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CONSTRUCTIVE APPROXIMATION

The Asymptotic Behavior of Conformal Modules of Quadrilaterals with Applications to the Estimation of Resistance Values

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Dedicated to Professor Dieter Gaier on the occasion of his 70*th birthday*

Abstract. We consider the conformal mapping of "strip-like" domains and derive a number of asymptotic results for computing the conformal modules of an associated class of quadrilaterals. These results are then used for the following two purposes: (a) to estimate the error of certain engineering formulas for measuring resistance values of integrated circuit networks; and (b) to compute the modules of complicated quadrilaterals of the type that occur frequently in engineering applications.

1. Introduction

The conformal module $m(Q)$ of a quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$, consisting of a Jordan domain Ω and four points z_1, z_2, z_3, z_4 in counterclockwise order on $\partial\Omega$, is defined as follows: Let R_L denote a rectangle of length L and height 1 of the form $R_L := \{w : 0 < \Re w < L,\ 0 < \Im w < 1\}$. Then $m(Q)$ is the unique value of L for which Q is conformally equivalent to the rectangular quadrilateral $\{R_L; i, 0, L, L+i\}$. That is, for $L = m(Q)$ and for this value only there exists a unique conformal map $\Omega \to R_L$ which takes the four points z_1, z_2, z_3, z_4, respectively, onto the four corners $i, 0, L, L+i$ of R_L.

In this paper we consider quadrilaterals $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ of the form illustrated in Figure 1.1. Such quadrilaterals are characterized by the following:

(i) The defining domain Ω is bounded to the left, right, and center, respectively, by:
- two segments s_1 and s_2 of the straight lines $\Im z = 1$ and $\Im z = 0$ and a Jordan arc γ_1 that joins s_1 to s_2 and lies (apart from its two endpoints) entirely within the strip defined by $\Im z = 0$ and $\Im z = 1$;
- two parallel straight-line segments s_3 and s_4, inclined at an angle θ to $\Im z = 0$ and at a distance κ apart, and a Jordan arc γ_2 that joins s_3 to s_4 and lies (apart

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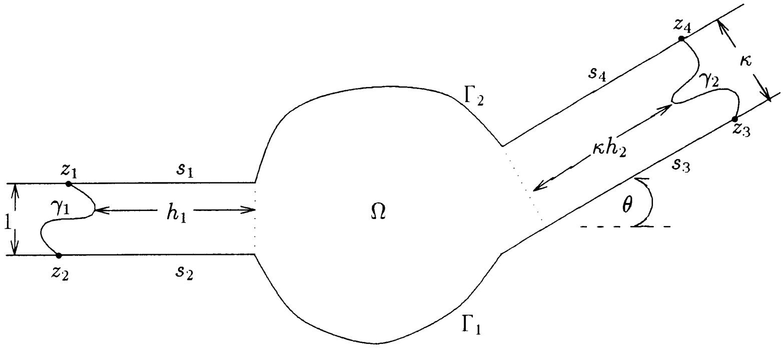

Fig. 1.1

from its two endpoints) entirely within the strip defined by the extensions of s_3 and s_4; and

- two Jordan arcs Γ_1 and Γ_2 that join, respectively, the segments s_2 with s_3 and s_4 with s_1.

(ii) The four specified points z_1, z_2, z_3, z_4 are the four corners where the two arcs γ_1 and γ_2 meet, respectively, the straight-line segments s_1, s_2 and s_3, s_4.

Let h_1 and $\kappa\, h_2$ be, respectively, the lengths of the "left" and "right" rectangular parts of Ω, as illustrated in Figure 1.1. Then, the main purpose for our paper is to study a problem (which was brought to our attention by Professor Dieter Gaier) concerning the asymptotic behavior of the difference

$$m(Q) - (h_1 + h_2), \tag{1.1}$$

as both h_1 and h_2 tend to infinity. In this connection, our main results are given in Section 3, where it is shown, in particular, that the limit of (1.1) as $h_1, h_2 \to \infty$ is a simple expression involving the so-called *exponential radii* of the arcs γ_1, γ_2 and the *angular derivatives* of the infinite domain obtained by extending the two rectangular end parts of Ω to infinity.

The other material of the paper is organized as follows:

Section 2 contains a number of preliminary results that are needed for our work in subsequent sections. These are mainly results from the theory of a domain decomposition method for computing the conformal modules of long quadrilaterals [6], [5], [17], [18], [19], and [20].

In Section 4 we study the quality of certain asymptotic formulas for approximating the conformal modules of a class of polygonal quadrilaterals. These formulas were derived (but without any error estimates) by Hall [9] for the purpose of measuring the resistance values of thin film patterns. Here, by making use of our theory in Section 3, we derive computable error estimates and show that the formulas of Hall are, in fact, remarkably accurate.

Finally, in Section 5, we illustrate further the practical significance of the results of Section 3 by using them (in conjunction with the domain decomposition method mentioned above) for computing the conformal modules of complicated quadrilaterals of the type that occur frequently in applications involving the measurement of resistance values of integrated circuit networks. In this connection we note that (from the practical point of view) the objectives of our work in this paper are the same as those of the domain decomposition method mentioned above. These objectives are:

(a) to overcome the crowding difficulties associated with the problem of computing the modules of long quadrilaterals, i.e., the difficulties associated with the conventional approach of seeking to determine $m(Q)$ by going via the unit disk or the half-plane (see, e.g., [16, pp. 67–68], [19, §1], and [24, p. 4]); and
(b) to take advantage of the fact that in many applications a complicated original quadrilateral can be decomposed into very simple component quadrilaterals.

2. Preliminary Results

Let γ be a Jordan arc that joins the lines $\Im z = 0$ and $\Im z = 1$ and lies entirely within the strip $\{z : 0 < \Im z < 1\}$, except for its endpoints. In Section 3 we shall refer frequently to the *exponential radius* r of such an arc γ. This is defined as follows (see [6, §1.2] and Figure 2.1):

Let γ^* be the arc obtained by translating γ along the real axis until it lies in $\Re z \geq 0$, with at least one point on $\Re z = 0$. Next, let Γ^* be the image of γ^* under the transformation $z \to e^{\pi z}$, and let $\overline{\Gamma^*}$ denote the reflection of Γ^* in the real axis. Finally, let Γ denote the symmetric curve $\Gamma := \Gamma^* \cup \overline{\Gamma^*}$. (Observe that Γ surrounds the unit circle and meets the unit circle in at least one point; see Figure 2.1.) Then, r is defined as the conformal radius of the domain $G := \operatorname{Int} \Gamma$ with respect to the origin 0. That is, the exponential radius of

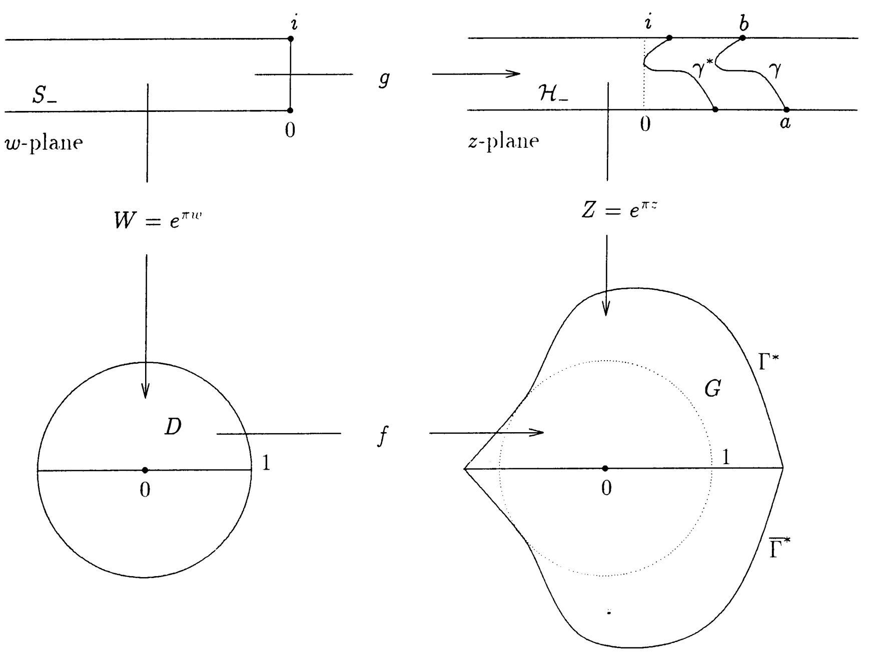

Fig. 2.1

the arc γ is the radius of the disk $|W| < r$ $(r \geq 1)$ onto which G can be mapped by a conformal map F such that $F(0) = 0$ and $F'(0) = 1$.

