Computational Methods and Function Theory
Volume 8 (2008), No. 2, 385–407

Asymptotics for Polynomial Zeros: Beware of Predictions from Plots

Edward B. Saff and Nikos S. Stylianopoulos

(Communicated by J. Milne Anderson and Philip J. Rippon)

Dedicated to Walter Hayman on the occasion of his eightieth birthday

Abstract. We consider five plots of zeros corresponding to four eponymous planar polynomials (Szegő, Bergman, Faber and OPUC), for degrees up to 60, and state five conjectures suggested by these plots regarding their asymptotic distribution of zeros. By using recent results on zero distribution of polynomials we show that all these "natural" conjectures are *false*. Our main purpose is to provide the theoretical tools that explain, in each case, why these accurate, low degree plots are misleading in the asymptotic sense.

Keywords. Szegő polynomials, Bergman polynomials, Faber polynomials, OPUC, zeros of polynomials, equilibrium measure.

2000 MSC. Primary 30C10; Secondary 30C15, 30C40, 31A05, 31A15.

1. Introduction: five plots

We are going to introduce five plots for zeros of sequences of polynomials and state conjectures suggested by these plots.

Example 1. Szegő polynomials for the symmetric lens.

Definition 1. In general, the Szegő polynomials $S_n(z)$ are defined for any rectifiable Jordan curve Γ in the complex plane $\mathbb{C}$ by

$$(1.1)\qquad \frac{1}{l}\int_\Gamma S_m(z)\overline{S_n(z)}\,|dz| = \delta_{m,n}, \qquad S_n(z) = \gamma_n z^n + \cdots,\ \gamma_n > 0,$$

where l denotes the length of Γ.

Received February 27, 2007, in revised form July 9, 2007.
Published online September 20, 2007.
The research of E. B. Saff was supported, in part, by the U. S. National Science Foundation under grant DMS-060382.

ISSN 1617-9447/$ 2.50

In Figure 1.1, we display the zeros of the Szegő polynomials $S_n(z)$, for $n = 30$, 40 and 50, for the boundary Λ of the symmetric lens domain G, consisting of two circular arcs of equal radii that meet at $\pm i$ with angles $\pi/4$. The high precision calculation of the zeros of $S_n(z)$ for $n \le 50$, suggests:

Conjecture 1. All zeros of $S_n(z)$ for the boundary Λ of the symmetric lens lie on the imaginary axis, for all n.

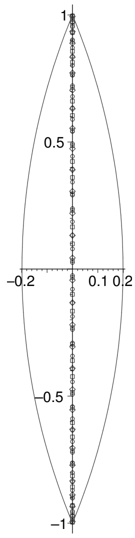

FIGURE 1.1. Symmetric lens Λ: Zeros of the Szegő polynomials S_n, for $n = 30$, 40 and 50.

Example 2. Bergman polynomials for the canonical pentagon.

Definition 2. For any bounded Jordan domain G, the Bergman polynomials $B_n(z)$ are orthonormal with respect to the area measure on G:

$$\int_G B_m(z)\overline{B_n(z)}\,dA(z) = \delta_{m,n}, \qquad B_n(z) = \lambda_n z^n + \cdots,\ \lambda_n > 0. \tag{1.2}$$

In Figure 1.2, we plot the zeros of the Bergman polynomials $B_n(z)$, for n up to 50, with G the interior of the canonical pentagon Π. The computations suggest:

Conjecture 2 (due to Eiermann and Stahl [7])**.** The only points of the boundary Π that are limit points of zeros of the $B_n(z)$, $n = 1, 2, \ldots$, are its five vertices.

Example 3. Bergman polynomials for the hypocycloid.

Consider the hypocycloid defined by

$$Y := \left\{ z\colon z = w + \frac{1}{2w^2},\ |w| = 1 \right\}.$$

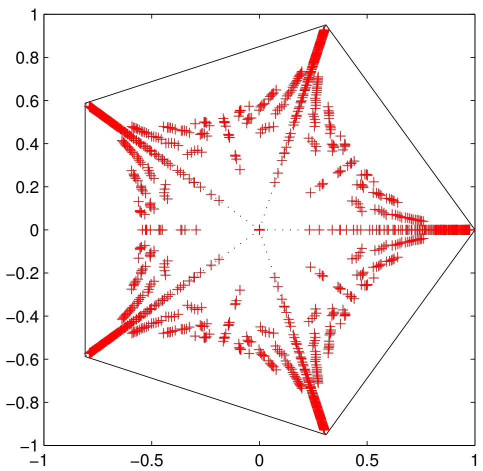

FIGURE 1.2. Canonical pentagon Π: Zeros of the Bergman polynomials B_n, for n up to 50.

In Figure 1.3, we plot the zeros of the Bergman polynomials $B_n(z)$, for degrees $n = 40$, 50 and 60, with $G = \text{int}(Y)$. These suggest, along with the computation of the zeros of $B_n(z)$, for $n \leq 60$, the following conjecture.

Conjecture 3. For all n, the zeros of $B_n(z)$ lie on the three radial lines $[0, 1.5]$, $e^{i\omega}[0, 1.5]$, $e^{i2\omega}[0, 1.5]$, $\omega = 2\pi/3$.

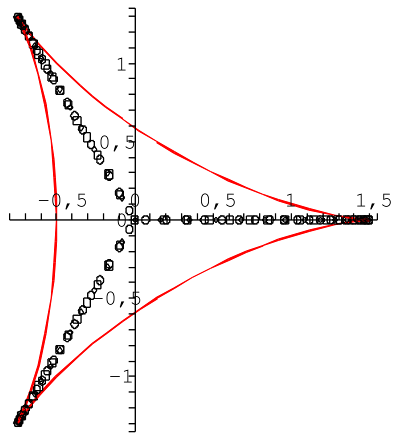

FIGURE 1.3. Hypocycloid Y: Zeros of the Bergman polynomials B_n, for $n = 40$, 50 and 60.

Example 4. Faber polynomials for the equilateral triangle.

For any compact set $E \subset \mathbb{C}$ with simply-connected complement $\Omega := \overline{\mathbb{C}} \setminus E$, the Faber polynomials of E are defined as follows.

Definition 4. Let Φ denote the Riemann mapping of Ω onto $\Delta := \{w\colon |w| > 1\}$, normalized so that, near infinity,

$$w = \Phi(z) = \frac{z}{c} + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \qquad c > 0. \tag{1.3}$$

Then, the n-th degree Faber polynomial $F_n(z)$ of E is the polynomial part of $\Phi^n(z)$, i.e.

$$F_n(z) - \Phi^n(z) = \mathcal{O}\left(\frac{1}{z}\right), \qquad \text{as } z \to \infty.$$

The constant c is called the (logarithmic) *capacity* of E and is denoted by $\text{cap}(E)$. Thus,

$$F_n(z) = \frac{1}{\text{cap}(E)^n} z^n + \cdots. \tag{1.4}$$

In Figure 1.4, we plot the zeros of the Faber polynomials $F_n(z)$ of degrees $n = 10$, 15 and 20, corresponding to the equilateral triangle T. These suggest the following conjecture.

Conjecture 4. All zeros of $F_n(z)$, $n = 1, 2, \ldots$, either lie on or are attracted to the radial lines $l_1 := e^{i\pi/3}[0,1]$, $l_2 := e^{i\pi}[0,1]$ and $l_3 := e^{-i\pi/3}[0,1]$ of T.

