha\pi$, of the unit circle.

Although the results of this paper apply only to quadrilaterals that have one of

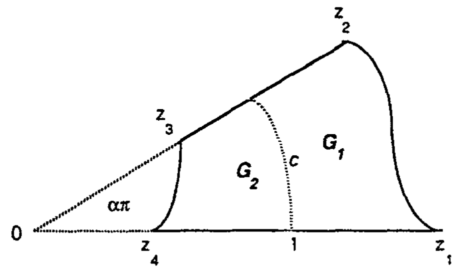

Fig. 1.2

the two special forms illustrated in Figs. 1.1 and 1.2, we note that the mapping of such quadrilaterals has received considerable attention recently; see, e.g., [1], [5], [11], [12], [15], and [17]. In this connection the decomposition method is of practical interest for the following two reasons:

(i) It can be used to overcome the "crowding" difficulties associated with the numerical conformal mapping of "thin" quadrilaterals of the forms illustrated in Figs. 1.1 and 1.2. (Full details of the crowding phenomenon and its damaging effects on numerical procedures for the mapping of "thin" quadrilaterals can be found in [12], [13], and [9]; see also p. 179 of [3], p. 428 of [8], and p. 4 of [16].)
(ii) Numerical methods for approximating the conformal maps of quadrilaterals of the form (1.4) or (1.5) are often substantially simpler than those for quadrilaterals of the more general form (1.3); see, e.g., [5] and Section 3.4 of [12].

The paper is organized as follows: In Section 2 we state without proof some preliminary results which are needed for our work in Section 3. These concern well-known properties of three integral operators that occur in the integral equations of the method of Garrick. In Section 3 we consider the Garrick formulations for the conformal maps of three closely related doubly connected domains. Hence, by making use of the theory given in Chapter V of [2], we derive a number of results that provide certain comparisons between the three conformal maps. Section 4 contains the main results of the paper. Here, we first identify certain well-known relationships between the conformal maps (1.7)–(1.9) and those considered in Section 3. Hence, by making use of the results of Section 3, we derive estimates of the errors (1.16) in the domain decomposition approximations (1.13)–(1.14). Finally, in Section 5 we present two numerical examples illustrating the theory of Section 4, and make a number of concluding remarks concerning this theory.

2. Preliminary Results

In Section 3 we make extensive use of the properties of three linear integral operators in the real function space

$$\mathbf{L}_2 := \{u\colon u \text{ is } 2\pi\text{-periodic and square integrable in } [0, 2\pi]\}. \tag{2.1}$$

These operators are denoted by $\mathbf{K}$, $\mathbf{R}_q$, and $\mathbf{S}_q$ and are defined as follows (see pp. 194–195 of [2]):

- $\mathbf{K}$ is the well-known operator for conjugation on the unit circle. That is, for $u \in \mathbf{L}_2$, the function $\mathbf{K}u$ is defined by the Cauchy principal value integral

$$\mathbf{K}[u(\varphi)] := \frac{1}{2\pi} PV \int_0^{2\pi} \cot\left(\frac{\varphi - t}{2}\right) u(t)\, dt.$$

- The operators $\mathbf{R}_q$ and $\mathbf{S}_q$ depend on a real parameter q, with $0 < q < 1$, and are defined by

$$\mathbf{R}_q[u(\varphi)] := \frac{1}{2\pi} \int_0^{2\pi} g_q(\varphi - t) u(t)\, dt, \qquad \mathbf{S}_q[u(\varphi)] := \frac{1}{2\pi} \int_0^{2\pi} h_q(\varphi - t) u(t)\, dt,$$

where the kernels g_q and h_q are given by the absolutely convergent series

$$g_q(\varphi) = 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}} \sin k\varphi, \qquad h_q(\varphi) = -4 \sum_{k=1}^{\infty} \frac{q^{k}}{1 - q^{2k}} \sin k\varphi.$$

The properties of the above three operators are studied in detail on pp. 195–205 of [2], where in particular the following basic results can be found:

- If $u \in \mathbf{L}_2$, then

$$\text{(2.2)} \qquad \mathbf{K}u,\ \mathbf{R}_q u,\ \mathbf{S}_q u \in \mathbf{L}_2,$$

and

$$\text{(2.3a)} \qquad \|\mathbf{K}u\| \le \|u\|, \qquad \|\mathbf{R}_q u\| \le \frac{2q^2}{1 - q^2}\|u\|, \qquad \|\mathbf{S}_q u\| \le \frac{2q}{1 - q^2}\|u\|.$$

Also, for $0 < q_2 < q_1 < 1$,

$$\text{(2.3b)} \qquad \|(\mathbf{R}_{q_1} - \mathbf{R}_{q_2})u\| \le \frac{2(q_1^2 - q_2^2)}{(1 - q_1^2)(1 - q_2^2)}\|u\|.$$

$\left(\text{Throughout this paper we take } \|u\| := \left\{\frac{1}{2\pi}\int_0^{2\pi} u^2(t)\, dt\right\}^{1/2}.\right)$

- Let $\mathbf{W}$ denote the space

(2.4)

$$\mathbf{W} := \{u\colon u \text{ is } 2\pi\text{-periodic and absolutely continuous in } [0, 2\pi] \text{ and } u' \in \mathbf{L}_2\}.$$

If $u \in \mathbf{W}$, then

$$\text{(2.5a)} \qquad \mathbf{K}u,\ \mathbf{R}_q u,\ \mathbf{S}_q u \in \mathbf{W}$$

and

$$\text{(2.5b)} \qquad (\mathbf{K}u)' = \mathbf{K}u', \qquad (\mathbf{R}_q u)' = \mathbf{R}_q u', \qquad (\mathbf{S}_q u)' = \mathbf{S}_q u'.$$

We also need the following:

- If $u \in \mathbf{W}$ and

$$\int_0^{2\pi} u(t)\,dt = 0,$$

then u satisfies the Warschawski inequality

$$|u(\varphi)|^2 \leq 2\pi \|u\| \|u'\|; \tag{2.6}$$

see p. 18 of [18] and p. 68 of [2]. In addition we have Wirtinger's inequality [7, p. 185], i.e.,

$$\|u\| \leq \|u'\|. \tag{2.7}$$

- Let $\mathbf{T}$ denote any of the three operators $\mathbf{K}$, $\mathbf{R}_q$, or $\mathbf{S}_q$. Then, for any function $u \in \mathbf{W}$,

$$\|\mathbf{T}u\| \leq \|\mathbf{T}\| \|u'\|. \tag{2.8}$$

(This follows at once from (2.7) and (2.5), by observing that

$$\int_0^{2\pi} \mathbf{T}u\,dt = 0;$$

see, e.g., equation (2.6) of [5].)

The significance of the results (2.2)-(2.8) in connection with our work in Section 3 is that the method of Garrick for the mapping of doubly connected domains can be formulated in terms of the operators $\mathbf{K}$, $\mathbf{R}_q$, and $\mathbf{S}_q$. The details are as follows:

Let Γ_1 and Γ_2 be two Jordan curves in the Z-plane which are starlike with respect to $Z = 0$ and are given in polar coordinates by

$$\Gamma_j := \{Z: Z = \rho_j(\theta)e^{i\theta}, 0 \leq \theta \leq 2\pi\}, \qquad j = 1, 2, \tag{2.9}$$

where $0 < \rho_2(\theta) < \rho_1(\theta)$, for $\theta \in [0, 2\pi]$. Also, let Ω be the doubly connected domain bounded externally and internally by Γ_1 and Γ_2, respectively, i.e.,

$$\Omega := (\operatorname{Int} \Gamma_1) \cap (\operatorname{Ext} \Gamma_2). \tag{2.10}$$

Then, for a certain value q, $0 < q < 1$, the domain Ω is conformally equivalent to the annulus

$$A_q := \{W: q < |W| < 1\}, \tag{2.11}$$

and the reciprocal $M := 1/q$ of the inner radius is called the conformal module of Ω.

Let f denote the conformal map $A_q \to \Omega$. Then the following are well known:

- f can be extended continuously to $\bar{A}_q$.
- On the boundaries $|W| = 1$ and $|W| = q$ of A_q the function f is given by two continuous boundary correspondence functions Θ and $\hat{\Theta}$ which are defined by

$$f(e^{i\varphi}) = \rho_1(\Theta(\varphi))e^{i\Theta(\varphi)}, \qquad f(qe^{i\varphi}) = \rho_2(\hat{\Theta}(\varphi))e^{i\hat{\Theta}(\varphi)}, \qquad \varphi \in [0, 2\pi]. \tag{2.12}$$

- The requirement that $|W| = 1$ is mapped onto Γ_1 defines f uniquely, apart from an arbitrary rotation in the W-plane. Here we normalize the mapping by requiring that

$$\int_0^{2\pi} \{\Theta(\varphi) - \varphi\}\, d\varphi = \int_0^{2\pi} \{\hat{\Theta}(\varphi) - \varphi\}\, d\varphi = 0. \tag{2.13}$$

- The outer and inner boundary correspondence functions Θ and $\hat{\Theta}$ and the inner radius q of A_q satisfy the Garrick integral equations:

$$\Theta(\varphi) = \varphi + (\mathbf{K} + \mathbf{R}_q)[\log \rho_1(\Theta(\varphi))] + \mathbf{S}_q[\log \rho_2(\hat{\Theta}(\varphi))], \tag{2.14a}$$

$$\hat{\Theta}(\varphi) = \varphi - \mathbf{S}_q[\log \rho_1(\Theta(\varphi))] - (\mathbf{K} + \mathbf{R}_q)[\log \rho_2(\hat{\Theta}(\varphi))], \tag{2.14b}$$

and

$$\log q = \frac{1}{2\pi} \int_0^{2\pi} \{\log \rho_2(\hat{\Theta}(\varphi)) - \log \rho_1(\Theta(\varphi))\}\, d\varphi; \tag{2.14c}$$

see pp. 198–199 of [2] and Section 3 of [5].