The exponential radius of an arc γ admits another interpretation as follows: With reference to Figure 2.1, let $\mathcal{H}_-$ denote the strip domain that lies to the left of the arc γ and let a and b denote, respectively, the points where γ intersects the lines $\Im z = 0$ and $\Im z = 1$. Also, let S_- denote the strip $S_- := \{w : \Re w < 0,\ 0 < \Im w < 1\}$ and let g be the conformal map $S_- \to \mathcal{H}_-$ normalized by the conditions

$$g(0) = a, \qquad g(i) = b, \qquad \text{and} \qquad \lim_{\substack{w\to\infty \\ w\in S_-}} g(w) = \infty. \tag{2.1}$$

Finally, let $\sigma := \min\{\Re z : z \in \gamma\}$. Then, the exponential radius r of the arc γ can also be defined by means of the relation

$$\lim_{\substack{\Re w\to-\infty \\ w\in S_-}} \{g(w) - w\} = \frac{1}{\pi}\log r + \sigma. \tag{2.2}$$

This can be deduced from the following observations:

- The transformation $w \to e^{\pi w}$ takes S_- onto the upper half of the unit disk $D := \{W : |W| < 1\}$.
- The function

$$f(W) := F^{[-1]}(rW) \tag{2.3}$$

  maps conformally D onto $G := \operatorname{Int}\Gamma$ (see Figure 2.1).
- The conformal maps g and f are related by means of

$$f(W) = \exp\left\{\pi g\left(\frac{1}{\pi}\log W\right) - \pi\sigma\right\}. \tag{2.4}$$

-

$$\log r = \log f'(0) = \lim_{W\to 0}\log\frac{f(W)}{W} = \pi \lim_{\substack{\Re w\to-\infty \\ w\in S_-}} \{g(w) - w\} - \pi\sigma. \tag{2.5}$$

In what follows we present a number of results from the theory of a domain decomposition method for computing approximations to the conformal modules of long quadrilaterals [5], [6], [17], [18], [19], and [20]. This method consists of decomposing, by means of appropriate crosscuts l_j, $j = 1, 2, \ldots,$ a given quadrilateral Q into two or more component quadrilaterals Q_j, $j = 1, 2, \ldots,$ and approximating the conformal module $m(Q)$ by the sum $\sum_j m(Q_j)$ of the modules of the component quadrilaterals. (An example of such a decomposition is illustrated in Figure 2.2, where the original quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ has been decomposed by means of two crosscuts l_1 and l_2 into the three component quadrilaterals $Q_1 := \{\Omega_1; z_1, z_2, a, d\}$, $Q_2 := \{\Omega_2; d, a, b, c\}$, and $Q_3 := \{\Omega_3; c, b, z_3, z_4\}$.) Note that

$$m(Q) \geq \sum_j m(Q_j) \tag{2.6}$$

and equality occurs only when the images of all the crosscuts l_j under the conformal map $\Omega \to R_{m(Q)} := \{w : 0 < \Re w < m(Q),\ 0 < \Im w < 1\}$ are straight lines parallel to

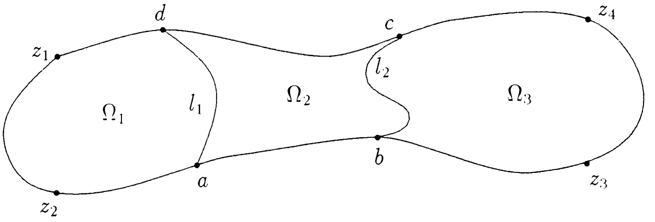

Fig. 2.2

the imaginary axis. This follows from the well-known composition law for modules of curve families; see, e.g., [1, pp. 54–56].

The results of the first two theorems stated below are consequences of results due to Gaier and Hayman [5] and [6], while the third theorem is given as a "Note Added in Proof" in [6, p. 467]. All three theorems concern a special class of decompositions of the form illustrated in Figure 2.3, where:

(i) the defining domain Ω of the original quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ is bounded by two segments of the lines $\Im z = 0$ and $\Im z = 1$ and two Jordan arcs γ_1, γ_2 that lie (apart from their endpoints) entirely within the strip $\{z : 0 < \Im z < 1\}$;

(ii) the points z_1, z_2, z_3, z_4 are the four corners where the arcs γ_1, γ_2 meet the lines $\Im z = 0$ and $\Im z = 1$;

(iii) the crosscut l of subdivision is a straight line parallel to the imaginary axis.

Theorem 2.1. *With reference to Figure* 2.3, *let h_1 and h_2 denote, respectively, the distances of the crosscut l from the arcs γ_1 and γ_2. Also, let $\overline{\gamma}_1$ denote the reflection of γ_1 in the imaginary axis and let r_1 and r_2 be, respectively, the exponential radii of the arcs $\overline{\gamma}_1$ and γ_2. Finally, let*

$$R := r_1 r_2 e^{\pi(h_1+h_2)}.$$

Then:

(i)

$$\left| m(Q) - \left\{ h_1 + h_2 + \frac{1}{\pi}(\log r_1 + \log r_2) \right\} \right| \leq \frac{8.37}{\pi}\frac{1}{R} \tag{2.7}$$

$$\leq \frac{8.5}{\pi} e^{-\pi m(Q)}, \tag{2.8}$$

provided that $R \geq e^{2\pi}$.

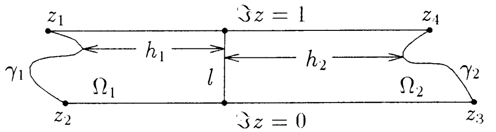

Fig. 2.3

(ii) *For* $j = 1$ *and* $j = 2$, *the expression* $m(Q_j) - h_j - (1/\pi)\log r_j$ *is a nonpositive increasing function of* h_j *and*

$$-0.191e^{-2\pi h_j} \le m(Q_j) - h_j - \frac{1}{\pi}\log r_j \le 0, \qquad j = 1, 2, \tag{2.9}$$

provided that $h_j \ge 1$. *Also*,

$$-\frac{8.37}{2\pi}e^{-2\pi m(Q_j)} \le m(Q_j) - h_j - \frac{1}{\pi}\log r_j \le 0, \qquad j = 1, 2, \tag{2.10}$$

provided that $m(Q_j) \ge 1$.

Theorem 2.2. *For the decomposition illustrated in Figure* 2.3 *we have*

$$0 \le m(Q) - \{m(Q_1) + m(Q_2)\} \le 5.33e^{-2\pi m^*}, \tag{2.11}$$

provided that $m^* := \min\{m(Q_1), m(Q_2)\} \ge 1$.

As was previously remarked, the results of Theorems 2.1 and 2.2 are consequences of results due to Gaier and Hayman [5] and [6]. The details of their derivations are as follows:

- Equations (2.7) and (2.10) can be derived by following the proof of Theorem 4 in [5, pp. 830–832] and using, in the right-hand side of [5, Eq. (4.1)], the result of [5, Theorem 2] (rather than that of [5, Theorem 3], as was done in [5]).
- Equation (2.8) follows at once from (2.7) and the assumption that $R \ge e^{2\pi}$, by observing that Theorems 1 and 2 of [5] imply that,

$$\frac{1}{R} < e^{-\pi m(Q)}\left(1 + \frac{8}{R} + \frac{163}{R^2}\right).$$

(In connection with the above we note that the results (2.7)–(2.8) can also be expressed in the form

$$\left|m(Q) - \left\{h_1 + h_2 + \frac{1}{\pi}(\log r_1 + \log r_2)\right\}\right| \le 0.381e^{-\pi(h_1+h_2)},$$

provided that $h_1 + h_2 \ge 2$. This is a direct consequence of the estimate (1.14) of [5, Theorem 4]; see [18, p. 218].)
- Equation (2.9) follows from the two estimates (1.15) and (1.16) in [5, Theorem 4] and the monotonicity result of [6, Theorem 4]; see also [18, p. 218].
- Since $\min\{m(Q_1), m(Q_2)\} \ge 1$, it follows from (2.9) that

$$2 \le m(Q_1) + m(Q_2) \le h_1 + \frac{1}{\pi}\log r_1 + h_2 + \frac{1}{\pi}\log r_2 = \frac{1}{\pi}\log R$$

and hence that $R \ge e^{2\pi}$. Because of this, (2.11) follows by using the two estimates (2.7) and (2.10) in the proof of Theorem 2 in [6, p. 462] (rather than the estimates given by [6, Eqs (1.3)–(1.4)] and [6, Eqs (1.8)–(1.9)] as was done in [6]).

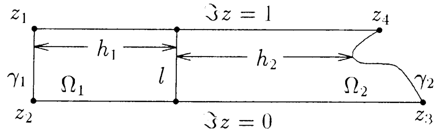

Fig. 2.4

Theorem 2.3 ([6, p. 467]). *If in Figure* 2.3 *the boundary arc* γ_1 *is a straight line parallel to the imaginary axis* (*so that* $m(Q_1) = h_1$), *i.e., if the decomposition is of the form illustrated in Figure* 2.4, *then*

$$0 \le m(Q) - \{h_1 + m(Q_2)\} \le \frac{4}{\pi} e^{-2\pi m(Q_2)}, \tag{2.12}$$

provided that $m(Q_2) \ge 1$. *Here* $4/\pi$ *cannot be replaced by a smaller number.*

We also note the following two inequalities which hold for quadrilaterals Q and Q_1, Q_2 of the form illustrated in Figure 2.3 (see, e.g., [11, pp. 35–37] and [1, pp. 53–56]):

$$h_1 + h_2 \le m(Q) \le h_1 + h_2 + 1 \tag{2.13}$$

and

$$h_j \le m(Q_j) \le h_j + \tfrac{1}{2}, \qquad j = 1, 2. \tag{2.14}$$

The next three theorems concern the decomposition of more general quadrilaterals. Theorem 2.4 is taken directly from [20], while Theorems 2.5 and 2.6 are improved versions of results given in [18] and [19]. More specifically:

- Theorem 2.5 improves the bound given in [18, Theorem 3.1]. This is achieved by making use of (2.8) and (2.10), in place of [18, Eqs (2.1), (2.2)].
- Theorem 2.6 improves the bound given in [19, Cor. 2.7]. This is achieved by making use of the results of Theorems 2.4 and 2.5 in place of [19, Cor. 2.6] and [19, Theorem 2.1].

Theorem 2.4 ([20, Res. 5]). *Consider a quadrilateral* $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ *of the form illustrated in Figure* 2.5, *and assume that the defining domain* Ω *can be decomposed by a straight-line crosscut* l *into* Ω_1 *and* Ω_2, *so that* Ω_2 *is the reflection in* l *of some subdomain of* Ω_1. *Then, for the decomposition defined by* l,

$$0 \le m(Q) - \{m(Q_1) + m(Q_2)\} \le \frac{4}{\pi} e^{-2\pi m(Q_2)}, \tag{2.15}$$

provided that $m(Q_2) \ge 1$.

Theorem 2.5. *With the notations of Figure* 2.2,

$$|m(Q) - \{m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2)\}| \le 2.71 e^{-\pi m(Q_2)}, \tag{2.16}$$

provided that $m(Q_2) \ge 3$.