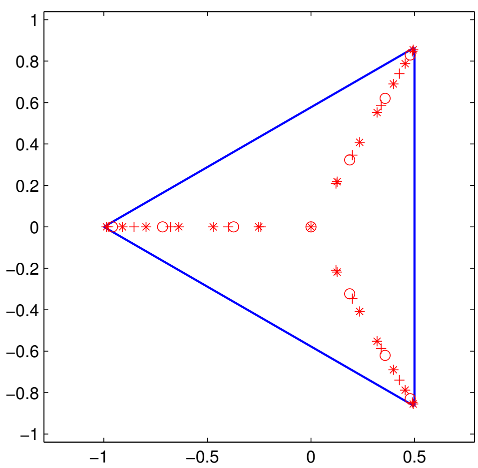

FIGURE 1.4. Equilateral triangle T: Zeros of the Faber polynomials F_n, for $n = 10$, 15 and 20.

Example 5. Orthogonal polynomials on the unit circle (OPUC) with respect to a measure $d\mu$ (also known as Szegő polynomials w.r.t. $d\mu$).

Definition 5. For a positive finite Borel measure μ with infinite support on $C := \{z\colon |z| = 1\}$, the orthonormal polynomials with respect to $d\mu$ are the polynomials $\varphi_n(z)$, $n = 0, 1, \ldots$, that satisfy

$$\int \varphi_m(z)\overline{\varphi_n(z)}d\mu(z) = \delta_{m,n}, \qquad \varphi_n(z) = \kappa_n z^n + \cdots, \; \kappa_n > 0, \; z = e^{i\theta}. \tag{1.5}$$

In Figure 1.5, we plot the zeros of the polynomials $\varphi_n(z)$, for $n = 40$, 50 and 60, corresponding to the measure

$$d\mu(z) = \left|e^{1/(z-1)^2}\right| d\theta, \qquad z = e^{i\theta}. \tag{1.6}$$

This suggests the following conjecture.

Conjecture 5. As $n \to \infty$, the zeros of $\varphi_n(z)$ tend to a proper subarc of the unit circle that omits $z = 1$.

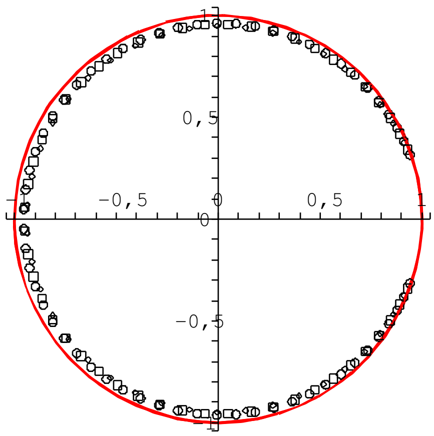

FIGURE 1.5. Unit circle C: Zeros of the OPUC φ_n, for $n = 40$, 50 and 60.

The surprising fact is that *all the above conjectures are false!* We wish to emphasize that all the plots are accurate to high precision, so that they represent the truth for the values of n described. The conjectures they suggest fail in the asymptotic sense as $n \to \infty$. In the next two sections we provide the asymptotic theory that disproves the conjectures, as well as give explanations as to why these lower degree plots have an appearance different from the asymptotic truth.

2. Fundamental theorems for the distribution of zeros

From now on we denote by G a bounded simply-connected domain with Jordan boundary Γ and let Ω be the complement $\overline{\mathbb{C}} \setminus \overline{G}$ with respect to the extended complex plane $\overline{\mathbb{C}}$. For simplicity, we state the theorems only for the case when Γ is a *piecewise analytic Jordan curve.* We begin with some definitions that are needed for the statements of our results. Let Q_n be a polynomial of degree n with zeros $z_1, z_2, \ldots, z_n$. The *normalized counting measure of the zeros* of Q_n is defined by

$$\nu(Q_n) := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k},$$

where δ_z denotes the unit point mass at the point z. In other words, for any subset A of $\mathbb{C}$,

$$\nu(Q_n)(A) = \frac{\text{number of zeros of } Q_n \text{ in } A}{n}.$$

Next, given a sequence $\{\sigma_n\}$ of Borel measures, we say that $\{\sigma_n\}$ *converges in the weak* sense* to a measure σ, symbolically $\sigma_n \xrightarrow{*} \sigma$, if

$$\int f d\sigma_n \longrightarrow \int f d\sigma, \quad n \to \infty,$$

for every function f continuous on $\overline{\mathbb{C}}$.

The exterior conformal map $\Phi \colon \Omega \to \Delta$ defined by (1.3) with $\Omega := \text{ext}(\Gamma)$, can be naturally extended to a homeomorphism between the corresponding closed domains, so that

$$\Phi(\Gamma) = \mathbb{T} := \{w \colon |w| = 1\}.$$

Then, the normalized angular measure $d\theta/2\pi$ on $\mathbb{T}$ gives rise, in a natural way, to a unit measure μ_Γ on Γ. Namely, for any Borel set $A \subset \Gamma$,

$$\mu_\Gamma(A) := \frac{1}{2\pi} \int_{\Phi(A)} d\theta.$$

This measure is called the *equilibrium measure for* Γ. See [2, 26, 28] for the definition of the equilibrium measure of more general compact sets.

Below we collect together some asymptotic properties of B_n, S_n and F_n that we are going to need for our analysis. Regarding Bergman polynomials, it is well known that (cf. [22, Lem. 4.3])

$$\lim_{n\to\infty} \|B_n\|_{\overline{G}}^{1/n} = 1. \tag{2.1}$$

(Here and in what follows $\|\cdot\|$ denotes the uniform norm on the subscripted set.) Proceeding as in the proof of (2.1) given in [22, pp. 335–336], and using the well known estimate

$$|f(z_0)|^2 \leq \frac{1}{2\pi r} \int_\Gamma |f(z)|^2 \, |dz|,$$

which is valid for any function f analytic in G, with square integrable boundary values on Γ, and for any z_0 in G, such that the disk $|z_0 - z| < r$ is contained in G (see e.g. [8, p. 117]), one can easily see that exactly the same property holds also for the Szegő polynomials of Definition 1, i.e.

$$\lim_{n\to\infty} \|S_n\|_{\overline{G}}^{1/n} = 1. \tag{2.2}$$

It turns out that (2.1) and (2.2) imply the following pairs of n-th root asymptotics:

$$\lim_{n\to\infty} \lambda_n^{1/n} = \frac{1}{\mathrm{cap}(\Gamma)}, \qquad \lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\mathrm{cap}(\Gamma)}; \tag{2.3}$$

$$\limsup_{n\to\infty} |B_n(z)|^{1/n} = 1 \quad \text{and} \quad \limsup_{n\to\infty} |S_n(z)|^{1/n} = 1, \tag{2.4}$$

for quasi-every† z on Γ;

$$\limsup_{n\to\infty} |B_n(z)|^{1/n} \le |\Phi(z)| \quad \text{and} \quad \limsup_{n\to\infty} |S_n(z)|^{1/n} \le |\Phi(z)|, \tag{2.5}$$

uniformly in $\overline{\Omega} \setminus \{\infty\}$;

$$\lim_{n\to\infty} |B_n(z)|^{1/n} = |\Phi(z)| \quad \text{and} \quad \lim_{n\to\infty} |S_n(z)|^{1/n} = |\Phi(z)|, \tag{2.6}$$

locally uniformly on $\overline{\mathbb{C}} \setminus \mathrm{Co}(\overline{G})$, where $\mathrm{Co}(\overline{G})$ denotes the convex hull of $\overline{G}$.