3. On the Conformal Maps of Three Doubly Connected Domains

Let the curves Γ_j, $j = 1, 2$, be given by (2.9) with

$$\rho_1(\theta) > 1 \quad \text{and} \quad 0 < \rho_2(\theta) < 1, \qquad \theta \in [0, 2\pi], \tag{3.1}$$

and as in Section 2, let f, Θ, and $\hat{\Theta}$ denote respectively the conformal map

$$f: A_q \to \Omega := (\operatorname{Int} \Gamma_1) \cap (\operatorname{Ext} \Gamma_2), \tag{3.2}$$

and the associated outer and inner boundary correspondence functions defined by (2.12). Also, let C_1 denote the unit circle

$$C_1 := \{Z: |Z| = 1\}, \tag{3.3}$$

and let q_1^{-1} and q_2^{-1} be respectively the conformal modules of the two doubly connected domains

$$\Omega_1 := (\operatorname{Int} \Gamma_1) \cap (\operatorname{Ext} C_1) \tag{3.4}$$

and

$$\Omega_2 := (\operatorname{Int} C_1) \cap (\operatorname{Ext} \Gamma_2). \tag{3.5}$$

Finally, let f_j, $j = 1, 2$, denote the conformal maps

$$f_j: A_{q_j} \to \Omega_j, \qquad j = 1, 2, \tag{3.6}$$

where

$$A_{q_j} := \{W: q_j < |W| < 1\}, \qquad j = 1, 2, \tag{3.7}$$

and let Θ_j and $\hat{\Theta}_j$, $j = 1, 2$, be the associated outer and inner boundary correspondence functions, i.e.,

$$f_1(e^{i\varphi}) = \rho_1(\Theta_1(\varphi))e^{i\Theta_1(\varphi)}, \qquad f_1(q_1 e^{i\varphi}) = e^{i\hat{\Theta}_1(\varphi)}, \tag{3.8a}$$

and

$$f_2(e^{i\varphi}) = e^{i\Theta_2(\varphi)}, \qquad f_2(q_2 e^{i\varphi}) = \rho_2(\hat{\Theta}_2(\varphi))e^{i\hat{\Theta}_2(\varphi)}. \tag{3.8b}$$

We recall that the conformal map f is normalized by the conditions (2.13) and, by analogy, we normalize the conformal maps f_j, $j = 1, 2$, respectively by

$$\int_0^{2\pi} \{\Theta_j(\varphi) - \varphi\}\, d\varphi = \int_0^{2\pi} \{\hat{\Theta}_j(\varphi) - \varphi\}\, d\varphi = 0, \qquad j = 1, 2. \tag{3.9}$$

We also recall that the boundary correspondence functions Θ, $\hat{\Theta}$ and the radius q, associated with f, satisfy the Garrick equations (2.14). Similarly, the boundary correspondence functions Θ_j, $\hat{\Theta}_j$, $j = 1, 2$, and the radii q_j, $j = 1, 2$, associated with f_j, $j = 1, 2$, satisfy the simplified Garrick equations:

$$\Theta_1(\varphi) = \varphi + (\mathbf{K} + \mathbf{R}_{q_1})[\log \rho_1(\Theta_1(\varphi))], \tag{3.10a}$$

$$\hat{\Theta}_1(\varphi) = \varphi - \mathbf{S}_{q_1}[\log \rho_1(\Theta_1(\varphi))], \tag{3.10b}$$

$$\log q_1 = -\frac{1}{2\pi} \int_0^{2\pi} \log \rho_1(\Theta_1(\varphi))\, d\varphi, \tag{3.10c}$$

and

$$\Theta_2(\varphi) = \varphi + \mathbf{S}_{q_2}[\log \rho_2(\hat{\Theta}_2(\varphi))], \tag{3.11a}$$

$$\hat{\Theta}_2(\varphi) = \varphi - (\mathbf{K} + \mathbf{R}_{q_2})[\log \rho_2(\hat{\Theta}_2(\varphi))], \tag{3.11b}$$

$$\log q_2 = \frac{1}{2\pi} \int_0^{2\pi} \log \rho_2(\hat{\Theta}_2(\varphi))\, d\varphi. \tag{3.11c}$$

(The above equations follow from (2.14), by setting respectively $\rho_2(\theta) = 1$ and $\rho_1(\theta) = 1$.)

In this section we derive a number of results that provide estimates for the quantities:

$$|\log q - (\log q_1 + \log q_2)|,$$

$$\max_{\varphi \in [0, 2\pi]} |\Theta(\varphi) - \Theta_1(\varphi)|, \qquad \max_{\varphi \in [0, 2\pi]} |\log \rho_1(\Theta(\varphi)) - \log \rho_1(\Theta_1(\varphi))|,$$

$$\max_{\varphi \in [0, 2\pi]} |\hat{\Theta}(\varphi) - \hat{\Theta}_2(\varphi)|, \qquad \max_{\varphi \in [0, 2\pi]} |\log \rho_2(\hat{\Theta}(\varphi)) - \log \rho_2(\hat{\Theta}_2(\varphi))|,$$

$$\max\{|\log f(W) - \log f_1(W)|: W \in \bar{A}_{q_1}\},$$
$$\max\{|\log f(qW/q_2) - \log f_2(W)|: W \in \bar{A}_{q_2}\},$$

and also for the real and imaginary parts of the function $\log\{f(W)/W\}$, $W \in A_q$. All these estimates are given in terms of the radii q and q_j, $j = 1, 2$, and are derived by making extensive use of the theory of the Garrick method given in Chapter V of [2]. The significance of the results of this section in connection with the domain decomposition method become apparent in Section 4, once certain well-known relationships between the conformal maps (3.2), (3.6), and (1.7)–(1.9) are identified.

Our results are established by assuming that the two boundary curves $\Gamma_j, j = 1, 2$, satisfy the conditions stated below.

Assumptions A3.1. The curves

$$\Gamma_j := \{Z : Z = \rho_j(\theta)e^{i\theta}, 0 \le \theta \le 2\pi\}, \qquad j = 1, 2,$$

satisfy the following:

(i) $\rho_1(\theta) > 1$ and $0 < \rho_2(\theta) < 1$, $\theta \in [0, 2\pi]$.

(ii) $\rho_j(\theta), j = 1, 2$, are absolutely continuous in $[0, 2\pi]$, and

$$d_j := \operatorname*{ess\,sup}_{0 \le \theta \le 2\pi} |\rho_j'(\theta)/\rho_j(\theta)| < \infty. \tag{3.12}$$

(iii) If

$$m_1 := \max_{0 \le \theta \le 2\pi} \{\rho_1^{-1}(\theta)\} \quad \text{and} \quad m_2 := \max_{0 \le \theta \le 2\pi} \{\rho_2(\theta)\}, \tag{3.13a}$$

then

$$\varepsilon_j := d_j \left\{\frac{1 + m_j}{1 - m_j}\right\} < 1, \qquad j = 1, 2. \tag{3.13b}$$

We note that the above assumptions resemble closely those that constitute the so-called $\varepsilon\delta$-condition associated with the theory of the method of Garrick; see p. 200 of [2] and p. 266 of [5]. We also note the following elementary results which are needed for our analysis:

- Assumption A3.1(ii) implies that

$$1|\log \rho_1(\Theta(\varphi)) - \log \rho_1(\Theta_1(\varphi))| = \left| \int_{\Theta_1(\varphi)}^{\Theta(\varphi)} \rho_1'(t)/\rho_1(t)\, dt \right| \le d_1 |\Theta(\varphi) - \Theta_1(\varphi)|, \qquad \varphi \in [0, 2\pi]. \tag{3.14a}$$

Hence, also

$$\|\log \rho_1(\Theta) - \log \rho_1(\Theta_1)\| \le d_1 \|\Theta - \Theta_1\|. \tag{3.14b}$$

Similarly,

$$|\log \rho_2(\hat{\Theta}(\varphi)) - \log \rho_2(\hat{\Theta}_2(\varphi))| \le d_2 |\hat{\Theta}(\varphi) - \hat{\Theta}_2(\varphi)|, \qquad \varphi \in [0, 2\pi], \tag{3.14c}$$

and

$$\|\log \rho_2(\hat{\Theta}) - \log \rho_2(\hat{\Theta}_2)\| \le d_2 \|\hat{\Theta} - \hat{\Theta}_2\|. \tag{3.14d}$$

- Since $q < q_j$, $j = 1, 2$, the Garrick equations (3.10c) and (3.11c) imply that

$$0 < q < q_j < m_j < 1, \qquad j = 1, 2. \tag{3.15a}$$

Hence, also

$$d_j \left\{\frac{1 + q}{1 - q}\right\} < d_j \left\{\frac{1 + q_j}{1 - q_j}\right\} \le \varepsilon_j, \qquad j = 1, 2. \tag{3.15b}$$

Lemma 3.1. *Let* Ψ *denote any of the boundary correspondence functions* Θ, $\hat{\Theta}$ *and* Θ_j, $\hat{\Theta}_j$, $j = 1, 2$. *If the curves* Γ_j, $j = 1, 2$, *satisfy Assumptions A3.1, then*:

(i) $\Psi' \in \mathbf{L}_2$, *i.e.*, $\Psi(\varphi) - \varphi \in \mathbf{W}$.
(ii)

$$\|\Psi'\| \le 1/(1 - \varepsilon^2)^{1/2} \tag{3.16a}$$

and

$$\|\Psi' - 1\| \le \varepsilon/(1 - \varepsilon^2)^{1/2}, \tag{3.16b}$$

where

- $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$, *when* $\Psi := \Theta, \hat{\Theta}$,

and

- $\varepsilon := \varepsilon_j$, *when* $\Psi := \Theta_j, \hat{\Theta}_j$, $j = 1, 2$.

Proof. (i) This follows from the Garrick equations (2.14) and (3.10)-(3.11), by modifying in an obvious manner the proof of Satz 3.5(b) on pp. 204–205 of [2].

(ii) The differentiation of (2.14a) gives

$$\Theta'(\varphi) - 1 = (\mathbf{K} + \mathbf{R}_q)\left[\frac{\rho_1'}{\rho_1}(\Theta(\varphi)) \cdot \Theta'(\varphi)\right] + \mathbf{S}_q\left[\frac{\rho_2'}{\rho_2}(\hat{\Theta}(\varphi)) \cdot \hat{\Theta}'(\varphi)\right].$$

This follows from (2.5), because $\log \rho_1(\Theta) \in \mathbf{W}$ and $\log \rho_2(\hat{\Theta}) \in \mathbf{W}$. Hence, by using (2.3) and (3.12) we find that

$$\|\Theta' - 1\| \le \left(\frac{1 + q^2}{1 - q^2}\right) d_1 \|\Theta'\| + \left(\frac{2q}{1 - q^2}\right) d_2 \|\hat{\Theta}'\|. \tag{3.17a}$$

Similarly, the differentiation of (2.14b) leads to

$$\|\Theta' - 1\| \le \left(\frac{2q}{1 - q^2}\right) d_1 \|\Theta'\| + \left(\frac{1 + q^2}{1 - q^2}\right) d_2 \|\hat{\Theta}'\|. \tag{3.17b}$$

Therefore, if $\|\Theta'\| \ge \|\hat{\Theta}'\|$, then

$$\|\Theta' - 1\| \le \varepsilon \|\Theta'\| \quad \text{and} \quad \|\hat{\Theta}' - 1\| \le \varepsilon \|\Theta'\|, \tag{3.18a}$$

and if $\|\Theta'\| \le \|\hat{\Theta}'\|$, then

$$\|\Theta' - 1\| \le \varepsilon \|\hat{\Theta}'\| \quad \text{and} \quad \|\hat{\Theta}' - 1\| \le \varepsilon \|\hat{\Theta}'\|, \tag{3.18b}$$

where $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$; see (3.13) and (3.15).