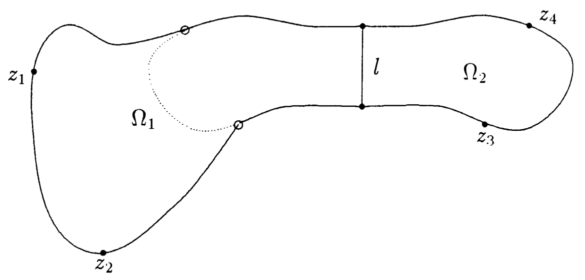

Fig. 2.5

The notations $Q_{1,2}$ and $Q_{2,3}$ in (2.16) refer to the quadrilaterals $Q_{1,2} := \{\Omega_{1,2}; z_1, z_2, b, c\}$ and $Q_{2,3} := \{\Omega_{2,3}; d, a, z_3, z_4\}$, where $\overline{\Omega}_{1,2} := \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\overline{\Omega}_{2,3} := \overline{\Omega}_2 \cup \overline{\Omega}_3$. We shall use throughout similar multisubscript notations to denote the additional subdomains and associated quadrilaterals that arise when the decomposition under consideration involves more than one crosscut. For example, with reference to Figure 2.6, $Q_{1,2,3}$ will denote the quadrilateral $Q_{1,2,3} := \{\Omega_{1,2,3}; z_1, z_2, b, c\}$, where $\overline{\Omega}_{1,2,3} := \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$.

Theorem 2.6. *Consider a quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ of the form illustrated in Figure 2.6, and assume that the defining domain Ω can be decomposed by means of a straight-line crosscut l and two other crosscuts l_1 and l_2 into four subdomains Ω_1, Ω_2, Ω_3 and Ω_4, so that Ω_3 is the reflection in l of Ω_2. Then, for the decomposition of Q defined by l,*

$$0 \leq m(Q) - \{m(Q_{1,2}) + m(Q_{3,4})\} \leq 5.26 e^{-2\pi m(Q_2)}, \tag{2.17}$$

provided that $m(Q_2) \geq 1.5$.

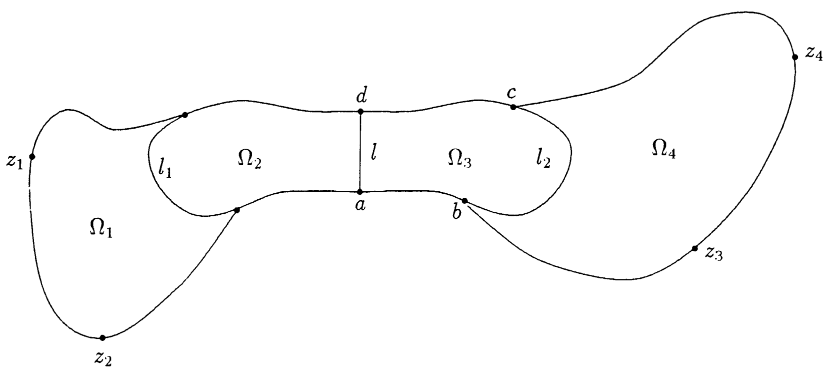

Fig. 2.6

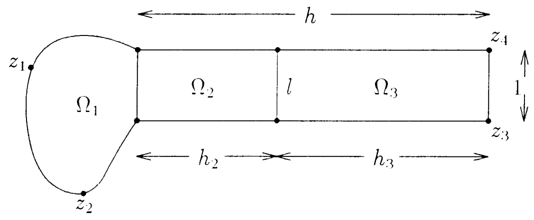

Fig. 2.7

The last theorem of this section concerns decompositions of the form illustrated in Figure 2.7, where:

(i) the defining domain Ω of the original quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ is the union of a rectangle of height 1 and length h (to the right) and a Jordan domain Ω_1 (to the left);
(ii) the decomposition is defined by a straight-line crosscut l that divides the rectangular part of Ω into two subrectangles Ω_2 and Ω_3 of lengths h_2 and h_3, respectively. (Note that $m(Q_2) = h_2$ and $m(Q_3) = h_3$.)

Theorem 2.7. *For the decomposition illustrated in Figure* 2.7 *we have*

$$0 \le m(Q) - \{m(Q_{1,2}) + h_3\} \le 1.28e^{-2\pi h_2}, \tag{2.18}$$

provided that $h_2 \ge 1$.

Proof. If $h_2 = h_3$, i.e., if $m(Q_2) = m(Q_3)$, then the result follows at once from Theorem 2.4 because $4/\pi < 1.28$. We shall consider separately the two cases $h_2 > h_3$ and $h_2 < h_3$.

(i) $h_2 > h_3$. Attach a rectangle Ω_4 of height 1 and length $h_2 - h_3$ to the right of Ω_3, so that $\Omega_{3,4}$ is a rectangle of length h_2 and $m(Q_{3,4}) = h_2 = m(Q_2)$. Then, from Theorem 2.4,

$$0 \le m(Q_{1,2,3,4}) - \{m(Q_{1,2}) + m(Q_{3,4})\} \le \frac{4}{\pi}e^{-2\pi h_2}.$$

The result follows because, from [20, Res. 7],

$$m(Q) - \{m(Q_{1,2}) + m(Q_3)\} \le m(Q_{1,2,3,4}) - \{m(Q_{1,2}) + m(Q_{3,4})\}. \tag{2.19}$$

(ii) $h_2 < h_3$. If $h_3 = kh_2$ for some $k \in \mathbf{N}$ ($k > 1$), then by dividing Ω_3 into k subrectangles each of length h_2 and applying (in an obvious manner) k times the procedure used in (i), we find that

$$0 \le m(Q) - \{m(Q_{1,2}) + m(Q_3)\} \le \alpha(h_2)e^{-2\pi h_2},$$

where

$$\alpha(h_2) := \frac{4}{\pi}\{1 + e^{-2\pi h_2} + \cdots + e^{-2(k-1)\pi h_2}\}.$$

The result follows because

$$\alpha(h_2) < \frac{4}{\pi(1-e^{-2\pi})} < 1.28.$$

If $h_3 \neq kh_2$, then we attach a rectangle Ω_4 to the right of Ω_3, so that $\Omega_{3,4}$ is a rectangle of height 1 and length kh_2, $k \in \mathbf{N}$, and apply the argument used above (for the case $h_3 = kh_2$) to the quadrilateral $Q_{1,2,3,4}$. This gives

$$0 \le m(Q_{1,2,3,4}) - \{m(Q_{1,2}) + m(Q_{3,4})\} \le 1.28e^{-2\pi h_2},$$

and the required result follows from (2.19). ■

3. Main Results

We consider first quadrilaterals $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ of the form illustrated in Figure 3.1(b), where:

(i) the defining domain Ω is bounded by the straight lines $s_1 := \{z : \Im z = 1,\ -h \le \Re z \le 0\}$, $s_2 := \{z : \Im z = 0,\ -h \le \Re z \le 0\}$ (where $h > 0$), and $\gamma := \{z : \Re z = -h,\ 0 \le \Im z \le 1\}$, and a Jordan arc Γ that meets s_1 and s_2 at the points i and 0; and

(ii) $z_1 := -h + i$, $z_2 := -h$, and z_3, z_4 are two distinct points on Γ.

We assume that Γ is such that the segment $\{z : \Re z = 0,\ 0 < \Im z < 1\}$ of the imaginary axis lies entirely within Ω, and seek to determine the asymptotic behavior of $m(Q) - h$ as $h \to \infty$. For this we proceed as follows:

Let H_- denote the infinite domain obtained by extending the rectangular part of Ω to infinity, let S_- denote the strip $S_- := \{w : \Re w < 0,\ 0 < \Im w < 1\}$, and let g be the conformal map $S_- \to H_-$ normalized by the conditions

$$g(0) = z_3, \qquad g(i) = z_4, \qquad \text{and} \qquad \lim_{\substack{w\to\infty \\ w\in S_-}} g(w) = \infty. \tag{3.1}$$

Then, for some $\xi > 0$, the function g maps conformally $\{w : \Re w < -\xi,\ 0 < \Im w < 1\}$ onto a strip domain $\mathcal{H}_-$ of the form illustrated in Figure 2.1. Therefore, from (2.2),

$$\lim_{\substack{\Re w\to-\infty \\ w\in S_-}} \{g(w) - w\} = \beta, \tag{3.2}$$

where β is real and finite.

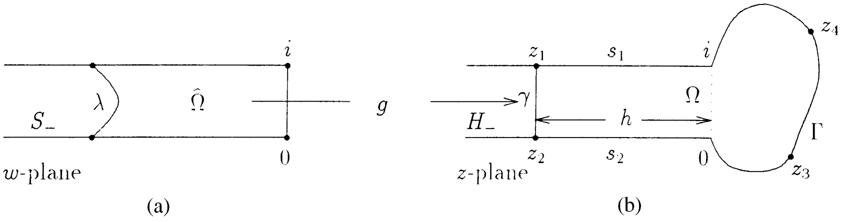

Fig. 3.1

Theorem 3.1. *With reference to Figure* 3.1 *and the notations introduced above, the expression* $m(Q) - h - \beta$ *is a nonpositive increasing function of* h *and*

$$\lim_{h\to\infty} \{m(Q) - h - \beta\} = 0. \tag{3.3}$$

Further,

$$-1.28e^{-2\pi h} \le m(Q) - h - \beta \le 0, \tag{3.4}$$

provided that $h \ge 1$.

Proof. The monotonicity of $m(Q) - h - \beta$ can be established easily by using an argument similar to that used in the proof of [6, Theorem 4, p. 464]. To show (3.3), we proceed as follows:

Let $\widehat{Q} := g^{[-1]}(Q)$ (i.e., $\widehat{Q}$ is defined by the preimages, under the mapping g, of the domain Ω and the points z_1, z_2, z_3, z_4) and denote by λ the preimage of the straight line γ; see Figure 3.1(a). Then, from (3.2),

$$\lim_{\substack{\Re z\to-\infty \\ z\in H_-}} \{z - g^{[-1]}(z)\} = \beta.$$

Hence, in particular, for any $z \in \gamma$,

$$\lim_{h\to\infty} \{-h - \Re g^{[-1]}(z) - \beta\} = 0.$$

This means that for any $\varepsilon > 0$ there exists $M = M(\varepsilon)$ such that for $h > M$,

$$-(h + \beta) - \varepsilon < \Re_{w\in\lambda} w < -(h + \beta) + \varepsilon.$$

The above, in conjunction with (2.6) and the conformal invariance property of modules, gives

$$(h + \beta) - \varepsilon < m(\widehat{Q}) = m(Q) < (h + \beta) + \varepsilon,$$

which implies (3.3).