The asymptotics (2.3)–(2.6) follow at once from Theorems 3.1.1, 3.2.1 and 3.2.3 of [32], by observing that the Green function $g_\Omega(z,\infty)$ of Ω with pole at infinity is given by $\log|\Phi(z)|$ and that $|\Phi(z)| \to 1$, as $z \to z' \in \partial\Omega$. (In the terminology of potential theory this means that Ω is a regular domain.)

Regarding Faber polynomials we note the following n-th root asymptotics:

$$\limsup_{n\to\infty} |F_n(z)|^{1/n} \le 1, \qquad z \in \Gamma; \tag{2.7}$$

$$\lim_{n\to\infty} |F_n(z)|^{1/n} = |\Phi(z)|, \tag{2.8}$$

locally uniformly in Ω;

$$\lim_{n\to\infty} \|F_n\|_{\overline{G}}^{1/n} = 1. \tag{2.9}$$

The asymptotics (2.7)–(2.9) are well known. The proof of (2.7) and (2.8) can be found, for example, in [33, pp. 42–43] and (2.9) follows easily from (2.7) and (1.4) by using the fact that for any monic polynomial p_n of degree n there holds $\|p_n\|_{\overline{G}} \ge \mathrm{cap}(\Gamma)^n$ (cf. e.g. [2, p. 16], [28, p. 46]).

The connection between the normalized interior mapping function $\varphi_\zeta \colon G \to \mathbb{D}$, $\zeta \in G$, with $\varphi_\zeta(\zeta) = 0$ and $\varphi'_\zeta(\zeta) > 0$, and the orthogonal polynomials $B_n(z)$ and

†By "quasi-every z on a set S" we mean "for all $z \in S \setminus E$, where E has logarithmic capacity zero".

$S_n(z)$, comes via the Bergman and the Szegő kernels, denoted here by $K_B(z,\zeta)$ and $K_S(z,\zeta)$ respectively; see e.g. [8, Kap. II], [31, Ch. 4] and [34, Ch. XVI] for details. More precisely,

$$\begin{aligned} K_B(z,\zeta) &= \frac{1}{\pi}\,\varphi_\zeta'(\zeta)\,\varphi_\zeta'(z),\\ K_B(z,\zeta) &= \sum_{n=0}^{\infty}\overline{B_n(\zeta)}B_n(z), \end{aligned}\qquad z,\zeta\in G \tag{2.10}$$

and

$$\begin{aligned} K_S(z,\zeta) &= \frac{l}{2\pi}\,\sqrt{\varphi_\zeta'(\zeta)}\,\sqrt{\varphi_\zeta'(z)},\\ K_S(z,\zeta) &= \sum_{n=0}^{\infty}\overline{S_n(\zeta)}S_n(z), \end{aligned}\qquad z,\zeta\in G. \tag{2.11}$$

Assume now that $w=\varphi(z)$ is any conformal mapping from G onto the unit disc $\mathbb{D}:=\{w\colon |w|<1\}$. We can now formulate our first result for piecewise analytic Γ.

Theorem 2.1. *A necessary and sufficient condition that there exists a subsequence of normalized zero-counting measures* $\{\nu(B_n)\}_{n=0}^{\infty}$ *(resp.* $\{\nu(S_n)\}_{n=0}^{\infty}$*) that converges in the weak* sense to the equilibrium distribution* μ_Γ *is that* φ *(resp.* $\sqrt{\varphi'}$*) has a singularity on the boundary* Γ *of* G*.*

Note that the fact φ or $\sqrt{\varphi'}$ has a singularity on Γ is independent of the choice of φ, since any two conformal mappings of G onto $\mathbb{D}$ are related by a Möbius transformation.

Proof. The result for $\nu(B_n)$ is given explicitly in [15, Thm. 2.1]. The result for $\nu(S_n)$ can be established along the same lines. More specifically, it follows easily from (2.2) and (2.3) and the proof of Theorem 2.1 in [15] that the result in question will follow from the assertion: $\sqrt{\varphi'}$ *has a singularity on* Γ *if and only if for some* $\zeta\in G$,

$$\limsup_{n\to\infty}|S_n(\zeta)|^{1/n}=1. \tag{2.12}$$

This latter assertion can be established, in turn, by working as in the proof of [22, Thm. 2.1] and replacing the estimates for the Bergman polynomials by the corresponding ones for the Szegő polynomials. In particular, by using (2.2) and (2.11). (See also [35, § 6.6] and [21, Thm. 2.1].) ∎

Corollary 2.1. *If* φ *(respectively* $\sqrt{\varphi'}$*) has a singularity on* Γ*, then every point of* Γ *is a limit point of zeros of the sequence* $\{B_n(z)\}_{n=0}^{\infty}$ *(respectively* $\{S_n(z)\}_{n=0}^{\infty}$*).*

We now apply the result of Corollary 2.1 to the plots of Figures 1.1, 1.2 and 1.3. In Example 1, the associated interior mapping $\varphi_\zeta\colon \operatorname{int}(\Lambda)\to\mathbb{D}$ is easily seen to have a behavior like $(z^2+1)^4$ near $\pm i$. More precisely, it follows from [21, pp. 216–217] that $\varphi_\zeta(z)$ is a rational function with poles in Ω and

$$\varphi'_\zeta(z) = (z^2+1)^3\, h_\zeta(z), \tag{2.13}$$

where $h_\zeta(z)$ is analytic and non-zero on $\overline{G}$. Therefore, while φ_ζ has no singularity on Λ, the function $\sqrt{\varphi'_\zeta}$ does indeed have an algebraic singularity at $\pm i$. Hence, from the corollary above, all the points on Λ must attract zeros of $\{S_n(z)\}_{n=0}^\infty$. Thus, Conjecture 1 is false.

Similarly, in each of Examples 2 and 3, the associated interior conformal mapping φ_ζ will fail to be analytic at the vertices. For the case of the canonical pentagon of Example 2, this is due to the fact that φ_ζ has around a vertex z_j, $j=1,\dots,5$, an asymptotic expansion of the form

$$\varphi_\zeta(z) = \varphi_\zeta(z_j) + \sum_{k=1}^\infty a_k(z-z_j)^{5k/3}, \qquad a_1 \neq 0,$$

as can be easily seen by reversing the series of the local representation of the Schwarz-Christoffel transformation $\varphi_\zeta^{-1}\colon \mathbb{D}\to G$ (cf. e.g. [5, p. 170]).

For the case of the hypocycloid Y in Example 3, this follows from Theorem 3.1 and Corollary 3.2 of [4], where it is shown that $\varphi_\zeta \in C^\infty(Y)$ and every derivative of φ_ζ vanishes at the three vertices of Y. (This forbids φ_ζ from being analytic around any vertex; the contrary assumption will imply that φ_ζ is constant.) Thus, again by Corollary 2.1, every point of the boundary attracts zeros of the corresponding Bergman polynomials.

Consequently, Conjectures 2 and 3 are false. In contrast, for the case where G is the interior of an equilateral triangle or a square, it was shown in [18] that all the zeros of all the Bergman polynomials lie on the radial lines, as was conjectured in [7].