The two cases $\Psi := \Theta$ and $\Psi := \hat{\Theta}$ of (3.16) follow from inequalities (3.18), by recalling that $\varepsilon < 1$ and observing that

$$\|\Psi'\|^2 = \|\Psi' - 1\|^2 + 1;$$

see p. 70 of [2]. The other cases $\Psi := \Theta_j$ and $\Psi := \hat{\Theta}_j$, $j = 1, 2$, of (3.16) can be derived in a similar manner by differentiating the simplified Garrick equations (3.10a, b) and (3.11a, b). ■

Remark 3.1. The bounds for $\|\Theta' - 1\|$ and $\|\hat{\Theta}' - 1\|$, given by (3.16b), can be replaced respectively by

$$\|\Theta' - 1\| \le (\varepsilon_1 + 2\varepsilon_2 q)/(1 - \varepsilon^2)^{1/2} \tag{3.19a}$$

and

$$\|\hat{\Theta}' - 1\| \le (\varepsilon_2 + 2\varepsilon_1 q)/(1 - \varepsilon^2)^{1/2}, \tag{3.19b}$$

where as before $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$. These follow from inequalities (3.17), by substituting the bounds for $\|\Theta'\|$ and $\|\hat{\Theta}'\|$ given by (3.16a). Hence, from (3.19) and the bounds for $\|\Theta_1' - 1\|$ and $\|\hat{\Theta}_2' - 1\|$ given by (3.16b), we have that

$$\|\Theta' - \Theta_1'\| \le 2(\varepsilon_1 + \varepsilon_2 q)/(1 - \varepsilon^2)^{1/2} \tag{3.20a}$$

and

$$\|\hat{\Theta}' - \hat{\Theta}_2'\| \le 2(\varepsilon_2 + \varepsilon_1 q)/(1 - \varepsilon^2)^{1/2}. \tag{3.20b}$$

Lemma 3.2. *If the curves Γ_j, $j = 1, 2$, satisfy Assumptions A3.1, then*

$$\|\Theta - \Theta_1\| \le \alpha(\varepsilon_1, \varepsilon) \cdot \{\varepsilon_1 q_1^2 + \varepsilon_2 q\} \tag{3.21a}$$

and

$$\|\hat{\Theta} - \hat{\Theta}_2\| \le \alpha(\varepsilon_2, \varepsilon) \cdot \{\varepsilon_2 q_2^2 + \varepsilon_1 q\}, \tag{3.21b}$$

where

$$\alpha(\varepsilon_j, \varepsilon) := 2/\{(1 - \varepsilon_j)(1 - \varepsilon^2)^{1/2}\}, \qquad j = 1, 2, \tag{3.22}$$

and $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$.

Proof. The Garrick equations (2.14a) and (3.10a) imply that

$$\Theta(\varphi) - \Theta_1(\varphi) = (\mathbf{K} + \mathbf{R}_q)[\log \rho_1(\Theta(\varphi)) - \log \rho_1(\Theta_1(\varphi))]$$
$$-(\mathbf{R}_{q_1} - \mathbf{R}_q)[\log \rho_1(\Theta_1(\varphi))] + \mathbf{S}_q[\log \rho_2(\hat{\Theta}(\varphi))].$$

Hence,

$$\|\Theta - \Theta_1\| \le \|\mathbf{K} + \mathbf{R}_q\| d_1 \|\Theta - \Theta_1\|$$
$$+ \|\mathbf{R}_{q_1} - \mathbf{R}_q\| \left\| \frac{\rho_1'}{\rho_1}(\Theta_1) \cdot \Theta_1' \right\| + \|\mathbf{S}_q\| \left\| \frac{\rho_2'}{\rho_2}(\hat{\Theta}) \cdot \hat{\Theta}' \right\|,$$

where we made use of (2.8) and (3.14b). Therefore, by using (2.3), (3.12), (3.15), and (3.16a) we find that

$$\|\Theta - \Theta_1\| \le \left(\frac{1 + q^2}{1 - q^2}\right) d_1 \|\Theta - \Theta_1\|$$
$$+ \frac{2(q_1^2 - q^2)}{(1 - q_1^2)(1 - q^2)} \cdot \frac{d_1}{(1 - \varepsilon^2)^{1/2}} + \frac{2q}{(1 - q^2)} \cdot \frac{d_2}{(1 - \varepsilon^2)^{1/2}}$$
$$\le \varepsilon_1 \|\Theta - \Theta_1\| + \frac{2}{(1 - \varepsilon^2)^{1/2}} \{\varepsilon_1 q_1^2 + \varepsilon_2 q\}.$$

Since $\varepsilon_1 < 1$, this yields inequality (3.21a). Inequality (3.21b) is derived in a similar manner from (2.14b) and (3.11b). ∎

Theorem 3.1. *If the curves $\Gamma_j, j = 1, 2$, satisfy Assumptions A3.1, then*

$$(3.23)\qquad |\log q - (\log q_1 + \log q_2)| \leq d_1\alpha(\varepsilon_1, \varepsilon)\cdot\{\varepsilon_1 q_1^2 + \varepsilon_2 q\} + d_2\alpha(\varepsilon_2, \varepsilon)\cdot\{\varepsilon_2 q_2^2 + \varepsilon_1 q\},$$

where $\alpha(\cdot\,,\,\cdot)$ is given by (3.22).

Proof. Equations (2.14c), (3.10c), and (3.11c) in conjunction with the Schwarz inequality and inequalities (3.14b, d) give

$$|\log q - (\log q_1 + \log q_2)| \leq \|\log \rho_1(\Theta) - \log \rho_1(\Theta_1)\| + \|\log \rho_2(\hat{\Theta}) - \log \rho_2(\hat{\Theta}_2)\| \leq d_1\|\Theta - \Theta_1\| + d_2\|\hat{\Theta} - \hat{\Theta}_2\|.$$

The theorem then follows by substituting the bounds for $\|\Theta - \Theta_1\|$ and $\|\hat{\Theta} - \hat{\Theta}_2\|$ given in Lemma 3.2. ∎

Theorem 3.2. *If the curves $\Gamma_j, j = 1, 2$, satisfy Assumptions A3.1, then, for $\varphi \in [0, 2\pi]$,*

$$(3.24a)\qquad |\Theta(\varphi) - \Theta_1(\varphi)| \leq \sqrt{\pi}\beta(\varepsilon_1, \varepsilon)\cdot\{\varepsilon_1 + \varepsilon_2 q\}^{1/2}\cdot\{\varepsilon_1 q_1^2 + \varepsilon_2 q\}^{1/2}$$

and

$$(3.24b)\qquad |\hat{\Theta}(\varphi) - \hat{\Theta}_2(\varphi)| \leq \sqrt{\pi}\beta(\varepsilon_2, \varepsilon)\cdot\{\varepsilon_2 + \varepsilon_1 q\}^{1/2}\cdot\{\varepsilon_2 q_2^2 + \varepsilon_1 q\}^{1/2},$$

where

$$(3.25)\qquad \beta(\varepsilon_j, \varepsilon) := \sqrt{8}/\{1 - \varepsilon_j)(1 - \varepsilon^2)\}^{1/2}, \qquad j = 1, 2,$$

and $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$.

Proof. The first part of Lemma 3.1 and the normalizing conditions (2.13), (3.9), imply respectively that $\Theta - \Theta_1, \hat{\Theta} - \hat{\Theta}_2 \in \mathbf{W}$, and

$$\int_0^{2\pi} \{\Theta(\varphi) - \Theta_1(\varphi)\}\, d\varphi = \int_0^{2\pi} \{\hat{\Theta}(\varphi) - \hat{\Theta}_2(\varphi)\}\, d\varphi = 0.$$

Thus, the Warschawski inequality (2.6) is applicable to the functions $\Theta - \Theta_1$ and $\hat{\Theta} - \hat{\Theta}_2$. The theorem follows by applying this inequality to each of the two functions, and using the bounds for $\|\Theta' - \Theta_1'\|$, $\|\hat{\Theta}' - \hat{\Theta}_2'\|$ and $\|\Theta - \Theta_1\|$, $\|\hat{\Theta} - \hat{\Theta}_2\|$ given by (3.20) and (3.21). ∎

Remark 3.2. For any $\varphi \in [0, 2\pi]$, $|\log \rho_1(\Theta(\varphi)) - \log \rho_1(\Theta_1(\varphi))|$ can be bounded by the right-hand side of (3.24a) multiplied by d_1, and $|\log \rho_2(\hat{\Theta}(\varphi)) - \log \rho_2(\hat{\Theta}_2(\varphi))|$ can be bounded by the right-hand side of 3.24b) multiplied by d_2. This follows at once from inequalities (3.14a) and (3.14c).

Remark 3.3. If the outer boundary curve Γ_1 of Ω is a circle of radius r_1, i.e., if $\rho_1(\theta) = r_1 > 1$, then the domain Ω_1 reduces to a circular annulus of inner radius 1 and outer radius r_1. Thus, in this case, $f_1(W) = r_1 W$ and hence $q_1 = 1/r_1$ and $\Theta_1(\varphi) = \hat{\Theta}_1(\varphi) = \varphi$. Therefore, since $d_1 = \varepsilon_1 = 0$ and $\varepsilon = \varepsilon_2$, the results of Theorem 3.1 and 3.2 simplify to the following:

$$|\log q + \log r_1 - \log q_2| \leq \alpha(\varepsilon_2, \varepsilon_2) \cdot \varepsilon_2 d_2 q_2^2, \tag{3.26}$$

and

$$|\Theta(\varphi) - \varphi| \leq \sqrt{\pi}\beta(0, \varepsilon_2) \cdot \varepsilon_2 q, \tag{3.27a}$$

$$|\hat{\Theta}(\varphi) - \hat{\Theta}_2(\varphi)| \leq \sqrt{\pi}\beta(\varepsilon_2, \varepsilon_2) \cdot \varepsilon_2 q_2, \tag{3.27b}$$

where $\alpha(\cdot\,,\cdot)$ and $\beta(\cdot\,,\cdot)$ are given by (3.22) and (3.25). Similarly, if the inner boundary curve Γ_2 is a circle of radius r_2, i.e., if $\rho_2(\theta) = r_2 < 1$, then the results of the two theorems simplify as follows:

$$|\log q - \log q_1 - \log r_2| \leq \alpha(\varepsilon_1, \varepsilon_1) \cdot \varepsilon_1 d_1 q_1^2, \tag{3.28}$$

and

$$|\Theta(\varphi) - \Theta_1 \varphi| \leq \sqrt{\pi}\beta(\varepsilon_1, \varepsilon_1) \cdot \varepsilon_1 q_1, \tag{3.29a}$$

$$|\hat{\Theta}(\varphi) - \varphi| \leq \sqrt{\pi}\beta(0, \varepsilon_1) \cdot \varepsilon_1 q. \tag{3.29b}$$