Finally, for the proof of (3.4), we attach a rectangle R of length h' and height 1 to the left of γ and denote by Q' the quadrilateral defined by the extended domain Ω', where $\overline{\Omega'} := \overline{R} \cup \overline{\Omega}$. (That is, Q' is exactly of the same form as Q except for the length of its rectangular part, which is now $h' + h$, and for the two points z_1, z_2, which are now the points $-(h + h') + i$ and $-(h + h')$, respectively.)

Let

$$\delta(h') := m(Q') - (h' + h) - \beta \qquad \text{and} \qquad \varepsilon(h) := m(Q') - \{h' + m(Q)\}.$$

Then, from (3.3) and Theorem 2.7,

$$\lim_{h'\to\infty} \delta(h') = 0 \qquad \text{and} \qquad 0 \le \varepsilon(h) \le 1.28e^{-2\pi h}.$$

The required estimate (3.4) follows from the above, by noting that

$$m(Q) - h - \beta = \delta(h') - \varepsilon(h),$$

and letting $h' \to \infty$. ∎

Remark 3.1. Theorem 3.1 implies that

$$m(Q) - h - \beta = \mathcal{O}(e^{-2\pi h}). \tag{3.5}$$

In fact, the order $\mathcal{O}(e^{-2\pi h})$ is best possible as the following example shows:
Consider the case where Ω is the canonical trapezium

$$\Omega := \{z : 0 < \Im z < 1,\ -h < \Re z < \Im z\}, \qquad h > 0, \tag{3.6}$$

and denote by T_h the quadrilateral defined by Ω and its four corners, i.e.,

$$T_h := \{\Omega; -h + i, -h, 0, 1 + i\}. \tag{3.7}$$

In this case the conformal module $m(T_h)$ is known, for any value of h, in terms of elliptic integrals and

$$m(T_h) = h + \left(\tfrac{1}{2} - \frac{1}{\pi}\log 2\right) - \frac{4}{\pi}e^{-\pi}e^{-2\pi h} + \mathcal{O}(e^{-3\pi h}); \tag{3.8}$$

see [6, p. 456]. Further, it can be concluded easily from [14, p. 157] that the function

$$g(w) = \frac{1}{\pi}\{\arccos(e^{\pi w}) - \operatorname{arccosh}(-e^{-\pi w})\} + i,$$

maps conformally the strip S_- onto the strip $\{z : 0 < \Im z < 1,\ \Re z < \Im z\}$, with

$$g(0) = 0, \qquad g(i) = 1 + i, \qquad \lim_{\substack{w\to\infty\\ w\in S_-}} g(w) = \infty.$$

Hence,

$$\beta = \lim_{\substack{\Re w\to-\infty\\ w\in S_-}} \{g(w) - w\} = \frac{1}{2} - \frac{1}{\pi}\log 2 \tag{3.9}$$

and, from (3.8),

$$m(T_h) - h - \beta = -\frac{4}{\pi}e^{-\pi}e^{-2\pi h} + \mathcal{O}(e^{-3\pi h}). \tag{3.10}$$

Next, let $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ be a quadrilateral of the form illustrated in Figure 3.2, where γ is a Jordan arc that lies (apart from its two endpoints) entirely within the strip $\{z : \Re z < 0,\ 0 < \Im z < 1\}$. Such quadrilaterals are of more general form than those of Figure 3.1(b), in the sense that the boundary segment γ of the defining domain Ω is now allowed to be a curved Jordan arc rather than just a straight line.

Theorem 3.2. *With reference to Figure* 3.2, *let* h *denote the distance of* γ *from the imaginary axis*. *Also, let* $\overline{\gamma}$ *denote the reflection of the arc* γ *in the imaginary axis and let* r *be the exponential radius of* $\overline{\gamma}$. *Then,*

$$-1.48e^{-\pi h} \le m(Q) - h - \beta - \frac{1}{\pi}\log r \le 5.26e^{-\pi h}, \tag{3.11}$$

provided that $h \ge 3$.

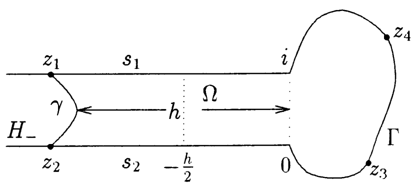

Fig. 3.2

Proof. Consider the decomposition of Q defined by the straight line $\{z : \Re z = -h/2,\ 0 \le \Im z \le 1\}$ and let Q_1 and Q_2 be the two component quadrilaterals

$$Q_1 := \left\{\Omega_1; z_1, z_2, -\frac{h}{2}, -\frac{h}{2} + i\right\} \qquad \text{and} \qquad Q_2 := \left\{\Omega_2; -\frac{h}{2} + i, -\frac{h}{2}, z_3, z_4\right\},$$

where

$$\Omega_1 := \left\{z : \Re z < -\frac{h}{2}\right\} \cap \Omega \qquad \text{and} \qquad \Omega_2 := \left\{z : \Re z > -\frac{h}{2}\right\} \cap \Omega.$$

Then, from Theorem 2.6,

$$0 \le m(Q) - \{m(Q_1) + m(Q_2)\} \le 5.26e^{-\pi h}.$$

The required result follows because Theorems 2.1(ii) and 3.1 give, respectively,

$$-0.191e^{-\pi h} \le m(Q_1) - \frac{h}{2} - \frac{1}{\pi}\log r \le 0$$

and

$$-1.28e^{-\pi h} \le m(Q_2) - \frac{h}{2} - \beta \le 0. \qquad \blacksquare$$

Remark 3.2. Theorem 3.2 implies that

(3.12) $$m(Q) - h - \beta - \frac{1}{\pi}\log r = \mathcal{O}(e^{-\pi h}).$$

Here again we can show that the indicated order is best possible by considering the quadrilateral

(3.13) $$Q := \{\Omega; -h - 1 + i, -h, 0, 1 + i\},$$

with

(3.14) $$\Omega := \{z : 0 < \Im z < 1,\ -h - \Im z < \Re z < \Im z\},$$

and observing that $m(Q) = 2m(T_{h/2})$, where T_h is the trapezoidal quadrilateral given by (3.6) and (3.7).

Remark 3.3. Theorem 3.2 extends the results of Gaier and Hayman [6, Theorem 1] to the case where the right-hand boundary arc of the defining domain Ω (i.e., the boundary arc γ_2 of a quadrilateral of the form illustrated in Figure 2.3) is not constrained to lie within the strip $\{z : 0 \le \Im z \le 1\}$. We note, however, that this extension is achieved at the cost of introducing larger constants in (3.11) than those involved in the result of [6, Theorem 1].

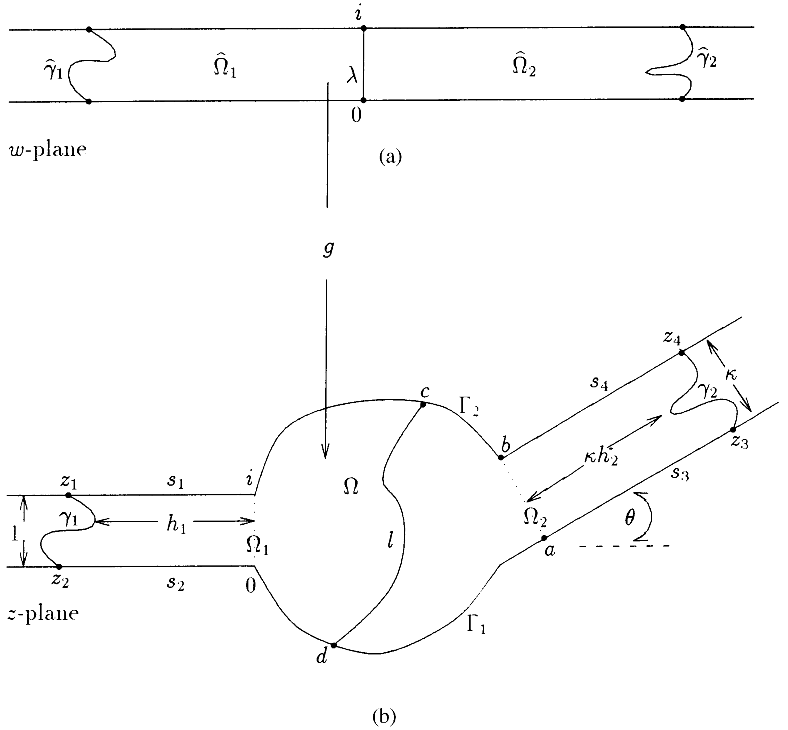

Fig. 3.3

We recall now our introductory remarks of Section 1 and consider quadrilaterals $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ of the form illustrated in Figure 1.1. For definiteness, we assume that the arc Γ_1 meets the lines s_2 and s_3 at the points 0 and a and the arc Γ_2 meets the lines s_1 and s_4 at the points i and $b := a + \kappa e^{i(\theta+\pi/2)}$, as is shown in Figure 3.3(b). We also assume that the arcs Γ_1 and Γ_2 are such that the two straight lines that join, respectively, the points 0 with i and a with b lie (apart from their endpoints) entirely within Ω. Our objective is to consider the asymptotic behavior of $m(Q) - (h_1 + h_2)$ as both h_1 and $h_2 \to \infty$ where, as in Section 1, h_1 and κh_2 denote, respectively, the distances of the arc γ_1 from the imaginary axis and of the arc γ_2 from the straight line joining the points a and b.