Remark 2.1. Andrievskii and Blatt [1, Thm. 5] were the first to prove that Conjecture 2 is false. Their result, however, does not apply to the case of Example 3, for which φ_ζ is C^∞ on the boundary.

Concerning Example 4, we appeal to the following result of Kuijlaars and Saff [14, Thm. 1.5].

Theorem 2.2. *If Γ is a piecewise analytic curve with a singularity other than an outward pointing cusp, then there is a subsequence of $\{\nu(F_n)\}_{n=0}^\infty$ that converges weak* to the equilibrium distribution μ_Γ of Γ. In such a case, every point of Γ attracts the zeros of $\{F_n(z)\}_{n=0}^\infty$.*

This result immediately shows that Conjecture 4 is false.

Regarding Example 5, we appeal to the following special case of a result of Mhaskar and Saff; see Theorem 2.3 and Remark 1 in [20]:

Theorem 2.3. *Let*

$$\Phi_n(z) := \frac{\varphi_n(z)}{\kappa_n} = z^n + \cdots, \qquad n = 0, 1, \ldots.$$

If the Verblunsky coefficients $\Phi_n(0)$ *satisfy*

$$\limsup_{n\to\infty} |\Phi_n(0)|^{1/n} = 1 \tag{2.14}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} |\Phi_k(0)| = 0, \tag{2.15}$$

then there exists a subsequence of $\{\nu(\varphi_n)\}_{n=0}^{\infty}$ *that converges in the weak* sense to the normalized Lebesgue measure* $d\theta/2\pi$ *on* $C := \{z\colon |z| = 1\}$.

By a result of Rakhmanov [25] (see also [17, Thm. 1]) the condition

$$\mu' > 0 \qquad \text{almost everywhere on } C,$$

implies

$$\lim_{n\to\infty} \Phi_n(0) = 0.$$

For the case of Example 5, μ' clearly satisfies this condition and thus (2.15) holds. Moreover, condition (2.14) holds since the contrary assumption would imply that the function $e^{1/(z-1)^2}$ is analytic at $z = 1$ (cf. [20, Thm. 2.2]). Consequently, by Theorem 2.3, every point of the unit circle C attracts the zeros of $\{\varphi_n(z)\}_{n=0}^{\infty}$, which disproves Conjecture 5.

3. Explanations for the misleading plots

In this section we provide reasons for the appearance of the plots whose short-term behavior is quite different from the asymptotic behavior.

In order to derive our first result we follow closely the construction in [3, §2.5], see also [24, pp. 87–89]. Assume that the boundary Γ of G consists of N analytic arcs that meet at corner points z_j, $j = 1, \ldots, N$, where they form exterior angles $\lambda_j\pi$, with $0 < \lambda_j < 2$. Then, the interior conformal map $\varphi_\zeta\colon G \to \mathbb{D}$ can be extended conformally, by means of the reflection principle, beyond Γ to a larger domain $\widetilde{G}$ such that the boundary $\partial\widetilde{G}$ of $\widetilde{G}$ consists of N analytic arcs γ_j, $j = 1, \ldots, N$, with end points at z_j and z_{j+1} (we set $z_{N+1} = z_1$). The extension is such that φ_ζ is analytic on γ_j except for the endpoints. On each γ_j, $j = 1, \ldots, N$, we choose a fixed point ζ_j and denote by γ_j^1 and γ_j^2 the two parts that ζ_j divides γ_j into, so that $\partial\widetilde{G} = \bigcup_{j=1}^{N}\bigcup_{i=1}^{2}\gamma_j^i$.

Consider now the inner product associated with the Bergman polynomials on G:

$$\langle f, g\rangle_G := \int_G f(z)\overline{g(z)}\, dA(z).$$

This defines the Bergman space $L^2_a(G)$, i.e. the Hilbert space of analytic functions in G with finite induced norm: $\|f\|_{L^2(G)} := \langle f, f\rangle_G^{1/2}$. The next theorem shows that the L^2-error in approximating the Bergman kernel function $K_B(z,\zeta)$ by "low" degree polynomials depends on the exterior angles of Γ, as well as on how far φ_ζ can be analytically extended in $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$.

Theorem 3.1. *With the notation and assumptions above, let* $\lambda := \min_{j=1,\dots,N}\{\lambda_j\}$ *and let* $\rho := \min_{j=1,\dots,N}\{|\Phi(\zeta_j)|\}$. *Then, for any* R, $1 < R < \rho$, *there exist a sequence of polynomials* $\{p_n(z)\}_{n=1}^{\infty}$, *with* $\deg(p_n) = n$, *and positive constants* κ_1 *and* κ_2 *such that*

$$\|K_B(\cdot,\zeta) - p_n\|_{L^2(G)} \le \kappa_1 \frac{1}{R^n} + \kappa_2 \frac{1}{n^{\lambda/(2-\lambda)}}, \qquad n = 1, 2, \dots . \tag{3.1}$$

Remark 3.1. Levin, Papamichael and Siderides were the first to suggest in [16] that the error in approximating φ_ζ on $\overline{G}$ by polynomials of low degree will depend on both the boundary and the pole singularities of φ_ζ. Hence, in order to improve the numerical performance of the so-called Bergman kernel method for approximating φ_ζ, they introduced a method which is based on orthonormalizing a system of functions consisting of monomials and singular terms that reflect both corner and pole singularities of φ_ζ. Thus, Theorem 3.1 provides, in view of (2.10), a theoretical justification for the need of using basis functions that reflect the pole singularities of φ_ζ in Ω. Error estimates providing theoretical justification for the use of singular basis functions reflecting corner singularities, along with the precise improvement in the convergence rates, are given in [19, Thm.3.1 & Thm. 3.2].

From the reproducing property of K_B, the orthogonality of B_{n+1} to any polynomial of degree at most n, and the Cauchy-Schwarz inequality we have

$$\begin{aligned} |B_{n+1}(\zeta)| &= |\langle B_{n+1}, K_B(\cdot,\zeta)\rangle_G| = |\langle B_{n+1}, K_B(\cdot,\zeta) - p_n\rangle| \\ &\le \|B_{n+1}\|_{L^2(G)}\, \|K_B(\cdot,\zeta) - p_n\|_{L^2(G)} \\ &= \|K_B(\cdot,\zeta) - p_n\|_{L^2(G)}. \end{aligned}$$

Hence, Theorem 3.1 yields:

Corollary 3.1. *For any* $\zeta \in G$, *there exist positive constants* κ_1 *and* κ_2 *such that*

$$|B_{n+1}(\zeta)| \le \kappa_1 \frac{1}{R^n} + \kappa_2 \frac{1}{n^{\lambda/(2-\lambda)}}, \qquad n = 1, 2, \dots . \tag{3.2}$$

Remark 3.2. Of course, we cannot expect to obtain sharp estimates for the values of the two constants κ_1 and κ_2 in (3.1). This limits, in a sense, the practical scope of Theorem 3.1 and its corollary, unless a fact such as that κ_1 and κ_2 are of comparable size is available. Nevertheless, Theorem 3.1 and Corollary 3.1 provide a plausible justification for the short-term (i.e. for values of n not too large) geometric decay of the $L^2(G)$-error in (3.1) and of the Bergman polynomials B_n in G, when such a case is, indeed, evident.