Furthermore, it is easy to see that the results (3.26)-(3.29) hold under the somewhat less restrictive assumptions obtained by replacing inequalities (3.13) of Assumption A3.1(iii) by

$$\varepsilon_2 := d_2\left\{\frac{1 + m_2^2}{1 - m_2^2}\right\} < 1, \qquad \text{when} \quad \rho_1(\theta) = r_1, \tag{3.30a}$$

$$\varepsilon_1 := d_1\left\{\frac{1 + m_1^2}{1 - m_1^2}\right\} < 1, \qquad \text{when} \quad \rho_2(\theta) = r_2. \tag{3.30b}$$

(This follows by modifying the analysis in an obvious manner, after first observing that in each of the two special cases under consideration the Garrick equations (2.14) take the simplified forms (3.10)-(3.11).) In addition, it is easy to see that the results (3.27) and (3.29) also hold in the limiting cases where $r_1 = 1$ or $r_2 = 1$. Thus, in particular, (3.29b) and (3.27a) imply respectively that

$$|\hat{\Theta}_1(\varphi) - \varphi| \leq \sqrt{\pi}\beta(0, \varepsilon_1)\varepsilon_1 q_1 \tag{3.31a}$$

and

$$|\Theta_2(\varphi) - \varphi| \leq \sqrt{\pi}\beta(0, \varepsilon_2)\varepsilon_2 q_2, \tag{3.31b}$$

where $\hat{\Theta}_1$ and Θ_2 are the boundary correspondence functions defined by (3.8).

Theorem 3.3. *For any p, where $q < p < 1$, let*

$$f(pe^{i\varphi}) = P(p, \varphi)e^{i\Phi(p,\,\varphi)}, \qquad \varphi \in [0, 2\pi]. \tag{3.32}$$

If the curves Γ_j, $j = 1, 2$, satisfy Assumptions A3.1 and, in addition, are both symmetric with respect to the real axis, then

$$(3.33) \qquad |\Phi(p, \varphi) - \varphi| \le \sqrt{\pi}\beta(0, \varepsilon)\{\varepsilon_1 p + \varepsilon_2(q/p)\},$$

and

$$(3.34a) \qquad |\log P(p, \varphi) - \log p + \log \kappa_j| \le \tfrac{1}{2}\sqrt{\pi}\beta(0, \varepsilon)\{\varepsilon_1 p + \varepsilon_2(q/p)\} + d_j\alpha(\varepsilon_j, \varepsilon)\{\varepsilon_j q_j^2 + \varepsilon_{3-j}q\}, \qquad j = 1, 2,$$

with

$$(3.34b) \qquad \kappa_1 := q_1 \quad \textit{and} \quad \kappa_2 := q/q_2,$$

where $\alpha(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are given by (3.22) *and* (3.25) *and* $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$.

Proof. The symmetry of the curves Γ_j, $j = 1, 2$, implies that the Fourier series of the functions $u(\varphi) := \log \rho_1(\Theta(\varphi))$ and $\hat{u}(\varphi) := \log \rho_2(\hat{\Theta}(\varphi))$ are of the form

$$u(\varphi) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos k\varphi \quad \text{and} \quad \hat{u}(\varphi) = \tfrac{1}{2}\hat{a}_0 + \sum_{k=1}^{\infty} \hat{a}_k \cos k\varphi.$$

The symmetry also implies that the function

$$F(W) := \log\{f(W)/W\}$$

has the Laurent series expansion

$$F(W) = \sum_{k=-\infty}^{\infty} c_k W^k, \qquad W \in A_q,$$

where the coefficients c_k are all real and are related to the Fourier coefficients a_k, $\hat{a}_k$ by

$$c_0 = \tfrac{1}{2}a_0 = \frac{1}{2\pi}\int_0^{2\pi} \log \rho_1(\Theta(\varphi))\, d\varphi,$$

$$c_k = \{a_k - \hat{a}_k q^k\}/(1 - q^{2k}) \quad \text{and} \quad c_{-k} = \{\hat{a}_k q^k - a_k q^{2k}\}/(1 - q^{2k}), \qquad k = 1, 2, \ldots;$$

see p. 270 of [5]. It follows that, for any fixed p, $q < p < 1$,

$$F(pe^{i\varphi}) = c_0 + U_p(\varphi) + iV_p(\varphi), \qquad \varphi \in [0, 2\pi],$$

where the functions U_p and V_p have the Fourier series representations

$$U_p(\varphi) = \sum_{k=1}^{\infty} \alpha_k \cos k\varphi \quad \text{and} \quad V_p(\varphi) = \sum_{k=1}^{\infty} \beta_k \sin k\varphi,$$

with

$$(3.35a) \qquad \alpha_k = \{a_k(p^{2k} - q^{2k}) + \hat{a}_k q^k(1 - p^{2k})\}/\{p^k(1 - q^{2k})\}$$

and

$$(3.35b) \qquad \beta_k = \{a_k(p^{2k} + q^{2k}) - \hat{a}_k q^k(1 + p^{2k})\}/\{p^k(1 - q^{2k})\}.$$

This implies that

$$\| U'_p \| = \left\{ \tfrac{1}{2} \sum_{k=1}^{\infty} k^2 \alpha_k^2 \right\}^{1/2} \quad \text{and} \quad \| V'_p \| = \left\{ \tfrac{1}{2} \sum_{k=1}^{\infty} k^2 \beta_k^2 \right\}^{1/2}.$$

Hence, by substituting the values (3.35) of α_k and β_k and applying the Minkowski inequality to each of the resulting right-hand sides, we find that

$$\begin{aligned} \| U'_p \| &\le p \left\{ \tfrac{1}{2} \sum_{k=1}^{\infty} k^2 a_k^2 \right\}^{1/2} + \left(\frac{q}{p}\right) \left\{ \tfrac{1}{2} \sum_{k=1}^{\infty} k^2 \hat{a}_k^2 \right\}^{1/2} \\ &\le \frac{1}{(1-\varepsilon^2)^{1/2}} \cdot \{\varepsilon_1 p + \varepsilon_2 (q/p)\} \end{aligned}$$

and

$$\begin{aligned} \| V'_p \| &\le \left\{ \frac{2p}{1-q^2} \right\} \cdot \left\{ \tfrac{1}{2} \sum_{k=1}^{\infty} k^2 a_k^2 \right\}^{1/2} + \left\{ \frac{2q}{p(1-q^2)} \right\} \cdot \left\{ \tfrac{1}{2} \sum_{k=1}^{\infty} k^2 \hat{a}_k^2 \right\}^{1/2} \\ &\le \frac{2}{(1-\varepsilon^2)^{1/2}} \cdot \left\{ \left(\frac{d_1}{1-q^2} \right) p + \left(\frac{d_2}{1-q^2} \right) \cdot \left(\frac{q}{p} \right) \right\} \\ &\le \frac{2}{(1-\varepsilon^2)^{1/2}} \cdot \left\{ \varepsilon_1 p + \varepsilon_2 \left(\frac{q}{p} \right) \right\}. \end{aligned}$$

(In deriving the above we made use of the two inequalities

$$\left\{ \tfrac{1}{2} \sum_{k=1}^{\infty} k^2 a_k^2 \right\}^{1/2} \le \frac{d_1}{(1-\varepsilon^2)^{1/2}} \quad \text{and} \quad \left\{ \tfrac{1}{2} \sum_{k=1}^{\infty} k^2 \hat{a}_k^2 \right\}^{1/2} \le \frac{d_2}{(1-\varepsilon^2)^{1/2}},$$

which are obtained by recalling that $u, \hat{u} \in \mathbf{W}$ and using (3.12) and (3.16a), i.e., $\sum_{k=1}^{\infty} k^2 a_k^2 = 2\| u' \|^2 = 2\| (\rho'_1/\rho_1)(\Theta) \cdot \Theta' \|^2$, etc.)

To complete the proof, we observe that

$$\int_0^{2\pi} U_p(\varphi)\, d\varphi = \int_0^{2\pi} V_p(\varphi)\, d\varphi = 0,$$

and recall Wirtinger's inequality (2.7). Hence, by applying the Warschawski inequality (2.6) to each of the functions U_p and V_p we find that

$$|U_p(\varphi)|^2 \le 2\pi \| U_p \| \, \| U'_p \| \le 2\pi \| U'_p \|^2 \le \frac{2\pi}{(1-\varepsilon^2)} \left\{ \varepsilon_1 p + \varepsilon_2 \left(\frac{q}{p} \right) \right\}^2 \tag{3.36a}$$

and

$$|V_p(\varphi)|^2 \le 2\pi \| V_p \| \, \| V'_p \| \le 2\pi \| V'_p \|^2 \le \frac{8\pi}{(1-\varepsilon^2)} \left\{ \varepsilon_1 p + \varepsilon_2 \left(\frac{q}{p} \right) \right\}^2. \tag{3.36b}$$

Inequality (3.33) then follows at once from (3.36b) because

$$V_p(\varphi) = \Phi(p, \varphi) - \varphi.$$

Similarly, inequality (3.34) follows easily from (3.36a) and (3.21) by observing that

$$\begin{aligned} U_p(\varphi) &= \log P(p, \varphi) - \log p - c_0 \\ &= \log P(p, \varphi) - \log p + \log q_1 - (c_0 + \log q_1) \\ &= \log P(p, \varphi) - \log p + \log(q/q_2) - (c_0 + \log(q/q_2)), \end{aligned}$$

where from (3.10c) and (3.14b)

$$|c_0 + \log q_1| \le d_1 \|\Theta - \Theta_1\|,$$

and from (2.14c), (3.11c), and (3.14d)

$$|c_0 + \log(q/q_2)| \le d_2 \|\hat{\Theta} - \Theta_2\|. \qquad \blacksquare$$

Remark 3.4. If, as in Remark 3.3, the outer boundary curve Γ_1 is a circle of radius $r_1 \ge 1$, then (3.33) and the case $j = 1$ of (3.34) simplify respectively to

$$|\Phi(p, \varphi) - \varphi| \le \sqrt{\pi}\beta(0, \varepsilon_2)\varepsilon_2 q/p \tag{3.37}$$

and

$$|\log P(p, \varphi) - \log p - \log r_1| \le \tfrac{1}{2}\sqrt{\pi}\beta(0, \varepsilon_2)\varepsilon_2 q/p. \tag{3.38}$$

In particular, in the limiting case $p = 1$ the function Φ coincides with the boundary correspondence function Θ and, as might be expected, (3.37) coincides with the result (3.27a) of Remark 3.3. Similarly, if the inner boundary curve Γ_2 is a circle of radius $r_2 \le 1$, then (3.33) and the case $j = 2$ of (3.34) simplfy to

$$|\Phi(p, \varphi) - \varphi| \le \sqrt{\pi}\beta(0, \varepsilon_1)\varepsilon_1 p \tag{3.39}$$

and

$$|\log P(p, \varphi) - \log p + \log(q/r_2)| \le \tfrac{1}{2}\sqrt{\pi}\beta(0, \varepsilon_1)\varepsilon_1 p, \tag{3.40}$$

and in the limiting case $p = q$, (3.39) coincides with (3.29b).