Let H denote the infinite domain obtained by extending the two rectangular parts of Ω to infinity and let S denote the strip $S := \{w : 0 < \Im w < 1\}$. Also, let g be the conformal map $S \to H$ normalized by the conditions

$$g(0) = d, \tag{3.15}$$

for some fixed point $d \in \Gamma_1$, and let

$$\lim_{\substack{\Re w \to -\infty \\ w \in S}} g(w) = \infty, \qquad \lim_{\substack{\Re w \to +\infty \\ w \in S}} g(w) = \infty, \tag{3.16}$$

so that, for some $\xi > 0$, the two rays $\{w : \Re w < -\xi,\ \Im w = 0\}$ and $\{w : \Re w > \xi,\ \Im w = 0\}$ map, respectively, into the infinite extensions of the straight lines s_2 and s_3. Finally, let l be the image under g of the straight line $\lambda := \{w : \Re w = 0,\ 0 < \Im w < 1\}$ and denote by H_- and H_+ the two components of $H \setminus l$, with $s_2 \in \overline{H}_-$ and $s_3 \in \overline{H}_+$. Note that H_- is a domain of the form illustrated in Figure 3.2 and that H_+ can be mapped conformally onto a domain of such form, by means of the transformation

$$z \to -e^{-i\theta}(z-a)/\kappa + i. \tag{3.17}$$

Note also that the transformation

$$w \to -w + i \tag{3.18}$$

maps $S_+ := \{w : \Re w > 0,\ 0 < \Im w < 1\}$ onto $S_- := \{w : \Re w < 0,\ 0 < \Im w < 1\}$. Therefore, since g maps conformally S_- onto H_-, with $\lim_{\substack{\Re w \to -\infty \\ w \in S_-}} g(w) = \infty$, it follows that

$$\lim_{\substack{\Re w \to -\infty \\ w \in S}} \{g(w) - w\} = \beta_1, \tag{3.19}$$

where β_1 is real and finite. Similarly, the observations about the domains H_+ and S_+ imply that

$$\lim_{\substack{\Re w \to +\infty \\ w \in S}} \{w - e^{-i\theta}(g(w) - a)/\kappa\} = \beta_2, \tag{3.20}$$

where β_2 is real and finite. (See also [21, pp. 483–485] and [2, pp. 300–303] and note that the numbers β_1 and β_2 are the angular derivatives of $g\colon S \to H$, corresponding to the two prime ends of H that contain the point at infinity.) The theorem below corresponds to the special case where the two boundary arcs γ_1 and γ_2 of the defining domain Ω are both line segments.

Theorem 3.3. *If in Figure* 3.3(b) *the two boundary segments γ_1 and γ_2 of Ω are both straight lines perpendicular to s_2 and s_3, respectively, then the expression $m(Q) - (h_1 + h_2) - (\beta_1 + \beta_2)$ is a nonpositive increasing function of h_1 and h_2, and*

$$\lim_{h_1, h_2 \to \infty} \{m(Q) - (h_1 + h_2) - (\beta_1 + \beta_2)\} = 0. \tag{3.21}$$

Further,

$$-1.28(e^{-2\pi h_1} + e^{-2\pi h_2}) \le m(Q) - (h_1 + h_2) - (\beta_1 + \beta_2) \le 0, \tag{3.22}$$

provided that $\min(h_1, h_2) \ge 1$.

Proof. Here again (as for the first part of Theorem 3.1) the monotonicity of $m(Q) - (h_1 + h_2) - (\beta_1 + \beta_2)$ can be established by using an argument similar to that used in the proof of [6, Theorem 4, p. 464]. To show (3.21) we recall the notations introduced above and proceed as follows:

With reference to Figure 3.3(b), the arc l intersects $\partial\Omega$ at the point $d = g(0) \in \Gamma_1$ and at some other point $c \in \Gamma_2$. Let Q_1 and Q_2 be the two component quadrilaterals of the decomposition of Q defined by l, i.e., $Q_1 := \{\Omega_1; z_1, z_2, d, c\}$ and $Q_2 := \{\Omega_2; c, d, z_3, z_4\}$, where $\Omega_1 := H_- \cap \Omega$ and $\Omega_2 := H_+ \cap \Omega$. Also, let

$$\delta_1(h_1) := m(Q_1) - (h_1 + \beta_1), \qquad \delta_2(h_2) := m(Q_2) - (h_2 + \beta_2).$$

Then, from Theorem 3.1,

$$\lim_{h_1\to\infty}\delta_1(h_1)=0 \qquad \text{and} \qquad \lim_{h_2\to\infty}\delta_2(h_2)=0. \tag{3.23}$$

Next, let $\widehat{\Omega}$, $\widehat{\Omega}_1$, and $\widehat{\Omega}_2$ be, respectively, the preimages of the domains Ω, Ω_1, and Ω_2, under the mapping $g\colon S\to H$, let $\widehat{Q}$, $\widehat{Q}_1$, and $\widehat{Q}_2$ be the corresponding quadrilaterals and let

$$\begin{aligned}\varepsilon(h_1,h_2) &:= m(Q)-\{m(Q_1)+m(Q_2)\}\\ &= m(\widehat{Q})-\{m(\widehat{Q}_1)+m(\widehat{Q}_2)\}.\end{aligned}$$

Then, since h_1, $h_2\to\infty$ implies that $m(\widehat{Q}_1)$, $m(\widehat{Q}_2)\to\infty$, the application of Theorem 2.2 to the decomposition of $\widehat{Q}$ defined by λ gives that

$$\lim_{h_1,h_2\to\infty}\varepsilon(h_1,h_2)=0. \tag{3.24}$$

The required result (3.21) follows by observing that

$$m(Q)-(h_1+h_2)-(\beta_1+\beta_2)=\varepsilon(h_1,h_2)+\delta_1(h_1)+\delta_2(h_2).$$

Finally, (3.22) can be established by using an argument similar to that used in the proof of (3.4). The details are as follows:

Recall our assumption that in Figure 3.3(b) γ_1 and γ_2 are both straight lines and attach a rectangle R_1 of height 1 and length h_1', to the left of γ_1, and a rectangle R_2 of corresponding dimensions κ and and $\kappa h_2'$, to the “right” of γ_2. Let Q_1' and Q' be the two quadrilaterals that are defined, respectively, by the domains Ω_1' and Ω', where $\overline{\Omega_1'}:=\overline{R}_1\cup\overline{\Omega}$ and $\overline{\Omega'}:=\overline{R}_2\cup\overline{\Omega_1'}=\overline{R}_1\cup\overline{\Omega}\cup\overline{R}_2$, and note that both Q_1' and Q' have exactly the same form as Q.

Let

$$\delta(h_1',h_2'):=m(Q')-(h_1'+h_1+\beta_1)-(h_2'+h_2+\beta_2),$$

and

$$\varepsilon_1(h_1):=m(Q_1')-\{h_1'+m(Q)\},\qquad \varepsilon_2(h_2):=m(Q')-\{h_2'+m(Q_1')\}.$$

Then, from (3.21),

$$\lim_{h_1',h_2'\to\infty}\delta(h_1',h_2')=0$$

and, from Theorem 2.7,

$$0\le\varepsilon_1(h_1)\le 1.28e^{-2\pi h_1}\qquad\text{and}\qquad 0\le\varepsilon_2(h_2)\le 1.28e^{-2\pi h_2}.$$

The required estimate (3.22) follows by noting that

$$m(Q)-(h_1+h_2)-(\beta_1+\beta_2)=\delta(h_1',h_2')-\varepsilon_1(h_1)-\varepsilon_2(h_2),$$

and letting h_1', $h_2'\to\infty$. ■

Theorem 3.4. *With reference to Figure* 3.3(b), *let γ_2^* be the image of the arc γ_2 under the transformation* (3.17), *let $\overline{\gamma_1}$ and $\overline{\gamma_2^*}$ denote, respectively, the reflections of γ_1 and γ_2^* in the imaginary axis and let r_1 and r_2 be the exponential radii of the arcs $\overline{\gamma_1}$ and $\overline{\gamma_2^*}$. Then,*

(3.25)
$$-1.48(e^{-\pi h_1}+e^{-\pi h_2}) \le m(Q)-(h_1+h_2)-(\beta_1+\beta_2)-\frac{1}{\pi}(\log r_1+\log r_2) \le 5.26(e^{-\pi h_1}+e^{-\pi h_2}),$$

provided that $\min(h_1,h_2)\ge 3$.

Proof. Let l_1 and l_2 be two straight-line crosscuts of Ω, such that l_1 is perpendicular to s_2 and at a distance $h_1/2$ from γ_1, and l_2 is perpendicular to s_3 and at a distance $\kappa h_2/2$ from γ_2. Consider the decomposition of Q defined by l_1 and l_2 and let Q_1, Q_2, and Q_3 be the three component quadrilaterals that correspond, respectively, to the three subdomains Ω_1, Ω_2, and Ω_3, where

$$\Omega_1 := \left\{z : \Re z < -\frac{h_1}{2}\right\}\cap\Omega, \qquad \Omega_3 := \left\{z : \Re(e^{-i\theta}(z-a)) > \frac{\kappa h_2}{2}\right\}\cap\Omega$$

and

$$\overline{\Omega}_1\cup\overline{\Omega}_2\cup\overline{\Omega}_3=\overline{\Omega}.$$

Also, let $Q_{2,3}$ be the component quadrilateral that corresponds to the subdomain $\Omega_{2,3}$, where

$$\overline{\Omega}_{2,3} := \overline{\Omega}_2\cup\overline{\Omega}_3.$$

Then, since $\min(h_1,h_2)\ge 3$, Theorem 2.6 implies that

$$0\le m(Q)-\{m(Q_1)+m(Q_{2,3})\}\le 5.26e^{-\pi h_1}$$

and

$$0\le m(Q_{2,3})-\{m(Q_2)+m(Q_3)\}\le 5.26e^{-\pi h_2}.$$

Hence, by combining the above,

(3.26) $$0\le m(Q)-\{m(Q_1)+m(Q_2)+m(Q_3)\}\le 5.26(e^{-\pi h_1}+e^{-\pi h_2}).$$

The required estimate (3.25) follows from (3.26) by observing the following:

- Theorem 2.1(ii) (applied to Q_1) gives
$$-0.191e^{-\pi h_1}\le m(Q_1)-\frac{h_1}{2}-\frac{1}{\pi}\log r_1\le 0.$$
- The same theorem (applied to the image of Q_3 under the mapping (3.17)) gives
$$-0.191e^{-\pi h_2}\le m(Q_3)-\frac{h_2}{2}-\frac{1}{\pi}\log r_2\le 0.$$
- Theorem 3.3 (applied to Q_2) gives
$$-1.28(e^{-\pi h_1}+e^{-\pi h_2})\le m(Q_2)-\left(\frac{h_1}{2}+\frac{h_2}{2}\right)-(\beta_1+\beta_2)\le 0.$$ ■

Remark 3.4. If $h^* := \min(h_1, h_2)$, then Theorem 3.4 implies that

$$m(Q) - (h_1 + h_2) - (\beta_1 + \beta_2) - \frac{1}{\pi}(\log r_1 + \log r_2) = \mathcal{O}(e^{-\pi h^*}). \tag{3.27}$$

Similarly, in the special case where γ_1 and γ_2 are straight lines perpendicular to s_2 and s_3, Theorem 3.3 gives

$$m(Q) - (h_1 + h_2) - (\beta_1 + \beta_2) = \mathcal{O}(e^{-2\pi h^*}). \tag{3.28}$$

In both cases, the indicated orders are best possible. This can be seen by considering:

- In the case of (3.27), the quadrilateral whose domain of definition is the union of:
  (a) the domain (3.14);
  (b) the reflection of the domain (3.14) in the line $\{z : \Re z = \Im z\}$;
  (c) the straight-line segment $\{z : 0 < \Re z < 1,\ \Re z = \Im z\}$.
- In the case of (3.28), the quadrilateral whose domain of definition is the union of:
  (a) the domain (3.6);
  (b) the reflection of the domain (3.6) in the line $\{z : \Re z = \Im z\}$;
  (c) the straight-line segment $\{z : 0 < \Re z < 1,\ \Re z = \Im z\}$.

4. Engineering Rules for Conformal Modules

In this section we make use of the theory of Section 3 in order to estimate the error of certain engineering formulas for approximating the conformal modules $m(Q)$ of "strip-like" quadrilaterals $Q := \{\Omega; z_1, z_2, z_3, z_4\}$. Such formulas appear frequently in the engineering literature, mainly in connection with VLSI[1] applications involving the measurement of resistance values of integrated circuit networks, and are often based on heuristic arguments (see, e.g., [7, pp. 43–45], [10], [12], [15, pp. 155–159], [22, pp. 223–224], [23, pp. 470–471], and [25, pp. 120–123]). The link between the problems of computing conformal modules and of measuring resistance values is provided by the following physical interpretation of $m(Q)$:

Consider a quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ and assume that Ω represents a thin plate of homogeneous electrically conducting material of specific resistance 1. Suppose also that constant voltages are applied on the two boundary segments (z_1, z_2) and (z_3, z_4), while the remainder of $\partial\Omega$ is insulated. Then, the conformal module $m(Q)$ of Q gives the resistance of the conducting plate.

We consider first the formulas (4.1)–(4.6) given below, for approximating, respectively, the conformal modules of the six quadrilaterals illustrated in Figure 4.1(a)–(f):

$$\text{(a)}\quad m(Q) \approx h + \frac{2}{\pi}\log\left(\csc\left(\frac{\pi y}{2}\right)\right). \tag{4.1}$$

$$\text{(b)}\quad m(Q) \approx h + x - \frac{2}{\pi}\log\left(\sinh\left(\frac{\pi x}{2}\right)\right). \tag{4.2}$$

$$\text{(c)}\quad m(Q) \approx h + \frac{1}{2} + \frac{1}{\pi}\log\left(\frac{b^4 - 1}{2}\right), \tag{4.3}$$

[1] VLSI stands for "Very Large Scale Integration." It is a subject that covers a broad area of study including semiconductor devices and processing, integrated electronic circuits, digital logic, design disciplines, and tools for creating complex systems.

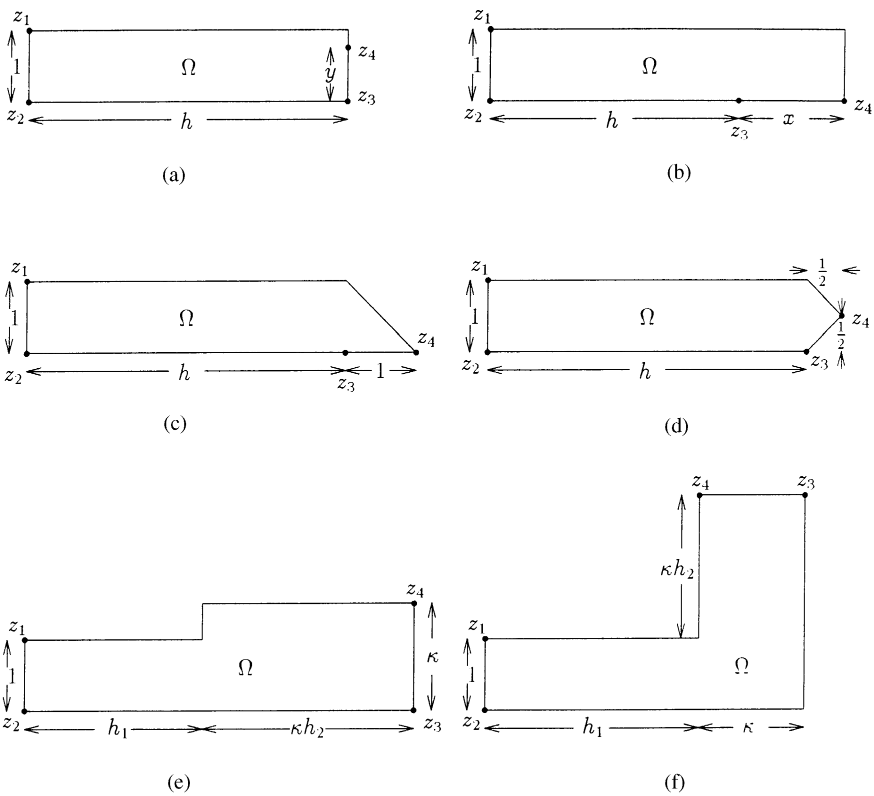

Fig. 4.1

where b is the solution of the equation

$$b = \tan\left(\tfrac{1}{2}\log\left(\frac{b+1}{b-1}\right)\right).$$

$$\text{(4.4)}\quad \text{(d)}\quad m(Q) \approx h + \frac{1}{4} + \frac{1}{2\pi}\log 2.$$

$$\text{(4.5)}\quad \text{(e)}\quad m(Q) \approx h_1 + h_2 + \frac{1}{\pi}\left(\frac{k^2+1}{k}\log\left(\frac{k+1}{k-1}\right) - 2\log\left(\frac{4k}{k^2-1}\right)\right), \quad \text{for } k > 1.$$

$$\text{(f)}\quad m(Q) \approx h_1 + h_2 + \frac{1}{k} - \frac{2}{\pi}\log\left(\frac{4k}{k^2+1}\right)$$

$$\text{(4.6)}\qquad + \frac{k^2-1}{k\pi}\arccos\left(\frac{k^2-1}{k^2+1}\right).$$

The above approximations were derived by Hall [9] by arguments based on the observation that when h (or $\min(h_1, h_2)$) is "large," then (with our notations of Section 3)

$$m(Q) \approx h + \beta,$$

for the quadrilaterals (a)–(d) of Figure 4.1, and

$$m(Q) \approx h_1 + h_2 + \beta_1 + \beta_2,$$

for the quadrilaterals (e) and (f) (see [9, pp. 279–280] and also [8, pp. 131–132]).

Regarding the constants β and β_1, β_2, Hall determined these by estimating the limits

$$\beta := \lim_{\substack{\Re w \to -\infty \\ w \in S_-}} \{g(w) - w\},$$

and

$$\beta_1 + \beta_2 := \lim_{\substack{\Re w \to -\infty \\ w \in S}} \{g(w) - w\} + \lim_{\substack{\Re w \to +\infty \\ w \in S}} \{w - e^{-i\theta}(g(w) - a)/\kappa\},$$

because in each of the six cases illustrated in Figure 4.1 the associated conformal map g is known exactly. In other words, the additive terms that appear to the right of h, in (4.1)–(4.4), and to the right of $h_1 + h_2$, in (4.5), (4.6) are, respectively, the values of β and $\beta_1 + \beta_2$ that correspond to the quadrilaterals of Figure 4.1. It therefore follows from Theorem 3.1 that

$$m(Q) = h + \beta + \mathcal{E}_h, \tag{4.7}$$

where, for $h \geq 1$,

$$-1.28e^{-2\pi h} \leq \mathcal{E}_h \leq 0, \tag{4.8}$$

for the quadrilaterals of Figure 4.1(a)–(d), while for those of Figures 4.1(e), (f) Theorem 3.3 gives

$$m(Q) = h_1 + h_2 + \beta_1 + \beta_2 + \mathcal{E}_{h_1,h_2}, \tag{4.9}$$

where, for $\min(h_1, h_2) \geq 1$,

$$-1.28(e^{-2\pi h_1} + e^{-2\pi h_2}) \leq \mathcal{E}_{h_1,h_2} \leq 0. \tag{4.10}$$

The above results indicate clearly that the approximations (4.1)–(4.6) are remarkably accurate. It appears, however, that this has not been generally recognized, probably due to the lack of precise error estimates. In fact, the approximations of Hall [9], and the associated conformal mapping method for deriving them, have received very little attention by those working in VLSI applications. Instead, it appears from the engineering literature that, in recent years, VLSI practitioners have relied extensively on much less accurate approximations which are based mainly on heuristic geometrical arguments. For example, if in Figure 4.1(f), $h_1 = 3$, $h_2 = 4$, and $\kappa = 1$, then (4.6) (i.e., the conformal mapping method of Hall [9]) gives the approximation

$$\begin{aligned} \widetilde{m}(Q) &= 7 + 1 - \frac{2}{\pi}\log 2 \\ &= 7.558\,728\,799\,694\,\ldots \end{aligned}$$

and (4.9) and (4.10) show that

$$m(Q) = \widetilde{m}(Q) + \mathcal{E},$$

where

$$-8.36 \times 10^{-9} \leq \mathcal{E} \leq 0.$$

By contrast, the more recent methodologies of [12], [23, p. 471], [10], [15, p. 158], and

[7, p. 44], give, respectively, the following:

$$\widetilde{m}(Q) = 7.50, \qquad \widetilde{m}(Q) = 7.65, \qquad \widetilde{m}(Q) = 7.47,$$

$$\widetilde{m}(Q) = 7.66, \qquad \text{and} \qquad \widetilde{m}(Q) = 7.55.$$

5. Numerical Example

Our purpose in this section is to illustrate how the formulas that we considered in the previous section (and hence, more generally, the theory of Section 3) can be used, in conjunction with the domain decomposition theory of Section 2, in order to compute (essentially by hand calculation) accurate approximations to the conformal modules of complicated quadrilaterals. We do this by considering a quadrilateral similar to those that occur frequently in VLSI applications, in connection with the measurement of resistances of integrated circuit networks (see, e.g., [3, p. 230]).

