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Strong Asymptotics for Bergman Polynomials over Domains with Corners and Applications

Nikos Stylianopoulos

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Abstract Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\varGamma := \partial G$ is a Jordan curve, and let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of G. This is defined as the unique sequence

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0,\ n = 0, 1, 2, \ldots,$$

of polynomials that are orthonormal with respect to the inner product

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}\, dA(z),$$

where dA stands for the area measure.

We establish the strong asymptotics for p_n and λ_n, $n \in \mathbb{N}$, under the assumption that $\varGamma$ is piecewise analytic. This complements an investigation started in 1923 by T. Carleman, who derived the strong asymptotics for $\varGamma$ analytic, and carried over by P.K. Suetin in the 1960s, who established them for smooth $\varGamma$. In order to do so, we use a new approach based on tools from quasiconformal mapping theory. The impact of the resulting theory is demonstrated in a number of applications, varying from coefficient estimates in the well-known class $\varSigma$ of univalent functions and a connection with operator theory, to the computation of capacities and a reconstruction algorithm from moments.

Keywords Bergman orthogonal polynomials · Faber polynomials · Strong asymptotics · Polynomial estimates · Quasiconformal mapping · Conformal mapping

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N. Stylianopoulos (✉)
Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus
e-mail: nikos@ucy.ac.cy

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1 Introduction and Main Results

Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\Gamma := \partial G$ is a Jordan curve, and let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of G. This is defined as the unique sequence of polynomials

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0,\ n = 0, 1, 2, \ldots, \tag{1.1}$$

that are orthonormal with respect to the inner product

$$\langle f, g\rangle_G := \int_G f(z)\overline{g(z)}\, dA(z),$$

where dA stands for the area measure. We denote by $L_a^2(G)$ the Hilbert space of functions f analytic in G for which

$$\|f\|_{L^2(G)} := \langle f, f\rangle_G^{1/2} < \infty,$$

and recall that the sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ forms a complete orthonormal system for $L_a^2(G)$.

Let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ denote the complement of $\overline{G}$ in $\overline{\mathbb{C}}$, and let Φ denote the conformal map $\Omega \to \Delta := \{w : |w| > 1\}$, normalized so that near infinity,

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots, \quad \gamma > 0. \tag{1.2}$$

Finally, let $\Psi := \Phi^{-1} : \Delta \to \Omega$ denote the inverse conformal map. Then,

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots, \quad |w| > 1, \tag{1.3}$$

where $b = 1/\gamma$ gives the (*logarithmic*) *capacity* $\operatorname{cap}(\Gamma)$ of Γ.

As in the bounded case, we define the inner product

$$\langle f, g\rangle_\Omega := \int_\Omega f(z)\overline{g(z)}\, dA(z)$$

on the (unbounded) complement Ω of $\overline{G}$ and denote by $L_a^2(\Omega)$ the Hilbert space of functions f analytic in Ω for which

$$\|f\|_{L^2(\Omega)} := \langle f, f\rangle_\Omega^{1/2} < \infty.$$

We note that $L_a^2(G)$ and $L_a^2(\Omega)$ are known as the *Bergman spaces* of G and Ω, respectively. It is easy to see that for $f \in L_a^2(\Omega)$ to hold, it is necessary that $f(z)$ has around infinity a Laurent series expansion starting with $1/z^2$.

The main purpose of the paper is to establish the strong asymptotics of the leading coefficients $\{\lambda_n\}_{n\in\mathbb{N}}$ and the Bergman polynomials $\{p_n\}_{n\in\mathbb{N}}$, in Ω, for nonsmooth boundary Γ. We do this under the assumption that Γ is *piecewise analytic without cusps*. This means that Γ consists of a finite set of analytic arcs that meet at exterior angles $\omega\pi$, with $0<\omega<2$. Thus, we allow Γ to have corners. In this sense, our results complement an investigation started by T. Carleman [11] in 1923, who derived the strong asymptotics under the assumption that Γ is analytic, and was carried over by P.K. Suetin [50] in the 1960s, who verified them for smooth Γ.