Remark 3.5. The additional symmetry condition, under which Theorem 3.3 was proved, was imposed because our work of Section 4 is concerned only with the case where both the curves Γ_j, $j = 1, 2$, are symmetric with respect to the real axis. However, the results of the theorem remain valid even when this condition is not fulfilled, except that in the nonsymmetric case the estimate in the right-hand side of (3.33) and the first term in the right-hand side of (3.34) must be multiplied by 2. (The details of the proof are the same, but in the nonsymmetric case the Laurent series expansion of the function F must be replaced by that given on p. 264 of [5].)

Remark 3.6. Estimates similar to those given by (3.33)-(3.34) can also be obtained under the less restrictive assumption that the functions ρ_j, $j = 1, 2$, are only continuous. For example, by modifying the details of the proof that come after the two equations (3.35), it is easy to show that

$$|\Phi(p, \varphi) - \varphi| \le \frac{2l}{1 - q^2}\left\{\frac{\mathscr{E}_1 p}{1 - p} + \frac{\mathscr{E}_2 q}{p - q}\right\}, \qquad q < p < 1,$$

and

$$|\log P(p, \varphi) - \log p - c_0| \le l\left\{\frac{\mathscr{E}_1 p}{1-p} + \frac{\mathscr{E}_2 q}{p-q}\right\}, \qquad q < p < 1,$$

where:

(i) c_0 has the same meaning as in Theorem 3.3.
(ii) $\mathscr{E}_j := \max_{\theta \in [0, 2\pi]} \{\log \rho_j(\theta)\} - \min_{\theta \in [0, 2\pi]} \{\log \rho_j(\theta)\}$, $j = 1, 2$.
(iii) $l = 1$ when the curves Γ_j, $j = 1, 2$, are both symmetric with respect to the real axis and $l = 2$ otherwise.

In the special case where $\rho_1(\theta) = 1$ estimates of the above form can also be deduced directly from the so-called distortion theorems of Gaier and Huckemann [4] and Menke [10]. For example, if $\rho_1(\theta) = 1$ and ρ_2 is continuous, then Theorem 2 (ii) of [10] implies that

$$|\log P(p, \varphi) - \log p| \le 2 \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{q}{p}\right)^k \left\{\frac{1-p^{2k}}{1+q^{2k}}\right\} \le 2\left\{\frac{q}{p-q}\right\}.$$

Theorem 3.4. *If the curves Γ_j, $j = 1, 2$, satisfy Assumptions A3.1 and, in addition, are both symmetric with respect to the real axis, then*

$$\max\{|\log f(W) - \log f_1(W)|: W \in \bar{A}_{q_1}\} \le \max\{M_1, N_1\} \tag{3.41a}$$

and

$$\max\{|\log f(qW/q_2) - \log f_2(W)|: W \in \bar{A}_{q_2}\} \le \max\{M_2, N_2\}, \tag{3.41b}$$

where

$$M_j := \sqrt{\pi}(1 + d_j^2)^{1/2}\, \beta(\varepsilon_j, \varepsilon)\{\varepsilon_j + \varepsilon_{3-j} q\}^{1/2}\{\varepsilon_j q_j^2 + \varepsilon_{3-j} q\}^{1/2}, \qquad j = 1, 2, \tag{3.42a}$$

$$N_j := \tfrac{1}{2}\sqrt{\pi}\beta(0, \varepsilon)\{5\varepsilon_j q_j + 3\varepsilon_{3-j}(q/q_j)\} + d_j \alpha(\varepsilon_j, \varepsilon)\{\varepsilon_j q_j^2 + \varepsilon_{3-j} q\}, \qquad j = 1, 2, \tag{3.42b}$$

and where $\alpha(\cdot, \cdot)$, $\beta(\cdot, \cdot)$ are given by (3.22), (3.25), *and* $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$.

Proof. Let

$$E := \max\{|\log f(W) - \log f_1(W)|: W \in \bar{A}_{q_1}\},$$

and observe that the function $\log f(W) - \log f_1(W)$ is regular and single-valued in A_{q_1} and continuous on $\bar{A}_{q_1}$. Therefore, by the principle of maximum modulus,

$$E \le \max\left\{\max_{\varphi \in [0, 2\pi]} |\log f(e^{i\varphi}) - \log f_1(e^{i\varphi})|,\ \max_{\varphi \in [0, 2\pi]} |\log f(q_1 e^{i\varphi}) - \log f_1(q_1 e^{i\varphi})|\right\},$$

where, for $\varphi \in [0, 2\pi]$,

$$
\begin{aligned}
(3.43a)\quad & |\log f(e^{i\varphi}) - \log f_1(e^{i\varphi})| \\
&= |\{\log \rho_1(\Theta(\varphi)) - \log \rho_1(\Theta_1(\varphi))\} + i\{\Theta(\varphi) - \Theta_1(\varphi)\}| \\
&\leq (1 + d_1^2)^{1/2} |\Theta(\varphi) - \Theta_1(\varphi)| \\
&\leq M_1,
\end{aligned}
$$

and, with the notation of Theorem 3.3,

$$
\begin{aligned}
(3.43b)\quad & |\log f(q_1 e^{i\varphi}) - \log f_1(q_1 e^{i\varphi})| \\
&= |\log P(q_1, \varphi) + i\{\Phi(q_1, \varphi) - \hat{\Theta}_1(\varphi)\}| \\
&\leq |\log P(q_1, \varphi)| + |\Phi(q_1, \varphi) - \varphi| + |\hat{\Theta}_1(\varphi) - \varphi| \\
&\leq N_1.
\end{aligned}
$$

(In deriving (3.43a) we made use of (3.14a) and (3.24a), and in deriving (3.43b) we made use of (3.33)-(3.34), with $p = q_1$ and $j = 1$, and of (3.31a).)

Inequality (3.41a) follows at once from the above. Inequality (3.41b) is established in a similar manner, by observing that the function $\log f(qW/q_2) - \log f_2(W)$ is regular and single-valued in A_{q_2}, and continuous on $\bar{A}_{q_2}$, and then showing that

$$|\log f(qe^{i\varphi}/q_2) - \log f_2(e^{i\varphi})| \leq N_2$$

and

$$|\log f(qe^{i\varphi}) - \log f_2(q_2 e^{i\varphi})| \leq M_2. \qquad \blacksquare$$

4. Decomposition of Quadrilaterals

We recall the notations (1.2)-(1.12) concerning the quadrilaterals Q and $Q_j, j = 1, 2$, defined by (1.3)-(1.5), their conformal modules

$$(4.1)\qquad h := m(Q) \quad \text{and} \quad h_j := m(Q_j),\ \ j = 1, 2,$$

and the three associated conformal maps

$$(4.2)\qquad g : R \to G, \quad g_1 : R_1 \to G_1, \quad \text{and} \quad g_2 : R_2 \to G_2,$$

where $R := R_h\{-h_1\}$, $R_1 := R_{h_1}\{-h_1\}$, and $R_2 := R_{h_2}\{0\}$ are the three rectangles defined by (1.10)-(1.12). We also recall that the decomposition method outlined in Section 1 consists of the following:

- Decomposing the quadrilateral Q into the two smaller quadrilaterals Q_1 and Q_2.
- Approximating the conformal module of Q by the sum of the conformal modules of Q_1 and Q_2, i.e., approximating h by

$$(4.3)\qquad \tilde{h} := h_1 + h_2.$$

- Approximating the rectangle R and the conformal map g respectively by $\tilde{R} := R_{\tilde{h}}\{-h_1\}$ and

$$(4.4) \qquad \tilde{g}(w) := \begin{cases} g_2(w): R_2 \to G_2 & \text{for} \quad w \in R_2, \\ g_1(w): R_1 \to G_1 & \text{for} \quad w \in R_1. \end{cases}$$

In this section we study the errors (1.16) of the domain decomposition approximations (4.3)-(4.4), and show that estimates of these errors can be deduced directly from our results of Section 3. We do this by first making the following elementary observations, which establish a well-known connection between the conformal maps (4.2) and those studied in Section 3; see, e.g., Section 5 of [5].

- By using the Schwarz reflection principle, the conformal map g can be extended to map the infinite strip $\{(\xi, \eta): -\infty < \xi < \infty, -h_1 < \eta < h - h_1\}$ onto the infinite domain bounded by the two curves $y = -\tau_1^{(p)}(x)$ and $y = \tau_2^{(p)}(x)$, where $\tau_j^{(p)}, j = 1, 2$, are the periodic functions defined by

$$\tau_j^{(p)}(\pm x) = \tau_j(x), \quad x \in [0, 1], \qquad \text{and} \qquad \tau_j^{(p)}(2 + x) = \tau_j^{(p)}(x).$$

  Similarly, the conformal maps g_1 and g_2 can be extended to map respectively the infinite strips $\{(\xi, \eta): -\infty < \xi < \infty, -h_1 < \eta < 0\}$ and $\{(\xi, \eta): -\infty < \xi < \infty, 0 < \eta < h_2\}$ onto the infinite domains bounded by the real axis and the curve $y = -\tau_1^{(p)}(x)$, and the real axis and the curve $y = \tau_2^{(p)}(x)$. The above also show that the functions $g(w) - w$ and $g_j(w) - w, j = 1, 2$, are periodic with period 2.
- The exponential function $Z = e^{i\pi z}$ maps the domain G conformally onto the upper half of the symmetric doubly connected domain

$$(4.5) \qquad \Omega := (\operatorname{Int} \Gamma_1) \cap (\operatorname{Ext} \Gamma_2),$$

  where

$$(4.6a) \qquad \Gamma_j := \{Z: Z = \rho_j(\theta) e^{i\theta}, 0 \le \theta \le 2\pi\}, \qquad j = 1, 2,$$

  with

$$(4.6b) \qquad \rho_j(\theta) := \exp\{(-1)^{j-1} \pi \tau_j(\theta/\pi)\}, \qquad \theta \in [0, \pi],$$

  and

$$\rho_j(\theta) = \rho_j(2\pi - \theta), \qquad \theta \in (\pi, 2\pi].$$

  Thereby the domains G_1 and G_2 go respectively onto the upper halves of the symmetric doubly-connected domains

$$(4.7) \qquad \Omega_1 := (\operatorname{Int} \Gamma_1) \cap (\operatorname{Ext} C_1)$$

  and

$$(4.8) \qquad \Omega_2 := (\operatorname{Int} C_1) \cap (\operatorname{Ext} \Gamma_2),$$

  where C_1 is the unit circle (3.3) and $\Gamma_j, j = 1, 2$, are the curves (4.6).