Consider the decomposition of the quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$ illustrated in Figure 5.1 and note that, because of the symmetry, $m(Q_4) = m(Q_5) = m(Q_6)$, $m(Q_7) = m(Q_8)$, and $m(Q_9) = m(Q_{10}) = m(Q_{11}) = m(Q_{12})$. Note also that the approximations to the conformal modules of the component quadrilaterals Q_j, $j = 1, 2, 3, 4$, Q_7, Q_9,

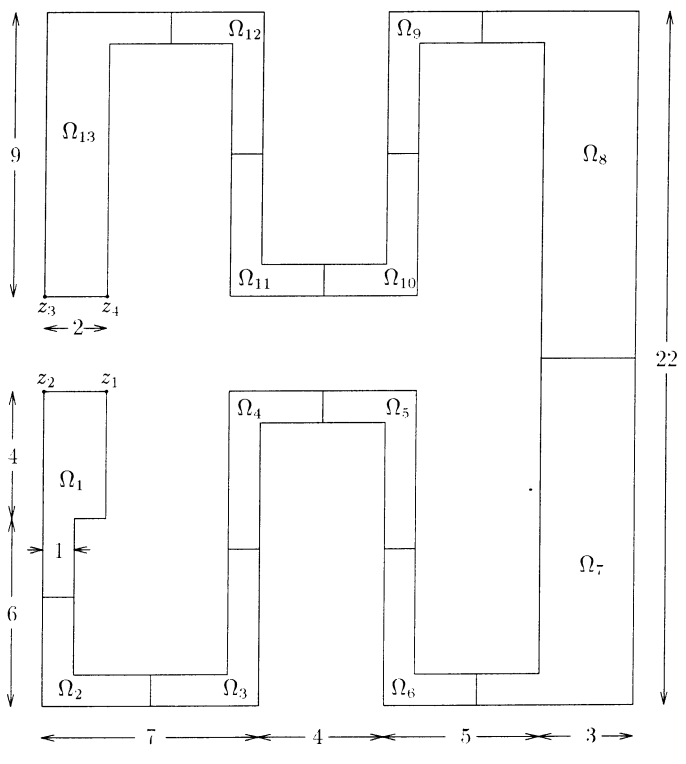

Fig. 5.1

and Q_{13} can be written down by means of (4.5) and (4.6). These approximations are given, together with the corresponding error estimates (see (4.9) and (4.10)), by the following:

$$(5.1)\qquad m(Q_1) = 4.5 + \frac{1}{\pi}\left(\frac{5}{2}\log 3 - 2\log\left(\frac{8}{3}\right)\right) + E_1,$$
$$-1.28(e^{-4\pi} + e^{-5\pi}) \le E_1 \le 0,$$

$$(5.2)\qquad m(Q_2) = 6 - \frac{2}{\pi}\log 2 + E_2, \qquad -2.56e^{-5\pi} \le E_2 \le 0,$$

$$(5.3)\qquad m(Q_3) = 7.5 - \frac{2}{\pi}\log 2 + E_3, \qquad -1.28(e^{-5\pi} + e^{-8\pi}) \le E_3 \le 0,$$

$$(5.4)\qquad m(Q_4) = 7 - \frac{2}{\pi}\log 2 + E_4, \qquad -1.28(e^{-4\pi} + e^{-8\pi}) \le E_4 \le 0,$$

$$m(Q_7) = \frac{17}{3} - \frac{2}{\pi}\log\left(\frac{6}{5}\right) + \frac{8}{3\pi}\arccos\left(\frac{4}{5}\right) + E_7,$$
$$(5.5)\qquad -1.28(e^{-4\pi} + e^{-(20/3)\pi}) \le E_7 \le 0,$$

$$(5.6)\qquad m(Q_9) = 6.5 - \frac{2}{\pi}\log 2 + E_9, \qquad -1.28(e^{-4\pi} + e^{-7\pi}) \le E_9 \le 0,$$

$$m(Q_{13}) = 6.5 - \frac{2}{\pi}\log\left(\frac{8}{5}\right) + \frac{3}{2\pi}\arccos\left(\frac{3}{5}\right) + E_{13},$$
$$(5.7)\qquad -1.28(e^{-4\pi} + e^{-8\pi}) \le E_{13} \le 0.$$

Therefore, the domain decomposition approximation to $m(Q)$ is

$$(5.8)\qquad \widetilde{m}(Q) := \sum_{j=1}^{13} m(Q_j) = 80.115\,564\,76 + E,$$

where

$$(5.9)\qquad -4.99 \times 10^{-5} < E \le 0.$$

In order to estimate the total error in the approximation $\widetilde{m}(Q)$ to $m(Q)$, we first recall the definition (3.6), (3.7) of the trapezoidal quadrilateral T_h and note that

$$(5.10)\qquad m(T_h) = h + \tfrac{1}{2} - \frac{1}{\pi}\log 2 + \mathcal{E}_h,$$

where, for $h \ge 1$,

$$(5.11)\qquad -1.28e^{-2\pi h} \le \mathcal{E}_h \le 0$$

(see (3.9), (3.10), and Theorem 3.1). Then, by applying repeatedly the results of Theorems 2.4 and 2.6, we have that:

$$0 \le m(Q) - \{m(Q_1) + m(Q_{2,\ldots,13})\} \le 5.26e^{-2\pi m(T_{2.5})},$$
$$0 \le m(Q_{2,\ldots,13}) - \{m(Q_2) + m(Q_{3,\ldots,13})\} \le \frac{4}{\pi}e^{-2\pi m(Q_2)},$$
$$0 \le m(Q_{3,\ldots,13}) - \{m(Q_3) + m(Q_{4,\ldots,13})\} \le 5.26e^{-8\pi},$$

$$0 \le m(Q_{4,\dots,13}) - \{m(Q_4) + m(Q_{5,\dots,13})\} \le \frac{4}{\pi} e^{-2\pi m(Q_4)},$$
$$0 \le m(Q_{5,\dots,13}) - \{m(Q_5) + m(Q_{6,\dots,13})\} \le 5.26 e^{-8\pi},$$
$$0 \le m(Q_{6,\dots,13}) - \{m(Q_6) + m(Q_{7,\dots,13})\} \le 5.26 e^{-2\pi m(T_2)},$$
$$0 \le m(Q_{7,\dots,13}) - \{m(Q_7) + m(Q_{8,\dots,13})\} \le \frac{4}{\pi} e^{-2\pi m(Q_7)},$$
$$0 \le m(Q_{8,\dots,13}) - \{m(Q_8) + m(Q_{9,\dots,13})\} \le 5.26 e^{-2\pi m(T_2)},$$
$$0 \le m(Q_{9,\dots,13}) - \{m(Q_9) + m(Q_{10,\dots,13})\} \le 5.26 e^{-7\pi},$$
$$0 \le m(Q_{10,\dots,13}) - \{m(Q_{10}) + m(Q_{11,\dots,13})\} \le \frac{4}{\pi} e^{-2\pi m(Q_{10})},$$
$$0 \le m(Q_{11,\dots,13}) - \{m(Q_{11}) + m(Q_{12,13})\} \le 5.26 e^{-7\pi},$$
$$0 \le m(Q_{12,13}) - \{m(Q_{12}) + m(Q_{13})\} \le 5.26 e^{-2\pi m(T_2)}.$$

Therefore,

$$0 \le m(Q) - \sum_{j=1}^{13} m(Q_j) \le \varepsilon, \tag{5.12}$$

where, by using (5.2), (5.4), (5.5), (5.6) and (5.10), (5.11),

$$0 \le \varepsilon < 9.66 \times 10^{-6}. \tag{5.13}$$

Hence, in view of (5.8) and (5.9),

$$m(Q) = 80.115\,564\,76 + \mathcal{E}, \tag{5.14}$$

where

$$-4.99 \times 10^{-5} < \mathcal{E} < 9.66 \times 10^{-6}, \tag{5.15}$$

i.e.,

$$80.115\,514 < m(Q) < 80.115\,575. \tag{5.16}$$

For comparison purposes, we now consider two of the heuristic methodologies that we have come across in the VLSI literature, and apply the corresponding algorithms to the problem of computing the conformal module of the same quadrilateral Q. We consider first the so-called "resistance extraction algorithm" of Horowitz and Dutton [12]. As is, in general, the case with VLSI methodologies, the application of this algorithm is restricted to polygonal quadrilaterals bounded by straight lines inclined at angles of 90^o and 45^o. In terms of our terminology, the algorithm of [12] involves the following three steps:

- (a) decomposing the original quadrilateral Q by introducing, at each concave corner c of the defining polygon, a straight-line crosscut perpendicular to the straight line that joins c with the adjacent corner that is farthest away from it (see [12, p. 147]);
- (b) approximating the conformal module of each of the resulting component quadrilaterals by following a set of heuristic rules which are described in [12]; and
- (c) approximating $m(Q)$ by the sum of the computed values for the conformal modules of the component quadrilaterals.