The techniques employed in both [11] and [50] are tied to the specific properties that characterize the mapping functions Φ and Ψ in cases when Γ is analytic, or smooth, and it turns out that they are not suitable for treating domains with corners. In order to overcome this, we have developed an approach that we believe to be novel. This approach involves, in particular, new techniques from the theory of quasiconformal mapping and a new sharp estimate concerning the growth of a polynomial in terms of its L^2-norm.

Our main results are the following three theorems.

Theorem 1.1 *Assume that Γ is piecewise analytic without cusps. Then, for any $n\in\mathbb{N}$, it holds that*

$$\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2}=1-\alpha_n, \tag{1.4}$$

where

$$0\le\alpha_n\le c_1(\Gamma)\,\frac{1}{n}. \tag{1.5}$$

Theorem 1.2 *Under the assumptions of Theorem 1.1, for any $n\in\mathbb{N}$, it holds that*

$$p_n(z)=\sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\bigl\{1+A_n(z)\bigr\},\quad z\in\Omega, \tag{1.6}$$

where

$$\bigl|A_n(z)\bigr|\le\frac{c_2(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}}+c_3(\Gamma)\,\frac{1}{n}. \tag{1.7}$$

Above and subsequently, we use $c(\Gamma)$, $c_1(\Gamma)$, $c_2(\Gamma)$, etc., to denote nonnegative constants that depend only on Γ. We also use $\operatorname{dist}(z,B)$ to denote the (Euclidian) distance of z from a set B and call the quantities α_n and $A_n(z)$, defined by (1.4) and (1.6), the *strong asymptotic errors* associated with λ_n and $p_n(z)$, respectively.

From (1.7) and the well-known distortion property of conformal mappings

$$\operatorname{dist}\bigl(\Phi(z),\partial\mathbb{D}\bigr)\le 4\operatorname{dist}(z,\Gamma)\,\bigl|\Phi'(z)\bigr|,\quad z\in\Omega, \tag{1.8}$$

see, e.g., [3, p. 23], we arrive at another estimate for $A_n(z)$, which does not involve the derivative of Φ, i.e.,

$$\bigl|A_n(z)\bigr|\le\frac{c_4(\Gamma)}{|\Phi(z)|-1}\,\frac{1}{\sqrt{n}}+c_3(\Gamma)\,\frac{1}{n},\quad z\in\Omega. \tag{1.9}$$

Our next result provides an interesting link between the Bergman polynomials and the problem of coefficient estimates in univalent functions theory. This result is established under the assumption that $\varGamma$ belongs to a broader class of Jordan curves than the one appearing in Theorem 1.1, namely the class of quasiconformal curves. We recall that a Jordan curve $\varGamma$ is *quasiconformal* if there exists a constant M such that

$$\operatorname{diam} \varGamma(z,\zeta) \le M|z-\zeta| \quad \text{for all } z,\zeta \in \varGamma,$$

where $\varGamma(z,\zeta)$ is the arc (of smaller diameter) of $\varGamma$ between z and ζ. In connection with the assumptions of Theorem 1.1, we also recall that a piecewise analytic Jordan curve is quasiconformal if and only if has no cusps. The assumption that $\varGamma$ is quasiconformal ensures the existence of an associated K-quasiconformal reflection $y(z)$ for some $K \ge 1$, which is characterized by the properties (A1)–(A3) stated in Remark 4.1 below. The existence of the quasiconformal reflection was established by Ahlfors in [1]. All our estimates that are derived under the assumption that $\varGamma$ is quasiconformal are given in terms of the constant

$$k := (K-1)/(K+1), \tag{1.10}$$

which we subsequently refer to as the *reflection factor of* $\varGamma$ (*associated with* y). We note that $0 \le k < 1$, with $k = 0$ if $\varGamma$ is a circle.

In the next theorem, we require, in addition, that $\varGamma$ is rectifiable. Note that there are examples of nonrectifiable quasiconformal curves; see, e.g., [32, p. 104]. However, any quasiconformal curve has zero area.