- Let q^{-1} and $q_j^{-1}, j = 1, 2$, be respectively the conformal modules of the doubly connected domains Ω and Ω_j, $j = 1, 2$, given by (4.5)–(4.8), and let f and $f_j, j = 1, 2$, denote the associated conformal maps

$$f: A_q \to \Omega \quad \text{and} \quad f_j: A_{q_j} \to \Omega_j, \quad j = 1, 2. \tag{4.9}$$

  Also, let $h := -\{\log q\}/\pi$ and $h_j := -\{\log q_j\}/\pi, j = 1, 2$. Then, the exponential function $W = e^{i\pi(w+ih_1)}$ maps the rectangles R and R_1 conformally onto the upper halves of the annuli A_q and A_{q_1}, respectively. Similarly, the function $W = e^{i\pi w}$ maps the rectangle R_2 conformally onto the upper half of A_{q_2}.

It follows from the Schwarz reflection principle that the conformal modules (4.1) and the mapping functions (4.2) are related to the modules q^{-1} and q_j^{-1} and the mapping functions (4.9) respectively by

$$q = e^{-\pi h}, \qquad q_j = e^{-\pi h_j}, \quad j = 1, 2, \tag{4.10}$$

and

$$e^{i\pi g(w)} = f\{e^{i\pi(w+ih_1)}\}, \tag{4.11a}$$

$$e^{i\pi g_1(w)} = f_1\{e^{i\pi(w+ih_1)}\}, \tag{4.11b}$$

$$e^{i\pi g_2(w)} = f_2\{e^{i\pi w}\}. \tag{4.11c}$$

In other words the problem of determining the three conformal maps (4.2) is essentially equivalent to that of determining three conformal maps of the type studied in Section 3.

Let

$$X(\xi) := \operatorname{Re} g(\xi - ih_1), \qquad \hat{X}(\xi) := \operatorname{Re} g(\xi + i(h - h_1)),$$

$$X_1(\xi) := \operatorname{Re} g_1(\xi - ih_1), \qquad \hat{X}_1(\xi) := \operatorname{Re} g_1(\xi),$$

and

$$X_2(\xi) := \operatorname{Re} g_2(\xi), \qquad \hat{X}_2(\xi) := \operatorname{Re} g_2(\xi + ih_2).$$

Also, let Θ, Θ_j, $j = 1, 2$, and $\hat{\Theta}, \hat{\Theta}_j, j = 1, 2$, be respectively the outer and inner boundary correspondence functions associated with the conformal maps f, f_j, $j = 1, 2$, of the three doubly connected domains (4.6)–(4.8). Then the relations (4.11) imply that

$$X(\xi) = \frac{1}{\pi}\Theta(\pi\xi), \qquad X_j(\xi) = \frac{1}{\pi}\Theta_j(\pi\xi), \quad j = 1, 2, \tag{4.12a}$$

$$\hat{X}(\xi) = \frac{1}{\pi}\hat{\Theta}(\pi\xi), \qquad \hat{X}_j(\xi) = \frac{1}{\pi}\hat{\Theta}_j(\pi\xi), \quad j = 1, 2, \tag{4.12b}$$

and

$$\tau_1(X(\xi)) = \frac{1}{\pi}\log \rho_1(\Theta(\pi\xi)), \qquad \tau_1(X_1(\xi)) = \frac{1}{\pi}\log \rho_1(\Theta_1(\pi\xi)), \tag{4.12c}$$

$$\tau_2(\hat{X}(\xi)) = -\frac{1}{\pi}\log \rho_2(\hat{\Theta}(\pi\xi)), \qquad \tau_2(\hat{X}_2(\xi) = -\frac{1}{\pi}\log \rho_2(\hat{\Theta}_2(\pi\xi)). \tag{4.12d}$$

It is now easy to express the main results of Section 3 in terms of the notations associated with the conformal maps (4.2). We do this below, after first observing that the conditions of Assumptions A3.1 can be expressed in terms of the functions $\tau_j, j = 1, 2$, as follows:

Assumptions A4.1. The functions $\tau_j, j = 1, 2$, satisfy the following:

(i) $\tau_j(x) > 0, j = 1, 2, x \in [0, 1]$.
(ii) $\tau_j, j = 1, 2$, are absolutely continuous in $[0, 1]$, and

(4.13) $$d_j := \operatorname*{ess\,sup}_{0 \le x \le 1} |\tau_j'(x)| < \infty.$$

(iii) If

(4.14a) $$m_j := \max_{0 \le x \le 1}\{e^{-\pi\tau_j(x)}\}, \qquad j = 1, 2,$$

then

(4.14b) $$\varepsilon_j := d_j\left\{\frac{1 + m_j}{1 - m_j}\right\} < 1, \qquad j = 1, 2.$$

Theorem 4.1. *If the functions* $\tau_j, j = 1, 2$, *satisfy Assumptions A*4.1, *then*

(4.15) $$E_h := h - (h_1 + h_2) \le \pi^{-1}d_1\alpha(\varepsilon_1, \varepsilon)\{\varepsilon_1 e^{-2\pi h_1} + \varepsilon_2 e^{-\pi h}\} + \pi^{-1}d_2\alpha(\varepsilon_2, \varepsilon)\{\varepsilon_2 e^{-2\pi h_2} + \varepsilon_1 e^{-\pi h}\},$$

where $\alpha(\cdot, \cdot)$ *is given by* (3.22) *and* $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$.

Proof. At once from Theorem 3.1, by recalling the relations (4.10). ■

Remark 4.1. Since $h \ge h_1 + h_2$, the theorem implies that

(4.16a) $$E_h := h - (h_1 + h_2) = O\{e^{-2\pi h^*}\},$$

where

(4.16b) $$h^* := \min(h_1, h_2).$$

Theorem 4.2. *If the functions* $\tau_j, j = 1, 2$, *satisfy Assumptions A*4.1, *then, for* $\xi \in [0, 1]$,

(4.17a) $$|X(\xi) - X_1(\xi)| \le \pi^{-1/2}\beta(\varepsilon_1, \varepsilon)\{\varepsilon_1 + \varepsilon_2 e^{-\pi h}\}^{1/2}\{\varepsilon_1 e^{-2\pi h_1} + \varepsilon_2 e^{-\pi h}\}^{1/2}$$

and

(4.17b) $$|\hat{X}(\xi) - \hat{X}_2(\xi)| \le \pi^{-1/2}\beta(\varepsilon_2, \varepsilon)\{\varepsilon_2 + \varepsilon_1 e^{-\pi h}\}^{1/2}\{\varepsilon_2 e^{-2\pi h_2} + \varepsilon_1 e^{-\pi h}\}^{1/2},$$

where $\beta(\cdot, \cdot)$ *is given by* (3.25) *and* $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$.

Proof. At once from Theorem 3.2, by recalling the relations (4.12a, b). ■

Remark 4.2. For any $\xi \in [0, 1]$, $|\tau_1(X(\xi)) - \tau_1(X_1(\xi))|$ and $|\tau_2(\hat{X}(\xi)) - \tau_2(\hat{X}_2(\xi))|$ can be bounded respectively by the right-hand side of (4.17a) multiplied by d_1 and the right-hand side of (4.17b) multiplied by d_2. This follows from the relations (4.12c, d), by recalling the comment made in Remark 3.2.

Theorem 4.3. *For any point* $\xi + i\eta \in R := R_h\{-h_1\}$, *let*

$$x(\xi, \eta) := \operatorname{Re} g(\xi + i\eta) \quad \textit{and} \quad y(\xi, \eta) := \operatorname{Im} g(\xi + i\eta).$$

If the functions $\tau_j, j = 1, 2$, *satisfy Assumptions A*4.1, *then*

$$|x(\xi, \eta) - \xi| \le \pi^{-1/2}\beta(0, \varepsilon)\{\varepsilon_1 e^{-\pi(h_1+\eta)} + \varepsilon_2 e^{-\pi(h-h_1-\eta)}\} \tag{4.18a}$$

and

$$\begin{aligned} |y(\xi, \eta) - \eta| \le {} & \tfrac{1}{2}\pi^{-1/2}\beta(0, \varepsilon)\{\varepsilon_1 e^{-\pi(h_1+\eta)} + \varepsilon_2 e^{-\pi(h-h_1-\eta)}\} \\ & + \pi^{-1} d_1 \alpha(\varepsilon_1, \varepsilon)\{\varepsilon_1 e^{-2\pi h_1} + \varepsilon_2 e^{-\pi h}\}, \end{aligned} \tag{4.18b}$$

where $\alpha(\cdot, \cdot)$ *and* $\beta(\cdot, \cdot)$ *are given by* (3.22) *and* (3.25) *and* $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$.

Proof. At once from (3.33) and the case $j = 1$ of (3.34), by observing that if

$$p = e^{-\pi(h_1+\eta)} \quad \text{and} \quad \varphi = \pi\xi,$$

then

$$\Phi(p, \varphi) = \pi x(\xi, \eta) \quad \text{and} \quad \log P(p, \varphi) = -\pi y(\xi, \eta);$$

see (4.11a). ■

Remark 4.3. In particular, the theorem implies that

$$|x(\xi, 0) - \xi| = O\{e^{-\pi h^*}\} \quad \text{and} \quad |y(\xi, 0)| = O\{e^{-\pi h^*}\},$$

where $h^* := \min(h_1, h_2)$. More generally, the theorem implies that if Q_1 and Q_2 are "long" quadrilaterals then, at points sufficiently far from the two sides $\eta = -h_1$ and $\eta = h - h_1$ of $R_h\{-h_1\}$, the conformal map g can be approximated closely by the identity map.