Proof of Theorem 3.1. Following the method in [3, §2.5] we have by Cauchy's integral formula for $z \in G$,

$$\varphi_\zeta(z) = \frac{1}{2\pi i}\int_{\partial\widetilde{G}} \frac{\varphi_\zeta(t)}{t-z}\,dt = \frac{1}{2\pi i}\sum_{j=1}^{N}\sum_{i=1}^{2}\int_{\gamma_j^i}\frac{\varphi_\zeta(t)}{t-z}\,dt$$
$$= \frac{1}{2\pi i}\sum_{j=1}^{N}\sum_{i=1}^{2}\int_{\gamma_j^i}\frac{\varphi_\zeta(t)-\varphi_\zeta(z_{j+i-1})}{t-z}\,dt + \frac{1}{2\pi i}\sum_{j=1}^{N}\varphi_\zeta(z_j)\log\frac{\zeta_j - z}{\zeta_{j-1}-z},$$

where we set $\zeta_0 = \zeta_N$. Therefore,

$$\varphi'_\zeta(z) = g_1(z) + g_2(z), \tag{3.3}$$

where

$$g_1(z) := \frac{1}{2\pi i}\sum_{j=1}^{N}\sum_{i=1}^{2}\int_{\gamma_j^i}\frac{\varphi_\zeta(t)-\varphi_\zeta(z_{j+i-1})}{(t-z)^2}\,dt$$

and

$$g_2(z) := \frac{1}{2\pi i}\sum_{j=1}^{N}\varphi_\zeta(z_j)\left(\frac{1}{\zeta_{j-1}-z} - \frac{1}{\zeta_j - z}\right).$$

By using the result of Lemma 6 in [3], in conjunction with the remark following Theorem 1 of the same paper, we see that there exist a sequence of polynomials q_n, with $\deg(q_n) = n$, and a constant $\kappa_3 > 0$, such that

$$\|g_1 - q_n\|_{L^2(G)} \le \kappa_3\,\frac{1}{n^{\lambda/(2-\lambda)}}, \qquad n = 1, 2, \dots . \tag{3.4}$$

Furthermore, since g_2 is analytic on $\overline{G}$, it follows from [35, Thm. 7, §4.7] that there exist a sequence of polynomials r_n, with $\deg(r_n) = n$, and a constant $\kappa_4 > 0$, such that

$$\|g_2 - r_n\|_{\overline{G}} \le \kappa_4\,\frac{1}{R^n}, \qquad n = 1, 2, \dots , \tag{3.5}$$

for any R with $1 < R < \rho$. The required result (3.1) follows from (2.10), (3.3), (3.4), (3.5) and the triangle inequality, because the $L^2(G)$-norm is bounded by the uniform norm on $\overline{G}$ times the square root of the area of G. ∎

We shall also need the following lemma.

Lemma 3.1. *Let $\{Q_n(z)\}_{n=1}^{\infty}$ be a sequence of polynomials of respective degrees $n = 1, 2, \dots$ with positive leading coefficients β_n, such that*

(3.6)
$$\lim_{n\to\infty} {\beta_n}^{1/n} = \frac{1}{\operatorname{cap}(\Gamma)} \quad \textit{and} \quad \limsup_{n\to\infty} |Q_n(z)|^{1/n} \le 1 \textit{ for quasi-every } z \in \Gamma,$$

where, as in the preceding section, we assume that Γ is piecewise analytic. Extend Φ by reflection across a part of Γ, so that Φ is analytic in $\Omega \cup B$, where B

is a compact set with simply-connected interior containing points in both G and Ω; see Figure 3.1. Further, assume that

$$\limsup_{n\to\infty} |Q_n(z)|^{1/n} \le |\Phi(z)|, \qquad \textit{for quasi-every } z \in G \cap B. \tag{3.7}$$

Then,

$$\nu(Q_n)(K) \xrightarrow{\ *\ } 0, \qquad n \to \infty,$$

for any closed subset K of $\Omega \cup B$. In particular,

$$\nu(Q_n)(B) \xrightarrow{\ *\ } 0, \qquad n \to \infty. \tag{3.8}$$

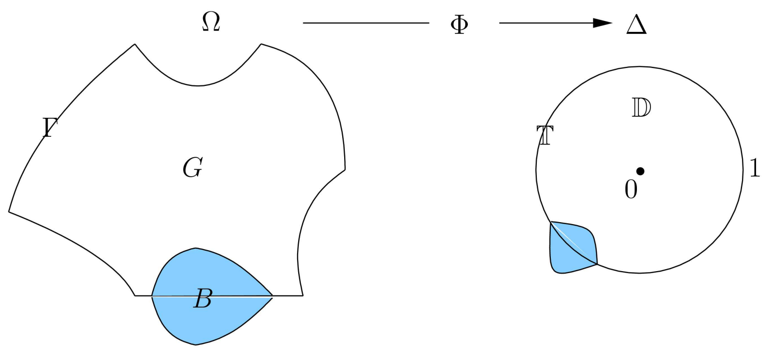

FIGURE 3.1. Illustration of Lemma 3.1.

Proof. The result follows easily from Lemma 4.3 in [21], by setting $E = \overline{G \setminus B}$, $g(z) = \text{cap}(\Gamma)\Phi(z)$, $q_n(z) = Q_n(z)/\beta_n$ and observing that (3.6), in conjunction with [28, Lem. III.4.4], implies $\nu(Q_n)(K) \xrightarrow{\ *\ } 0$, $n \to \infty$, for any closed subset K of Ω. ∎

Remark 3.3. Since $|\Phi(z)| < 1$, for $z \in G \cap B$, Lemma 3.1 requires that $Q_n(z)$ decays to zero geometrically fast, quasi-everywhere in $G \cap B$. In contrast, if there exist a subsequence $\mathcal{N}$ of $\mathbb{N}$ and a point $z_0 \in G$, such that

$$\lim_{n\to\infty,\, n\in\mathcal{N}} |Q_n(z_0)|^{1/n} = 1,$$

then

$$\nu(Q_n) \xrightarrow{\ *\ } \mu_\Gamma, \qquad n \to \infty, \quad n \in \mathcal{N};$$

see [28, Thm. III.4.1].

Remark 3.4. It follows from (1.4) and the asymptotic results given in (2.1)–(2.8) that Szegő, Bergman and Faber polynomials satisfy the condition (3.6). Hence, in order for property (3.8) to hold for these polynomials, it suffices to show that (3.7) is valid.

3.1. Szegő polynomials for the lens. Let G denote the lens of Example 1 and recall the definition of the exterior conformal map Φ in (1.3). The boundary $\Gamma = \Lambda$ of G consists of two circular arcs L_α and L_β (L_α being to the left of L_β) meeting at i and $-i$ with opening angle $\pi/4$. For any point $z \in \overline{G}$ the reflections z_α and z_β of z with respect to L_α and L_β are given by

$$z_\alpha = \frac{a\overline{z}+1}{\overline{z}-a}, \qquad z_\beta = \frac{-a\overline{z}+1}{\overline{z}+a}, \tag{3.9}$$

with $a = \cot(\pi/8)$. Define

$$G_\alpha := \mathrm{int}(L_\alpha \cup [-i,i]), \qquad G_\beta := \mathrm{int}(L_\beta \cup [-i,i]).$$

Then, by the reflection principle, the formula

$$\Phi(z) := \begin{cases} \dfrac{1}{\overline{\Phi(z_\alpha)}}, & \text{if } z \in G_\alpha \cup [-i,i], \\ \dfrac{1}{\overline{\Phi(z_\beta)}}, & \text{if } z \in G_\beta, \end{cases} \tag{3.10}$$

gives an analytic and conformal extension of Φ to $\mathbb{C} \setminus [-i,i]$ such that $|\Phi|$ is continuous in $\overline{\mathbb{C}}$; cf. [21, p. 194].