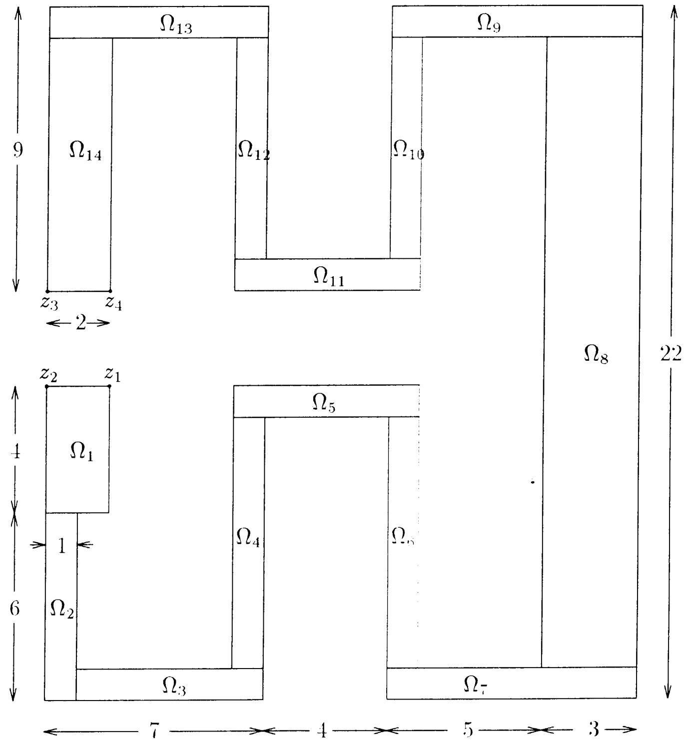

Fig. 5.2

In particular, for the quadrilateral under consideration, step (a) leads to the decomposition illustrated in Figure 5.2. Also, the set of rules of step (b) give the following approximations to the conformal modules of the component quadrilaterals (see also [20, §3]):

$$\begin{array}{lllll} \widetilde{m}(Q_1) = 2.3, & \widetilde{m}(Q_2) = 5.5, & \widetilde{m}(Q_3) = 5.5, & \widetilde{m}(Q_4) = 8, & \widetilde{m}(Q_5) = 5, \\ \widetilde{m}(Q_6) = 8, & \widetilde{m}(Q_7) = 5, & \widetilde{m}(Q_8) = \frac{20}{3}, & \widetilde{m}(Q_9) = 5, & \widetilde{m}(Q_{10}) = 7, \\ \widetilde{m}(Q_{11}) = 5, & \widetilde{m}(Q_{12}) = 7, & \widetilde{m}(Q_{13}) = 5, & \widetilde{m}(Q_{14}) = 4. & \end{array}$$

Thus, the algorithm of [12] leads to the approximation

$$\widetilde{m}(Q) = \sum_{j=1}^{14} \widetilde{m}(Q_j) = 78.967 \tag{5.17}$$

which, as can be seen from (5.16), is correct to only one significant figure.

Our second heuristic algorithm is due to Harbour and Drake [10]. This algorithm differs from that of Horowitz and Dutton [12] only with regard to step (b), i.e., to the determination of the modules of the component quadrilaterals. More specifically, Harbour and Drake consider only orthogonal polygonal quadrilaterals (i.e., quadrilaterals bounded by straight lines inclined at 90^o) which they decompose using the methodology of [12] (i.e., the process described in step (a) above). They then seek to reduce the error of the algorithm of [12] by determining the modules of the resulting rectangular component quadrilaterals accurately (from the available exact formulas) via the computation of elliptic integrals and functions. Thus, for the quadrilateral under consideration, the algorithm of [10] again uses the decomposition illustrated in Figure 5.2 and the exact values $m(Q_4) = m(Q_6) = 8$, $m(Q_8) = 20/3$, $m(Q_{10}) = m(Q_{12}) = 7$, and $m(Q_{14}) = 4$ for the modules of the component quadrilaterals Q_4, Q_6, Q_8, Q_{10}, Q_{12}, and Q_{14}. However, the algorithm of [10] now uses the "exact" values for the modules of the remaining component quadrilaterals Q_j, $j = 1, 2, 3$, Q_5, Q_7, Q_9, Q_{11}, and Q_{13}. To five significant figures, these values are as follows:

$$\begin{aligned} &m(Q_1) = 2.2206, \quad && m(Q_2) = 5.4694, \quad && m(Q_3) = 5.4694, \\ &m(Q_5) = 4.9388, && m(Q_7) = 4.9107, && m(Q_9) = 4.9107, \\ &m(Q_{11}) = 4.9388, && m(Q_{13}) = 4.9118 \end{aligned}$$

(see also (4.1), (4.2), and Theorem 2.6). Therefore, the algorithm of Harbour and Drake [10] leads to the approximation

$$\widetilde{m}(Q) := \sum_{j=1}^{14} m(Q_j) = 78.4369, \tag{5.18}$$

which (see (5.16)) is again correct to only one significant figure. Thus, although the conformal modules of all the component quadrilaterals have been determined accurately, because of the inappropriate choice of the crosscuts of subdivision, the final approximation to $m(Q)$ is again inaccurate. Ironically, a comparison of (5.16), (5.17), and (5.18) shows that the above approximation due to the algorithm of Harbour and Drake [10] is, in fact, somewhat less accurate than that obtained by means of the algorithm of Horowitz and Dutton [12].

Finally, we note that the MATLAB Schwarz–Christoffel Toolbox (*SC Toolbox*) of Driscoll [4] gives the approximation

$$\widetilde{m}(Q) = 80.115\,566 \tag{5.19}$$

to $m(Q)$. For the example under consideration the above computer package obtains the approximation (5.19) by making use of the modified Schwarz–Christoffel transformation of Howell and Trefethen [13]. (This transformation technique of [13] avoids the crowding difficulties, by using an infinite strip as the intermediate canonical domain for the mapping of the quadrilateral onto its conformally equivalent rectangle.) As can be seen, the approximation (5.19) lies within the range predicted by (5.16). We note, however, that the computation of (5.19) required a CPU time of 249 minutes on an IBM RS/6000.

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References

1. L. V. AHLFORS (1973): Conformal Invariants: Topics in Geometric Function Theory. New York: McGraw-Hill.
2. A. BAERNSTEIN II (1988): *Ahlfors and conformal invariants*. Ann. Acad. Sci. Fenn. Ser. A I Math., **13**:289–312.

3. T. F. BOGART JR. (1993): Electronic Devices and Circuits. New York: Macmillan.
4. T. A. DRISCOLL (1996): *A MATLAB toolbox for Schwarz–Christoffel mapping*. ACM Trans. Math. Software, **22**:168–186.
5. D. GAIER, W. K. HAYMAN (1990): *Moduli of long quadrilaterals and thick ring domains*. Rend. Mat. Appl., **10**:809–834.
6. D. GAIER, W. K. HAYMAN (1991): *On the computation of modules of long quadrilaterals*. Constr. Approx., **7**:453–467.
7. R. L. GEIGER, P. E. ALLEN, N. R. STRADER (1990): VLSI Design Techniques for Analog and Digital Circuits. New York: McGraw-Hill.
8. F. B. HAGEDORN, P. M. HALL (1963): *Right-angle bends in thin strip conductors*. J. Appl. Phys., **34**:128–133.
9. P. M. HALL (1967/68): *Resistance calculations for thin film patterns*. Thin Solid Films, **1**:277–295.
10. M. G. HARBOUR, J. M. DRAKE (1986): *Numerical method based on conformal transformations for calculating resistances in integrated circuits*. Internat. J. Electron., **60**:679–689.
11. W. K. HAYMAN (1948): *Remarks on Ahlfors' distortion theorem*. Quart. J. Math. (Oxford), **19**:33–53.
12. M. HOROWITZ, R. W. DUTTON (1983): *Resistance extraction from mask layout data*. IEEE Trans. Computer-Aided Design, **CAD-2**:145–150.
13. L. H. HOWELL, L. N. TREFETHEN (1990): *A modified Schwarz–Christoffel transformation for elongated regions*. SIAM J. Sci. Statist. Comput., **11**:928–949.
14. H. KOBER (1957): Dictionary of Conformal Representations. New York: Dover.
15. A. MUKHERJEE (1986): Introduction to nMOS and CMOS VLSI Systems Design. Englewood Cliffs, NJ: Prentice Hall.
16. N. PAPAMICHAEL (1989): *Numerical conformal mapping onto a rectangle with applications to the solution of Laplacian problems*. J. Comput. Appl. Math., **28**:63–83.
17. N. PAPAMICHAEL, N. S. STYLIANOPOULOS (1991): *A domain decomposition method for conformal mapping onto a rectangle*. Constr. Approx., **7**:349–379.
18. N. PAPAMICHAEL, N. S. STYLIANOPOULOS (1992): *A domain decomposition method for approximating the conformal modules of long quadrilaterals*. Numer. Math., **62**:213–234.
19. N. PAPAMICHAEL, N. S. STYLIANOPOULOS (1994): *On the theory and application of a domain decomposition method for computing conformal modules*. J. Comput. Appl. Math., **50**:33–50.
20. N. PAPAMICHAEL, N. S. STYLIANOPOULOS (1994/95): *Domain decomposition for conformal maps*. In: Computational Methods and Function Theory (R. M. Ali, St. Ruscheweyh, E. B. Saff, eds.). Singapore: World Scientific, pp. 267–291.
21. B. RODIN, S. E. WARSCHAWSKI (1976): *Extremal length and the boundary behavior of conformal mappings*. Ann. Acad. Sci. Fenn. Ser. A I Math., **2**:467–500.
22. R. SCHINZINGER, P. A. A. LAURA (1991): Conformal Mapping: Methods and Applications. Amsterdam: Elsevier.
23. S. M. SZE (1985): Semiconductor Devices, Physics and Technology. New York: Wiley.
24. L. N. TREFETHEN (1986): Numerical Conformal Mapping. Amsterdam: North-Holland.
25. N. H. E. WESTE, K. ESHRAGHIAN (1985): Principles of CMOS VLSI Design. Reading, MA: Addison-Wesley.

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