Theorem 4.4. *Let*

$$E_g^{(1)} := \max\{|g(w) - g_1(w)| : w \in \bar{R}_1\}$$

and

$$E_g^{(2)} := \max\{|g(w + iE_h) - g_2(w)| : w \in \bar{R}_2\},$$

where $E_h := h - (h_1 + h_2)$. *If the functions* τ_j, $j = 1, 2$, *satisfy Assumptions A*4.1, *then*

$$E_g^{(j)} \le \max\{M_j, N_j\}, \qquad j = 1, 2, \tag{4.19a}$$

where

$$\begin{aligned} M_j := {} & \pi^{-1/2}(1 + d_j^2)^{1/2}\beta(\varepsilon_j, \varepsilon) \\ & \times \{\varepsilon_j + \varepsilon_{3-j} e^{-\pi h}\}^{1/2}\{\varepsilon_j e^{-2\pi h_j} + \varepsilon_{3-j} e^{-\pi h}\}^{1/2}, \qquad j = 1, 2, \end{aligned} \tag{4.19b}$$

and

(4.19c) $$N_j := \tfrac{1}{2}\pi^{-1/2}\beta(0, \varepsilon)\{5\varepsilon_j e^{-\pi h_j} + 3\varepsilon_{3-j} e^{-\pi(h-h_j)}\} + \pi^{-1} d_j \alpha(\varepsilon_j, \varepsilon)\{\varepsilon_j e^{-2\pi h_j} + \varepsilon_{3-j} e^{-\pi h}\}, \qquad j = 1, 2,$$

and where $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ *are given by* (3.22), (3.25), *and* $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$.

Proof. At once from Theorem 3.4, by recalling the relations (4.11) and observing that

$$w \in R_1 \quad \Rightarrow \quad e^{i\pi(w + ih_1)} \in A_{q_1}$$

and

$$w \in R_2 \quad \Rightarrow \quad e^{i\pi w} \in A_{q_2}. \qquad \blacksquare$$

Remark 4.4. Theorem 4.4 implies that

(4.20) $$E_g^{(j)} = O\{e^{-\pi h^*}\}, \qquad j = 1, 2,$$

where, as in (4.16), $h^* := \min(h_1, h_2)$. In other words, the error in the domain decomposition approximation (4.4) is $O\{e^{-\pi h^*}\}$, whilst the error in (4.3) is $O\{e^{-2\pi h^*}\}$.

Remark 4.5. Considerable simplifications occur in the case where one of the two subdomains G_1 or G_2 is a rectangle. For example, if $\tau_1(x) = c > 0$, $x \in [0, 1]$, i.e., if

$$G_1 := \{(x, y): 0 < x < 1, -c < y < 0\} = R_c\{-c\},$$

then $g_1(w) = w$, $h_1 = c$, $d_1 = \varepsilon_1 = 0$, and the results of Theorems 4.1-4.4 simplify respectively as follows:

(4.21) $$E_h := h - (c + h_2) \leq \pi^{-1} d_2 \alpha(\varepsilon_2, \varepsilon_2)\varepsilon_2 e^{-2\pi h_2}.$$

(4.22a) $$|X(\xi) - \xi| \leq \pi^{-1/2}\beta(0, \varepsilon_2)\varepsilon_2 e^{-\pi h}$$

and

(4.22b) $$|\hat{X}(\xi) - \hat{X}_2(\xi)| \leq \pi^{-1/2}\beta(\varepsilon_2, \varepsilon_2)\varepsilon_2 e^{-\pi h_2}.$$

(4.23a) $$|x(\xi, \eta) - \xi| \leq \pi^{-1/2}\beta(0, \varepsilon_2)\varepsilon_2 e^{-\pi(h-c-\eta)}$$

and

(4.23b) $$|y(\xi, \eta) - \eta| \leq \tfrac{1}{2}\pi^{-1/2}\beta(0, \varepsilon_2)\varepsilon_2 e^{-\pi(h-c-\eta)}.$$

(4.24a) $$E_g^{(1)} := \max\{|g(w) - w|: w \in \bar{R}_c\{-c\}\} \leq \tfrac{3}{2}\pi^{-1/2}\beta(0, \varepsilon_2)\varepsilon_2 e^{-\pi(h-c)}$$

and

(4.24b) $$E_g^{(2)} := \max\{|g(w + iE_h) - g_2(w)|: w \in \bar{R}_2\} \leq \max\{M_2, N_2\},$$

where

$$M_2 := \pi^{-1/2}(1 + d_2^2)^{1/2}\beta(\varepsilon_2, \varepsilon_2)\varepsilon_2 e^{-\pi h_2},$$

and

$$N_2 := \tfrac{5}{2}\pi^{-1/2}\beta(0, \varepsilon_2)\varepsilon_2 e^{-\pi h_2} + \pi^{-1} d_2 \alpha(\varepsilon_2, \varepsilon_2)\varepsilon_2 e^{-2\pi h_2}.$$

Furthermore, all the above results hold under the less restrictive assumptions obtained by replacing inequalities (4.14) of Assumptions A4.1 by

$$\varepsilon_2 := d_2\left\{\frac{1+m_2^2}{1-m_2^2}\right\} < 1; \tag{4.25}$$

see Remark 3.3.

The results (4.23) are of particular interest. These results show that, for any point $w := \xi + i\eta \in R_c\{-c\}$,

$$|g(w) - w| = O\{e^{-\pi(h_2-\eta)}\}; \tag{4.26}$$

see also (4.22a) and (4.24a). In other words, if Q_2 is a "long" quadrilateral, then in the rectangle $R_c\{-c\}$ the conformal map g can be approximated closely by the identity map.

Remark 4.6. The observations concerning the identity map, which were made in Remarks 4.3 and 4.5, suggest the use of a more general decomposition procedure where the original quadrilateral Q is subdivided into a quadrilateral of the form (1.4) at the lower end, a rectangle in the middle, and a quadrilateral of the form (1.5) at the top. This procedure can be described as follows:

Let

$$G := \{(x, y): 0 < x < 1, -\tau_1(x) < y < \tau_2(x) + c\},$$

where $c < 0$, let

$$G_1 := \{(x, y): 0 < x < 1, -\tau_1(x) < y < 0\}$$

and

$$G_2 := \{(x, y): 0 < x < 1, c < y < \tau_2(x) + c\},$$

so that

$$\bar{G} = \bar{G}_1 \cup \bar{R}_c\{0\} \cup \bar{G}_2,$$

and let

$$z_1 = -i\tau_1(0), \qquad z_2 = 1 - i\tau_1(1),$$
$$z_3 = 1 + i(\tau_2(1) + c), \qquad z_4 = i(\tau_2(0) + c).$$

Then the procedure under consideration consists of the following:

- Subdividing the quadrilateral $Q := \{G; z_1, z_2, z_3, z_4\}$ into three smaller quadrilaterals, i.e., the quadrilaterals $Q_1 := \{G_1; z_1, z_2, 1, 0\}$ and $Q_2 := \{G_2; ic, 1 + ic, z_3, z_4\}$ at the lower and upper ends, and the rectangular quadrilateral

$$\{R_c\{0\}: 0, 1, 1 + ic, ic\}$$

  in the middle.

- Approximating the conformal module $h := m(Q)$ by

$$\tilde{h} := h_1 + h_2 + c, \tag{4.27}$$

where $h_j := m(Q_j), j = 1, 2$.

- Approximating the rectangle $R := R_{\tilde{h}}\{-h_1\}$ and the conformal map $g: R \to G$ respectively by $\tilde{R} := R_{\tilde{h}}\{-h_1\}$ and

$$\tilde{g}(w) := \begin{cases} g_2(w): R_{h_2}\{c\} \to G_2 & \text{for} \quad w \in R_{h_2}\{c\}, \\ w & \text{for} \quad w \in R_c\{0\}, \\ g_1(w): R_1 \to G_1 & \text{for} \quad w \in R_1. \end{cases} \tag{4.28}$$

Let $E_h, E_g^{(j)}, j = 1, 2$, and $E_g^{(c)}$ denote the errors in the approximations (4.27)–(4.28). That is,

$$E_h := h - (h_1 + h_2 + c),$$

$$E_g^{(1)} := \max\{|g(w) - g_1(w)|: w \in \bar{R}_1\},$$

$$E_g^{(2)} := \max\{|g(w + iE_h) - g_2(w)|: w \in \bar{R}_{h_2}\{c\}\},$$

and

$$E_g^{(c)} := \max\{|g(w) - w|: w \in \bar{R}_c\{0\}\}.$$

Then estimates of the above errors can be deduced easily from those given in Theorems 4.1, 4.3, and 4.4 and in Remark 4.5. For example, if the functions $\tau_j, j = 1, 2$, satisfy Assumptions A4.1, then by using (4.15) and (4.21) it is easy to show that, for any $c > 0$,

$$\begin{aligned} E_h \le\ & \pi^{-1} d_1 \alpha(\varepsilon_1, \varepsilon)\{\varepsilon_1 e^{-2\pi h_1} + \varepsilon_2 e^{-\pi h}\} \\ & + \pi^{-1} d_2 \alpha(\varepsilon_2, \varepsilon)\{\varepsilon_2 e^{-2\pi(h_2 + c)} + \varepsilon_1 e^{-\pi h}\} \\ & + \pi^{-1} d_2 \alpha(\varepsilon_2, \varepsilon_2)\varepsilon_2 e^{-2\pi h_2}. \end{aligned}$$

More generally, it is easy to show that if the functions $\tau_j, j = 1, 2$, satisfy Assumptions A4.1 then, for any $c > 0$,

$$E_h = O\{e^{-2\pi h^*}\},$$

$$E_g^{(j)} = O\{e^{-\pi h^*}\}, \quad j = 1, 2, \qquad \text{and} \qquad E_g^{(c)} = O\{e^{-\pi h^*}\},$$

where $h^* := \min(h_1, h_2)$.

Remark 4.7. Let $Q := \{G; z_1, z_2, z_3, z_4\}$ be of the form illustrated in Fig. 1.2. That is, let Q consist of a domain G bounded by a segment $l_1 := (z_4, z_1)$ of the real axis, a straight line $l_2 := (z_2, z_3)$ inclined at an angle $\alpha\pi$, $0 < \alpha \le 1$, to l_1, and two Jordan arcs $\gamma_1 := (z_1, z_2)$ and $\gamma_2 := (z_3, z_4)$ where

$$\gamma_j := \{z: z = \rho_j(\theta)e^{i\theta}, 0 \le \theta \le \alpha\pi\}, \qquad j = 1, 2,$$

with $\rho_1(\theta) > 1$ and $0 < \rho_2(\theta) < 1$.