We establish first the following estimate.

Proposition 3.1. *For any ζ in the lens G, there exist positive constants κ_1 and κ_2 such that*

$$|S_{m+n}(\zeta)| \le \kappa_1\,|\Phi(\zeta)|^n + \kappa_2\,\frac{1}{m^{7/2}}, \qquad m,n = 1,2,\dots. \tag{3.11}$$

Proof. Consider the inner product

$$\langle f, g\rangle_\Gamma := \frac{1}{l}\int_\Gamma f(z)\overline{g(z)}\,|dz|,$$

with induced norm $L^2(\Gamma)$ given by $\|f\|_{L^2(\Gamma)} := \langle f, f\rangle_\Gamma^{1/2}$ and the corresponding Hilbert space $L^2(\Gamma)$ of all analytic functions defined in G having boundary values with finite $L^2(\Gamma)$ norm.

Below we establish that there exists a sequence of polynomials π_k of respective degrees k such that

$$\|K_S(\cdot,\zeta) - \pi_{m+n}\|_{L^2(\Gamma)} \le \kappa_1|\Phi(\zeta)|^n + \kappa_2\frac{1}{m^{7/2}}, \qquad m,n = 1,2,\dots. \tag{3.12}$$

The required result will then follow, because from the reproducing property of K_S and the Cauchy-Schwarz inequality we have

$$\begin{aligned} |S_{m+n+1}(\zeta)| &= |\langle S_{m+n+1}, K_S(\cdot,\zeta)\rangle| = |\langle S_{m+n+1}, K_S(\cdot,\zeta) - \pi_{m+n}\rangle| \\ &\le \|S_{m+n+1}\|_{L^2(\Gamma)}\,\|K_S(\cdot,\zeta) - \pi_{m+n}\|_{L^2(\Gamma)} \\ &= \|K_S(\cdot,\zeta) - \pi_{m+n}\|_{L^2(\Gamma)}. \end{aligned}$$

From the relation between the Bergman and the Szegő kernels given in (2.10)–(2.11) and the discussion concerning the Bergman kernel of lens-shaped domains in [21, pp. 216–219], we infer that

$$K_S(z,\zeta) = \frac{4l}{\pi}\sqrt{(z^2+1)^3}\sqrt{g_\zeta(z)}, \tag{3.13}$$

where $g_\zeta(z)$ is a non-zero rational function having four double poles in Ω, with the nearest one to G lying on the level curve $\Gamma_\rho := \{z\colon |\Phi(z)| = \rho > 1\}$ of index $\rho = 1/|\Phi(\zeta)|$. Hence, there exists a sequence of polynomials p_n of respective degrees n and a positive constant M_1 such that

$$\|\sqrt{g_\zeta} - p_n\|_{\overline{G}} \le M_1|\Phi(\zeta)|^n, \qquad n = 1,2,\ldots; \tag{3.14}$$

this follows from [27, Thm. 2], by using the Faber polynomials $F_n(z)$ of $\overline{G}$ for $\omega_n(z)$ and noting:

(i) for any r, R, such that $1 < r < R$ and any $z \in \Omega$, with $|\Phi(z)| = R$,
$$F_n(z) = \Phi^n(z)\left\{1 + \mathcal{O}\left(\frac{r^n}{R^n}\right)\right\};$$
see [33, p. 43];

(ii) the convexity of G implies $\|F_n\|_{\overline{G}} \le 2$ and the fact that all the zeros of $F_n(z)$ are contained in G; see [13].

Since $\overline{G}$ is compact, we deduce that

$$\left\|\sqrt{(z^2+1)^3}\sqrt{g_\zeta} - \sqrt{(z^2+1)^3}\,p_n\right\|_{\overline{G}} \le M_2\,|\Phi(\zeta)|^n. \tag{3.15}$$

From (2.13) it follows that $\sqrt{\varphi'_\zeta}$ and $\sqrt{(z^2+1)^3}$ have similar singular behaviors on Γ. Thus, by arguing as in the proof of Theorem 1.2 in [24], in particular since the estimate (3.6) in [24, p. 89] holds also for $\sqrt{(z^2+1)^3}$, we see that there exist a sequence of polynomials Q_m of respective degrees m and a positive constant M_3 such that

$$\left\|\sqrt{(z^2+1)^3} - Q_m\right\|_{L^2(\Gamma)} \le M_3\frac{1}{m^{7/2}}, \qquad m = 1,2,\ldots. \tag{3.16}$$

Since the p_n's are uniformly bounded on $\overline{G}$ (see (3.14)), the latter inequality yields

$$\left\|\sqrt{(z^2+1)^3}\,p_n - Q_m\,p_n\right\|_{L^2(\Gamma)} \le M_4\frac{1}{m^{7/2}}, \quad m = 1,2,\ldots \tag{3.17}$$

and the required estimate (3.12) follows in view of (3.13) and (3.15), by using the triangle inequality and the fact that $\|\cdot\|_{L^2(\Gamma)}$ is bounded by the uniform norm on $\overline{G}$. ∎

Discussion. Since

$$\min_{\zeta\in G} |\Phi(\zeta)| = |\Phi(0)| \approx 0.797,$$

where the approximation to $\Phi(0)$ was obtained by means of the numerical conformal mapping package CONFPACK [12], and

$$(0.797)^{50} \approx 1.18 \times 10^{-5}, \qquad \frac{1}{50^{7/2}} \approx 1.13 \times 10^{-6},$$

it follows from Proposition 3.1 that for $n \le 100$, essentially,

$$|S_n(\zeta)| \le \kappa\, |\Phi(\zeta)|^n, \qquad \zeta \in G. \tag{3.18}$$

This suggests that $S_n(\zeta)$, for small values of n, "thinks" it belongs to a sequence for which (3.18) holds for any $n \in \mathbb{N}$. Hence it places its zeros according to Lemma 3.1, i.e. so that

$$\nu(S_n)(B) \xrightarrow{*} 0,$$

for any compact $B \subset \mathbb{C} \setminus [-i, i]$. Furthermore, since

$$|\Phi(0.1)| \approx 0.895 \qquad \text{and} \qquad |\Phi(0.15)| \approx 0.947,$$

the same argument as above indicates that $S_n(z)$ should start placing zeros around the points $\zeta = 0.1$ and $\zeta = 0.15$, only when $n > 300$ and $n > 750$, respectively.

To further illustrate the above, we test numerically the hypothesis

$$|S_n(\zeta)| \approx C|\Phi(\zeta)|^n, \tag{3.19}$$

for $\zeta = 0.0$, 0.1, 0.15, and some positive constant C. We do this by seeking to recover the value of $|\Phi(\zeta)|$ in (3.19) by means of the formula

$$\rho_n := \left(\frac{|S_{n+5}(\zeta)|}{|S_n(\zeta)|}\right)^{1/5}.$$

The numerical results for $n = 20(2)30$ are given in Table 3.1. These results indicate clearly the geometric decay of the Szegő polynomials for the tested values of ζ and n. They also indicate that for $\zeta = 0.0$, $\zeta = 0.1$, and $\zeta = 0.15$, the value of the constant C in (3.19) is around 0.78, 0.42 and 0.40, respectively.