Our result shows that the strong asymptotic error α_n cannot decay faster than $(n+1)|b_{n+1}|^2$, where b_{n+1} is the coefficient of $1/w^{n+1}$ in the Laurent series expansion (1.3) of $\varPsi(w)$.

Theorem 1.3 *Assume that $\varGamma$ is quasiconformal and rectifiable. Then, for any $n \in \mathbb{N}$, it holds that*

$$\alpha_n \ge \frac{\pi(1-k^2)}{A(G)}(n+1)|b_{n+1}|^2, \tag{1.11}$$

where $A(G)$ denotes the area of G and k is the reflection factor of $\varGamma$.

As was noted above, Theorem 1.3 provides a link between the problems of estimating the error in the strong asymptotics for λ_n and of estimating coefficients in the well-known class $\varSigma$, consisting of functions analytic and univalent in $\varDelta \setminus \{\infty\}$ that have a Laurent series expansion of the form (1.3) with $b = 1$. This latter problem is one of the best studied in geometric function theory; see, e.g., [38] and [16].

In another application, the result of Theorem 1.3 can be used to discuss the sharpness in the decay of order $O(1/n)$, predicted for the sequence $\{\alpha_n\}$ by Theorem 1.1. Ideally, in order to show by means of Theorem 1.3 that the estimate (1.5) is sharp, it would suffice to find a domain G, bounded by a piecewise analytic, and without cups, Jordan curve $\varGamma$, for which it would hold that $|b_n| \ge c_4/n$. In view of the estimate $|b_n| \le c_5/n$, for $n \in \mathbb{N}$, however, which was obtained by Gaier in [21] for $\varGamma$ piecewise Dini-smooth, this seems already tricky, because if $\varGamma$ is piecewise analytic (even with cusps), then it is also piecewise Dini-smooth. Moreover, in view of the estimate $|b_n| \le c_4/n^{1+\omega}$, $n \in \mathbb{N}$, with $0 < \omega < 2$, which is established in Sect. 7.8 below, the use of Theorem 1.3 for proving this kind of sharpness is of no help.

Nevertheless, Theorem 1.3 can be employed to show that the order $O(1/n)$ in (1.5) is best possible in a different sense:

Remark 1.1 For any $\epsilon > 0$, there exists a domain G, which is bounded by a piecewise analytic Jordan curve Γ, such that for the associated strong asymptotic error α_n it holds that

$$\alpha_n \ge c_5(\Gamma)\frac{1}{n^{1+\epsilon}} \tag{1.12}$$

for some positive constant $c_5(\Gamma)$ and infinitely many $n \in \mathbb{N}$.

This will be shown in a forthcoming article with the help of a domain G whose boundary consists of two symmetric, with respect to the imaginary axis, circular arcs that meet at i and $-i$, forming exterior angles π/N, with $N \in \mathbb{N}\setminus\{1\}$. More precisely, in this case, it can be shown that

$$|b_{2n+1}| \asymp \frac{1}{n^{1+1/N}}, \quad n \in \mathbb{N},$$

which, in view of Theorem 1.3, implies (1.12).

In addition to the above, we present two examples and certain numerical evidence supporting the hypothesis that the order $O(1/n)$ in (1.5) is, indeed, sharp.

The first example is based on a Jordan curve constructed by Clunie in [12], for which the sequence $\{n\,b_n\}_{n\in\mathbb{N}}$ is unbounded. More precisely, for $\epsilon = 1/50$, there exists some subsequence $\mathcal{N}$ of $\mathbb{N}$ such that

$$n|b_n| > n^{\epsilon}, \quad n \in \mathcal{N}. \tag{1.13}$$

It was shown by Gaier in [21, §4.2] that Clunie's curve is, eventually, quasiconformal.