It is easy to see that the domain decomposition method and the associated theory can also be applied to quadrilaterals of the above form, provided that the crosscut of subdivision is taken to be the arc $c := \{z: z = e^{i\theta}, 0 \le \theta \le \alpha\pi\}$ of the unit circle. For example, this can be seen by observing that the transformation

$$z \to \frac{1}{i\alpha\pi} \log z$$

maps the quadrilaterals Q and

$$\{G_1; z_1, z_2, e^{i\alpha\pi}, 1\}, \qquad \{G_2; 1, e^{i\alpha\pi}, z_3, z_4\},$$

illustrated in Fig. 1.2, onto three quadrilaterals of the form (1.3)-(1.5), with

$$\tau_j(x) = \frac{(-1)^{j-1}}{\alpha\pi} \log \rho_j(\alpha\pi x), \qquad j = 1, 2.$$

5. Numerical Examples and Discussion

Each of the two examples given below involves the mapping of a quadrilateral Q of the form (1.3) and, in each case, the decomposition is performed by subdividing Q into two quadrilaterals $Q_j, j = 1, 2$, of the form (1.4)-(1.5). In each example, we use the following notations for the presentation of the results:

- E_h and $E_g^{(j)}, j = 1, 2$: As before, these denote the actual errors (1.16) in the domain decomposition approximations to the module $h := m(Q)$ and the conformal map $g: R \to G$. More precisely, the values E_h and $E_g^{(j)}$ listed in the examples are reliable estimates of the actual errors. They are determined from accurate approximations to $h, h_j, j = 1, 2$, and $g, g_j, j = 1, 2$, which are computed by using the iterative algorithms described in [5]. In particular, $E_g^{(j)}$, $j = 1, 2$, are the maxima of two sets of values, which are obtained by sampling respectively the approximations to the functions $g(w) - g_1(w)$ and $g(w + iE_h) - g_2(w)$ at a number of test points on the boundary segments $\eta = -h_1, 0$ of $R_1 := R_{h_1}\{-h_1\}$ and $\eta = 0, h_2$ of $R_2 := R_{h_2}\{0\}$.
- $T(E_h)$ and $T(E_g^{(j)}), j = 1, 2$: These denote the theoretical estimates of the errors E_h and $E_g^{(j)}, j = 1, 2$, which are given respectively by the expressions on the right-hand sides of (4.15) and (4.19).

Example 5.1. Let Q and $Q_j, j = 1, 2$, be defined by (1.3)-(1.5) with

$$\tau_1(x) = 1.5 + 0.2 \operatorname{sech}^2(2.5x) + l$$

and

$$\tau_2(x) = 0.25x^4 - 0.375x^2 + 0.333x + 1.25 + l,$$

where $l \ge 0$.

In this case $d_1 = 0.3849$, $d_2 = 0.5830$, and the largest values of $\varepsilon_j, j = 1, 2$, i.e., $\varepsilon_1 = 0.3918$ and $\varepsilon_2 = 0.6064$, occur when $l = 0$. Therefore, the functions $\tau_j, j = 1, 2$, satisfy Assumptions A4.1, for all $l \ge 0$.

Table 5.1

(a)

l	h_1	h_2	h
0.00	1.565 514 72	1.333 348 92	2.898 870 58
0.25	1.815 515 54	1.583 350 99	3.398 867 97
0.50	2.065 515 71	1.833 351 42	3.898 867 43
0.75	2.315 515 74	2.083 351 51	4.398 867 31
1.00	2.565 515 75	2.333 351 53	4.898 867 29

(b)

l	E_h	$T(E_h)$	$E_g^{(1)}$	$T(E_g^{(1)})$	$E_g^{(2)}$	$T(E_g^{(2)})$
0.00	7.0E − 6	2.6E − 4	2.1E − 3	4.2E − 2	2.1E − 3	5.5E − 2
0.25	1.5E − 6	5.1E − 5	9.6E − 4	1.9E − 2	9.6E − 4	2.4E − 2
0.50	3.0E − 7	1.0E − 5	4.4E − 4	8.4E − 3	4.4E − 4	1.1E − 2
0.75	6.3E − 8	2.1E − 6	2.0E − 4	3.8E − 3	2.0E − 4	5.0E − 3
1.00	1.3E − 8	4.4E − 7	9.1E − 5	1.7E − 3	9.0E − 5	2.2E − 3

The numerical results corresponding to the values $l = 0.0(0.25)1.0$ are listed in Table 5.1(a) and (b). This table contains respectively the computed values of the conformal modules, which are expected to be correct to seven significant figures, and the values of the error estimates E_h, $T(E_h)$ and $E_g^{(j)}$, $T(E_g^{(j)}), j = 1, 2$.

Example 5.2. Let Q and $Q_j, j = 1, 2$, be defined by (1.3) – (1.5) with

$$\tau_1(x) = c > 0 \qquad \text{and} \qquad \tau_2(x) = 0.25x^4 - 0.5x^2 + 2.0 + l, \quad l \geq 0.$$

In this case Q is of the special form considered in Remark 4.5, i.e., $g_1(w) = w$, $h_1 := m(Q_1) = c$, and $d_1 = \varepsilon_1 = 0$. Also, $d_2 = 0.3849$ and, for all $l \geq 0$, $m_2 \leq 4.1 \times 10^{-3}$. Hence, (4.25) gives

$$\varepsilon_2 := d_2 \left\{ \frac{1 + m_2^2}{1 - m_2^2} \right\} < 0.385, \qquad \forall l \geq 0,$$

i.e., the simplified results (4.21)–(4.24) hold for all $l \geq 0$.

The numerical results corresponding to the values $c = 1$ and $l = 0.0(0.5)2.0$ are listed in Table 5.2(a) and (b). As in Example 5.1, the table contains respectively the computed values of the conformal modules h and h_2, and the values of the error estimates E_h, $T(E_h)$ and $E_g^{(j)}$, $T(E_g^{(j)}), j = 1, 2$.

We recall that $h_1 = 1$, and observe that the values of h and h_2 listed in Table 5.2(a) are expected to be correct to the number of figures quoted. (The algorithms of [5] achieve this remarkable accuracy because, in this case, the curve $\Gamma_2 \colon Z = \rho_2(\theta)e^{i\theta}$ corresponding to the arc $y = \tau_2(x)$ is analytic; see the comment made in Remark 3, p. 279, of [5].) We also observe that the estimates given in Table 5.2(b) remain unchanged for any value $c > 0$; see Remark 4.5.

Table 5.2

(a)

l	h_2	h
0.0	1.859 568 647 615	2.859 569 034 971
0.5	2.359 569 018 925	3.359 569 035 644
1.0	2.859 569 034 971	3.859 569 035 694
1.5	3.359 569 035 664	4.359 569 035 695
2.0	3.859 569 035 694	4.859 569 035 695

(b)

l	E_h	$T(E_h)$	$E_g^{(1)}$	$T(E_g^{(1)})$	$E_g^{(2)}$	$T(E_g^{(2)})$
0.0	3.9E − 7	1.4E − 6	5.0E − 4	2.9E − 3	5.0E − 4	4.8E − 3
0.5	1.7E − 8	6.1E − 8	1.0E − 4	6.0E − 4	1.0E − 4	1.0E − 3
1.0	7.2E − 10	2.6E − 9	2.2E − 5	1.3E − 4	2.1E − 5	2.1E − 4
1.5	3.1E − 11	1.1E − 10	4.5E − 6	2.6E − 5	4.5E − 6	4.4E − 5
2.0	1.4E − 12	4.9E − 12	9.3E − 7	5.4E − 6	9.3E − 7	9.0E − 6

We end this section by making the following concluding remarks:

Remark 5.1. The results of the two examples given above illustrate the remarkable accuracy that can be achieved by the domain decomposition method, even when the quadrilaterals involved are only moderately long. Furthermore, the results confirm the theory of Section 4 and show that the error estimates given in Theorems 4.1 and 4.4 reflect closely the actual errors in the domain decomposition approximations.

Remark 5.2. We recall the method used for computing the values $E_g^{(1)}$ and $E_g^{(2)}$ listed in Tables 5.1(b) and 5.2(b), and note that in both examples the maxima of $|g(w) - g_1(w)|$ and $|g(w + iE_h) - g_2(w)|$ occur on the common boundary segment $\eta = 0$ of R_1 and R_2. The errors on the sides $\eta = -h_1$ of R_1 and $\eta = h_2$ of R_2 are much smaller, indicating that the estimates of

$$E_X^{(1)} := \max_{0 \le \xi \le 1} |X(\xi) - X_1(\xi)| \quad \text{and} \quad E_X^{(2)} := \max_{0 \le \xi \le 1} |\hat{X}(\xi) - \hat{X}_2(\xi)|,$$

given in Theorem 4.2, are pessimistic. In fact, there is strong experimental evidence which suggests that $E_X^{(1)}$ and $E_X^{(2)}$ are both $O\{e^{-2\pi h^*}\}$, rather than $O\{e^{-\pi h^*}\}$ as predicted by (4.17). (Of course, exactly the same remark applies to the estimates referred to in Remark 4.2.)

The very close agreement between the values $E_g^{(1)}$ and $E_g^{(2)}$ listed in the tables is related to the above observations, and can be explained by the results of Theorem 4.3 and those given in Remark 4.5.

Remark 5.3. Since $h \geq h_1 + h_2$, the results of Theorems 4.1-4.4 provide computable error estimates, i.e., estimates that can be computed easily once the approximations to the conformal modules h_1 and h_2 are determined. In addition the results of the theorems can be used to provide *a priori* error estimates, i.e., estimates that can be determined before the approximations to h_1 and h_2 are computed. This can be done by observing that

$$h_j \geq \min_{0 \leq x \leq 1} \tau_j(x) := b_j, \quad j = 1, 2,$$

and

$$h \geq b_1 + b_2 := b,$$

and replacing the values of h and h_j, $j = 1, 2$, respectively by the lower bounds b and b_j, $j = 1, 2$.

Remark 5.4. Our final remark concerns Assumptions A4.1 under which the theoretical results of Section 4 were established. The most restrictive of these assumptions is, of course, condition (4.14) which requires that the quantities ε_j, $j = 1, 2$, are less than unity. In practice, (4.14) is more or less equivalent to requiring that the slopes of the two curves $y = \tau_j(x)$, $j = 1, 2$, are numerically less than unity in [0, 1]. This is so because the values m_j, $j = 1, 2$, given by (4.14a) are "small," even when the two quadrilaterals Q_j, $j = 1, 2$, are only moderately "long."

Condition (4.14) is certainly needed for our method of proof. However, the results of the example considered in Section 5 of [12] and those of several other numerical experiments given in [14] indicate clearly that

$$E_h = O\{e^{-2\pi h^*}\} \tag{5.1a}$$

and

$$E_g^{(j)} = O\{e^{-\pi h^*}\}, \qquad j = 1, 2, \tag{5.1b}$$

with $h^* = \min(h_1, h_2)$, even when (4.14) is not fulfilled. In fact there is very strong experimental evidence which suggests that the results (5.1) hold when the functions τ_j, $j = 1, 2$, satisfy only the first two conditions of Assumptions A4.1.

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