	$\zeta = 0.0$	$\lvert\Phi(\zeta)\rvert \approx 0.797$	$\zeta = 0.1$	$\lvert\Phi(\zeta)\rvert \approx 0.895$	$\zeta = 0.15$	$\lvert\Phi(\zeta)\rvert \approx 0.947$
n	$\lvert S_n(\zeta)\rvert$	ρ_n	$\lvert S_n(\zeta)\rvert$	ρ_n	$\lvert S_n(\zeta)\rvert$	ρ_n
20	8.4e-03	-	4.4e-02	-	1.4e-01	-
22	5.3e-03	0.80	3.5e-02	0.89	1.2e-01	0.95
24	3.4e-03	0.80	2.8e-02	0.89	1.1e-01	0.95
26	2.1e-03	0.80	2.3e-02	0.89	1.0e-01	0.95
28	1.4e-03	0.80	1.8e-02	0.89	9.0e-02	0.95
30	8.7e-04	0.80	1.5e-02	0.89	8.0e-02	0.95

TABLE 3.1. Rate of decay of S_n for the symmetric lens.

3.2. Bergman polynomials for the pentagon. In view of Theorem 3.1 and its corollary, this case is similar to Example 1, with singularities at the vertices of the pentagon yielding a decay of order $1/n^s$, with $s = 7/3$ (where the value of s is sharp; see [19, Thm. 3.2]) and the locations of the poles of the extension of the interior conformal map φ_ζ (relative to the level lines of the exterior map Φ) contributing a geometric term. For example, assume that B is the closed disk of radius 0.1 and center at the mid-point of a side L of the pentagon. Extend Φ by reflection into $G \cap B$ and let $\zeta \in G \cap B$. Then, the reflection ζ' of ζ w.r.t. L is the position of the nearest pole singularity of the extension φ_ζ in Ω. Since,

$$\min_{\zeta \in G \cap B} |\Phi(\zeta)| \approx 0.918,$$

where here the approximation to Φ was obtained by means of the MATLAB package SCToolbox [6], and

$$(0.918)^{100} \approx 2.03 \times 10^{-4}, \qquad \frac{1}{100^{7/3}} \approx 2.15 \times 10^{-5},$$

it follows from Corollary 3.1 and the relation $|\Phi(\zeta)| = 1/|\Phi(\zeta')|$, that for $n \leq 100$, essentially,

$$|B_n(\zeta)| \leq \kappa\, |\Phi(\zeta)|^n, \qquad \zeta \in G \cap B. \tag{3.20}$$

Thus, according to Lemma 3.1, if the trend for $n \leq 100$ were to continue, then (in a proportionate sense) there would be no zeros of B_n in $G \cap B$.

3.3. Bergman polynomials for the hypocycloid. Let $G = \mathrm{int}(Y)$, let, as above, $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ and note that Y is piecewise analytic with zero interior angles. By following the proof of Theorem 2.1 and Remark 2.3 in [4], we see that there exist a sequence of polynomials π_n, of respective degrees n, and positive constants κ_1, κ_2, c and r, with $0 < r < 1$, such that

$$\left\| \frac{\varphi_\zeta'}{\varphi_\zeta'(\zeta)} - \pi_n' \right\|_{L^2(G)} \leq \kappa_1 \frac{1}{R^n} + \kappa_2\, e^{-cn^r}, \qquad n = 1, 2, \ldots,$$

where, as in Theorem 3.1 the value of R depends on the nearest singularity of φ_ζ in Ω.

Hence, for any $\zeta \in G$ (see (2.10) and Corollary 3.1):

$$|B_n(\zeta)| \leq \kappa_3 \frac{1}{R^n} + \kappa_4 e^{-cn^r}, \qquad n = 1, 2, \ldots, \tag{3.21}$$

for some positive constants κ_3 and κ_4 that depend on ζ. Viewing the zero angles of Y as limits of small angles and keeping in mind (3.2), we see that the second error term in (3.21) bears a contribution of the form $1/n^s$ for some large number s. Thus, for ζ inside G, apart from the three radial lines, and n *not* sufficiently large we have

$$|B_n(\zeta)| \leq \kappa |\Phi(\zeta)|^n,$$

where $\Phi(\zeta)$ is defined in G by the reflection principle across the three segments of Y. Thus, in light of Lemma 3.1, the behavior of the zeros of the B_n's, as depicted in Figure 1.3, is certainly reasonable, as long as n is not too large.

Remark 3.5. For regions where an estimate of the type (3.21) holds with $r = 1$, it has been shown in several cases that the zeros tend to accumulate on a one-dimensional system of curves lying in G, except for their end points; see Figure 3.2 and [15], [21].

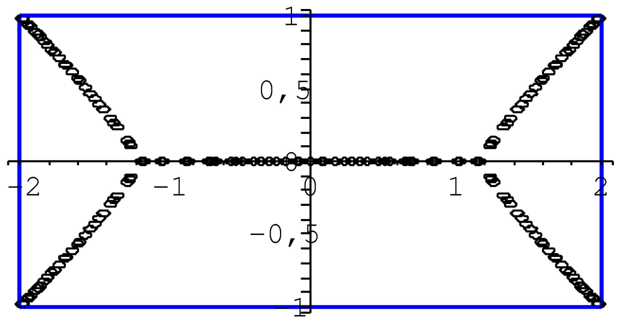

FIGURE 3.2. Rectangle 2×1: Zeros of Bergman polynomials B_n, for $n = 50$, 55 and 60.

Remark 3.6. The discussions in Sections 3.1–3.3 show that the plots of the zeros of the Bergman (Szegő) polynomials constitute a practical tool for deciding whether or not it is important to use pole-type singular basis functions, when approximating the conformal map φ_ζ by means of the Bergman (Szegő) kernel method: If φ_ζ ($\sqrt{\varphi'_\zeta}$) has a singularity on ∂G and the zeros stay well away from a compact set B of the type stated in Lemma 3.1, this is a strong indication of the presence of a dominant pole-type singularity of φ_ζ, with $\zeta \in B$, that needs to be taken into account in the choice of the basis functions. For the stability and convergence properties of the Bergman kernel method we refer to [23]. For the theory of the Szegő kernel method we refer to [24].

3.4. Faber polynomials for the equilateral triangle. First, we extend the exterior conformal map Φ inside G by reflection across the three sides of T, so that Φ becomes analytic in $\mathbb{C} \setminus \bigcup_{j=1}^{3} l_j$, and $|\Phi|$ is continuous in $\overline{\mathbb{C}}$. Fix a $\zeta \in G$ and let ρ be such that $|\Phi(\zeta)| < \rho < 1$. Then, by inspecting the proof of [9, Thm. 1] and choosing the curves γ_j in the path of integration inside G so that $|\Phi(z)| = \rho$, for $z \in \gamma_j$, we see that there exist positive constants κ_1 and κ_2, such that

$$|F_n(\zeta)| \leq \kappa_1 \rho^n + \kappa_2 \frac{1}{n^{5/3}}, \qquad n = 1, 2, \dots . \tag{3.22}$$

Again, for n *not* sufficiently large the geometric term dominates and, according to Lemma 3.1, "discourages" the zeros of F_n from getting to the boundary. (We note that $\min_{\zeta\in G}|\Phi(\zeta)| = |\Phi(0)| \approx 0.66$.)