The second example is generated by the function

$$\Psi(w) = w + \frac{1}{(m-1)w^{m-1}}, \quad |w| \ge 1. \tag{1.14}$$

For any $m \ge 3$, this function maps Δ conformally onto the exterior of a symmetric m-cusped hypocycloid H_m, which is a piecewise analytic Jordan curve with all exterior angles equal to 2π and thus not a quasiconformal curve. Nevertheless, for each $n \ge 2$, H_{n+1} provides an example of a cusped Jordan curve, where $b_n = 1/n$.

Regarding numerical evidence, we consider the case where G is the unit half-disk and display in Table 1 a range of computed values of α_n, for $n = 51, \dots, 60$. These were obtained from the exact value $\gamma = 1/\operatorname{cap}(\Gamma) = 3\sqrt{3}/4$ and the computed values of λ_n after constructing, in finite high precision and in the way indicated in Sect. 7.4 below, the Bergman polynomials up to degree 60. Thus, we expect all the figures quoted in the table to be correct. The reported values of α_n indicate that the strong asymptotic error for the leading coefficient decays monotonically to zero. In view

Table 1 The rate of decay of α_n for the unit half-disk

n	α_n	s
51	0.003 263 458 678	–
52	0.003 200 769 764	0.998 887
53	0.003 140 444 435	0.998 899
54	0.003 082 351 464	0.998 911
55	0.003 026 369 160	0.998 923
56	0.002 972 384 524	0.998 934
57	0.002 920 292 482	0.998 946
58	0.002 869 952 027	0.998 957
59	0.002 821 401 485	0.998 968
60	0.002 774 426 207	0.998 979

of the estimate (1.5), we test the hypothesis $\alpha_n \approx 1/n^s$. The values of s in the table indicate clearly that $s = 1$.

Exactly the same behavior was observed in a number of different nonsmooth cases involving various angles and shapes. Based on such evidence, we have conjectured the strong asymptotics for nonsmooth domains in [8, pp. 520–521].

The first ever result regarding strong asymptotics for $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{p_n\}_{n\in\mathbb{N}}$ was derived by Carleman in [11] for domains bounded by analytic Jordan curves. In this case, the conformal map Φ has an analytic and one-to-one continuation across Γ inside G.

Theorem (Carleman [11]; see also [20, pp. 12–14] and [36, Theorem 2.1.1]) *Let L_R denote the level curve $\{z : |\Phi(z)| = R\}$, and assume that $\varrho < 1$ is the smallest number for which Φ is conformal in the exterior of L_ϱ. Then, for any $n \in \mathbb{N}$,*

$$0 \le \alpha_n \le c_6(\Gamma)\,\varrho^{2n} \tag{1.15}$$

and

$$\left|A_n(z)\right| \le c_7(\Gamma)\sqrt{n}\,\varrho^n, \quad z \in \overline{\Omega}. \tag{1.16}$$

The next major step in removing the analyticity assumption on Γ was taken by P.K. Suetin in the 1960s. For his results, Suetin requires that the boundary curve Γ belongs to the smoothness class $C(p,\alpha)$. This means that Γ is defined by $z = g(s)$, where s denotes arclength, with $g^{(p)} \in \operatorname{Lip}\alpha$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$. In this case, both Φ and Ψ are p times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$, respectively, with $\Phi^{(p)}$ and $\Psi^{(p)}$ in $\operatorname{Lip}\alpha$. A typical result goes as follows:

Theorem (Suetin [50, Theorems 1.1 & 1.2]) *Assume that $\Gamma \in C(p+1,\alpha)$, with $p+\alpha > 1/2$. Then, for any $n \in \mathbb{N}$,*

$$0 \le \alpha_n \le c_8(\Gamma)\,\frac{1}{n^{2(p+\alpha)}} \tag{1.17}$$

and

$$\left|A_n(z)\right| \le c_9(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}. \tag{1.18}$$

The results of Carleman and Suetin given above, in conjunction with Theorem 1.3, immediately yield estimates for the decay of the coefficients b_n, depending on the degree of analyticity, or smoothness, of Γ. More precisely:

Corollary 1.1 *Under the assumptions of the theorem of Carleman it holds, for any* $n \in \mathbb{N}$, *that*

$$|b_n| \le c_{10}(\Gamma)\,\frac{\varrho^n}{\sqrt{n}}.$$

Under the assumptions of the theorem of Suetin, it holds, for any $n \in \mathbb{N}$, *that*

$$|b_n| \le c_{11}(\Gamma)\,\frac{1}{n^{p+\alpha+1/2}}.$$

Strong asymptotics for λ_n and p_n were also derived by E.R. Johnston in his Ph.D. thesis [27]. These asymptotics, however, were established under analytic assumptions on certain functions related with the conformal maps Φ and Ψ (as compared to the geometric assumptions on Γ in the theorems above), and they do not provide the order of decay of the associated errors. An account of Johnston's results can be found in [41].

In addition, we cite the following representative works about strong asymptotics for complex orthogonal polynomials generated by measures supported on 2-dimensional subsets of $\mathbb{C}$: (a) Szegő's book [51, Chap. XVI], for orthogonal polynomials with respect to the arclength measure (the so-called Szegő polynomials) on analytic Jordan curves; (b) Suetin's paper [49], for weighted Szegő polynomials on smooth Jordan curves; (c) Widom's paper [54], for weighted Szegő polynomials on systems of smooth Jordan curves and smooth Jordan arcs; (d) the recent paper [25] by Gustafsson, Putinar, Saff, and the author, for Bergman polynomials on systems of smooth Jordan domains.

The above list is by no means complete. Nevertheless, we haven't been able to trace in the literature a single result establishing strong asymptotics for orthogonal polynomials defined by measures supported on nonsmooth domains, curves, or arcs. In this connection, we note that the well-known approach that combines the Riemann–Hilbert reformulation of orthogonal polynomials of Fokas, Its, and Kitaev [18, 19], with the method of steepest descent, introduced by Deift and Zhou [14], cannot be applied, at least in its present form, to derive strong asymptotics for Bergman polynomials associated with nontrivial domains. This is so because this approach produces, invariably, orthogonal polynomials that satisfy a finite-term recurrence relation, and this is not the case with the Bergman polynomials, as Theorem 7.3 below shows.

By contrast, strong asymptotics for orthogonal polynomials on the real line and the unit circle is a well-studied subject. From the vast bibliography available, we cite the two volumes of B. Simon [45, 46], which contain a comprehensive treatment of the classical and the spectral theory of orthogonal polynomials on the unit circle, and the recent breakthrough paper of Lubinsky [33] on universality limits for kernel polynomials defined by positive Borel measures in $(-1, 1)$.

The paper is organized as follows: In Sect. 2, we study the properties of associated Faber polynomials and derive a number of preliminary results under the assumptions: (a) G is a bounded domain, and (b) Γ is a rectifiable Jordan curve. In addition, we state a number of results that are needed in the proofs of the three main theorems, under increasing assumptions on Γ, namely: (c) Γ is a quasiconformal curve, and (d) Γ is a piecewise analytic curve. The main result of Sect. 3 is a sharp estimate that relates the growth of a polynomial in Ω to its L^2-norm in G. This estimate is essential for establishing Theorem 1.2. Sections 4 and 5 are devoted to the proofs of the results stated in Sect. 2 regarding assumptions (c) and (d), respectively. Section 6 contain the proofs of the three main theorems of Sect. 1. Finally, in Sect. 7, we present briefly a number of applications of the strong asymptotics and the associated theory.

Theorems 1.1 and 1.2, along with Corollaries 7.1 and 7.2 and Theorems 7.1, 7.3, and 7.4, have been presented, without proofs, in [48].