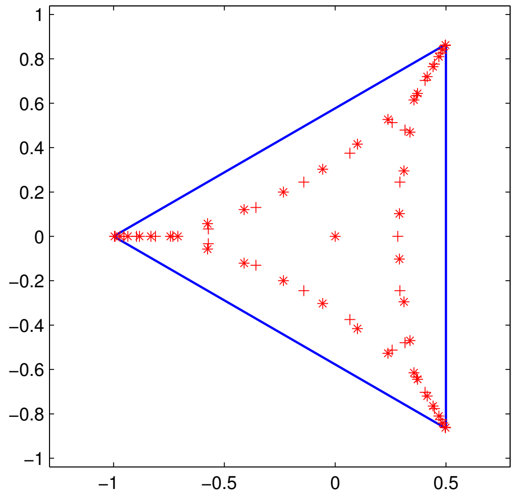

FIGURE 3.3. Equilateral triangle T: Zeros of the Faber polynomials F_n, for $n = 30$ and 40.

This behavior of zeros, for $n = 10$, 15 and 20, was evident in Figure 1.4. However, for $n = 30$ and $n = 40$, as shown in Figure 3.3, the zeros of F_n are "taking off" toward the boundary, but somewhat more reluctantly toward the centers of the three sides. As indicated in the discussion above, this should be attributed to the fact that around the zero-free area the Faber polynomials initially decay geometrically. To illustrate this statement, we test numerically the hypothesis

$$|F_n(\zeta)| \approx C|\Phi(\zeta)|^n, \tag{3.23}$$

for $\zeta = 0.4$ and 0.45. We do this, as in Section 3.1, by seeking to recover the value of $|\Phi(\zeta)|$ in (3.23) by means of the formula

$$\rho_n := \left(\frac{|F_{n+5}(\zeta)|}{|F_n(\zeta)|}\right)^{1/5}.$$

The numerical results for $n = 5(5)40$ are given in Table 3.2. (We note that, although the associated conformal map Φ and the Faber polynomials are known explicitly in this case, we found it convenient to compute the listed approximations of Φ and F_n by means of SCTOOLBOX [6].)

The results presented indicate clearly the geometric decay of the Faber polynomials for the tested values of ζ and n, thus accounting for the behavior of the

	$\zeta = 0.4$	$\lvert\Phi(\zeta)\rvert = 0.917$	$\zeta = 0.45$	$\lvert\Phi(\zeta)\rvert = 0.958$
n	$\lvert F_n(\zeta)\rvert$	ρ_n	$\lvert F_n(\zeta)\rvert$	ρ_n
5	5.5e-01	-	7.2e-01	-
10	4.4e-01	0.95	6.6e-01	0.98
15	2.9e-01	0.92	5.3e-01	0.96
20	1.8e-01	0.91	4.2e-01	0.95
25	1.1e-01	0.91	3.4e-01	0.96
30	7.2e-02	0.91	2.7e-01	0.96
35	5.0e-02	0.92	2.2e-01	0.96
40	3.3e-02	0.92	1.8e-01	0.96

TABLE 3.2. Rate of decay of F_n for the equilateral triangle.

zeros in the plot. They also indicate that for both $\zeta = 0.4$ and $\zeta = .45$, the value of constant C in (3.23) is around 1.

Remark 3.7. In contrast to Example 4, and in comparison to Example 3 it is interesting to note that the Faber polynomials for the hypocycloid Y do indeed have all their zeros on the radial lines $[0, 1.5]$, $e^{i\omega}[0, 1.5]$ and $e^{i2\omega}[0, 1.5]$, $\omega = 2\pi/3$, of Y, for all $n = 1, 2, \ldots$; see the plots and the results of [11].

3.5. OPUC with respect to the measure (1.6)**.** Here we offer two explanations for the zero free region near $z = 1$ in Figure 1.5 (recall that asymptotically all points of $C = \{z : |z| = 1\}$ will attract zeros). The first explanation is a rough heuristic one.

3.5.1. The weight function $w(z) = |e^{1/(z-1)^2}|$ has an infinite order zero at $z = 1$. Since $\Phi_n(z) = \varphi_n(z)/\kappa_n$ minimizes

$$\int_0^{2\pi} |p_n(z)|^2 w(z)\, d\theta, \qquad z = e^{i\theta}, \tag{3.24}$$

over all monic polynomials of degree n, there is no need to “waste” a zero near $z = 1$, when $w(z)$ is already quite small in comparison to its other values on C, in order to make the integral (3.24) small.

3.5.2. Let us assume for the moment that $d\mu$ is such that the resulting monic polynomials $\{\Phi_n(z)\}_{n=0}^{\infty}$ have constant Verblunsky coefficients, i.e. $\Phi_{n+1}(0) = \alpha$, $n = 0, 1, \ldots$, with $0 < |\alpha| < 1$. These are the so-called Geronimus polynomials. We need the following two results regarding Geronimus polynomials:

(i) The support of $d\mu$ consists of

$$C_\beta := \{e^{i\theta} : \beta \le \theta \le 2\pi - \beta\},$$

where $\beta := 2\arcsin(|\alpha|)$, with one possible mass point on $C \setminus C_\beta$, see [10, p. 3] and also [29, pp. 83–84].

(ii) z_0 is a limit point of the zeros of φ_n *if and only if* z_0 lies in the support of $d\mu$; see [30, Thm. 3.3].

In Table 3.3 we list the values of $\Phi_{n+1}(0)$, correct to six decimal places, for $n = 55, \ldots, 59$, corresponding to the measure $d\mu$ defined by (1.6). As predicted by the theory given in Section 2, these tend to zero. However the numbers in the table indicate a very slow convergence. As a result, $\Phi_n(z)$ for low values of n "thinks" it is actually a member of a sequence of Geronimus polynomials with defining parameter $\alpha \approx -0.127$. Hence it places its zeros according to the two results (i) and (ii), with $\beta = 2\arcsin(|\alpha|) \approx 0.25$, as a close inspection of the zero-free region in Figure 1.5 shows.

The same argument can be employed to explain the striking resemblance between the plots of zeros of Φ_n, for $n = 200$, in Figures 8.7 and 8.9 of [29, pp. 419–421], corresponding to the two cases $\Phi_n(0) = (n+1)^{-1/8}$ and $\Phi_n(0) = 1/2$, by simply noticing that $201^{-1/8} \approx 0.515$.

n	$\Phi_{n+1}(0)$
55	-0.129 883
56	-0.129 129
57	-0.128 392
58	-0.127 672
59	-0.126 968

Table 3.3. Verblunsky coefficients for Example 5.

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Edward B. Saff E-mail: edward.b.saff@Vanderbilt.Edu
Address: *Vanderbilt University, Department of Mathematics, 1326 Stevenson Center, 37240 Nashville, TN U.S.A.*
URL: http://www.math.vanderbilt.edu/~esaff/

Nikos S. Stylianopoulos E-mail: nikos@ucy.ac.cy
Address: *University of Cyprus, Department of Mathematics and Statistics, P.O. Box 20537, 1678 Nicosia, Cyprus.*
URL: http://www.ucy.ac.cy/nikos.html