2 Preliminary Results

The Faber polynomials $\{F_n\}_{n=0}^{\infty}$ of G are defined as the polynomial part of the expansion of $\Phi^n(z)$ near infinity. Therefore, from (1.2),

$$\Phi^n(z) = F_n(z) - E_n(z), \quad z \in \Omega, \tag{2.1}$$

where

$$F_n(z) = \gamma^n z^n + \cdots \tag{2.2}$$

is the Faber polynomial of degree n, and

$$E_n(z) = \frac{c_1^{(n)}}{z} + \frac{c_2^{(n)}}{z^2} + \frac{c_3^{(n)}}{z^3} + \cdots \tag{2.3}$$

is the singular part of $\Phi^n(z)$. According to the asymptotics established by Carleman, the Bergman polynomial $p_n(z)$ is related to $\Phi^n(z)\Phi'(z)$. Consequently, we consider the polynomial part of $\Phi^n(z)\Phi'(z)$, and we denote the resulting series by $\{G_n\}_{n=0}^{\infty}$. $G_n(z)$ is the so-called Faber polynomial of the 2nd kind (of degree n) and satisfies

$$\Phi^n(z)\Phi'(z) = G_n(z) - H_n(z), \quad z \in \Omega, \tag{2.4}$$

with

$$G_n(z) = \gamma^{n+1} z^n + \cdots \tag{2.5}$$

and

$$H_n(z) = \frac{a_2^{(n)}}{z^2} + \frac{a_3^{(n)}}{z^3} + \frac{a_4^{(n)}}{z^4} + \cdots, \tag{2.6}$$

valid in a neighborhood of infinity. It follows immediately from (2.1) and (2.4) that

$$G_n(z) = \frac{F'_{n+1}(z)}{n+1} \quad \text{and} \quad H_n(z) = \frac{E'_{n+1}(z)}{n+1}. \tag{2.7}$$

Lemma 2.1 *For any $n \in \mathbb{N}$, it holds that $H_n \in L_a^2(\Omega)$.*

Proof First we observe that the function $\Phi^n(z)\Phi'(z)$ is square integrable in the bounded doubly-connected domain D_R defined by the boundary curve Γ and the level line

$$L_R := \{z : |\Phi(z)| = R\} = \{\Psi(w) : |w| = R\}, \quad R > 1.$$

Indeed, by making the change of variables $w = \Phi(z)$, we have

$$\int_{D_R} |\Phi^n(z)\Phi'(z)|^2 \, dA(z) = \int_{1<|w|<R} |w|^{2n} \, dA(w) = \frac{\pi}{n+1}\{R^{2(n+1)} - 1\}.$$

Therefore,

$$\begin{aligned} \left[\int_{D_R} |H_n(z)|^2 \, dA(z)\right]^{1/2} &\le \left[\int_{D_R} |\Phi^n(z)\Phi'(z)|^2 \, dA(z)\right]^{1/2} \\ &\quad + \left[\int_{D_R} |G_n(z)|^2 \, dA(z)\right]^{1/2} \\ &< \infty. \end{aligned} \tag{2.8}$$

Next, from the splitting (2.4), we see that $H_n(z)$ is analytic in Ω and has a double zero at infinity. Assume that (2.6) is valid for $|z| > R_1$. Then $\limsup_{k\to\infty} |a_k^{(n)}|^{1/k} = R_1$ and, hence, the estimate

$$|a_k^{(n)}| \le c\, R_2^k$$

holds for some $R_2 > R_1$. Therefore, for any $R_3 > 1$, with $R_3 > R_2$, we have

$$\begin{aligned} \int_{|z|>R_3} |H_n(z)|^2 \, dA(z) &= \int_0^{2\pi} \int_{R_3}^{\infty} |H_n(re^{i\theta})|^2 r \, dr \, d\theta = \sum_{k=2}^{\infty} \frac{|a_k^{(n)}|^2}{(k-1)R_3^{2(k-1)}} \\ &\le c \sum_{k=2}^{\infty} \frac{R_2^{2k}}{(k-1)R_3^{2(k-1)}} < \infty. \end{aligned} \tag{2.9}$$

Now, choose R sufficiently large so that D_R contains the circle $\{z : |z| = R_3\}$. Then, the result $\|H_n\|_{L^2(\Omega)} < \infty$ follows at once from the two estimates (2.8) and (2.9). □

Remark 2.1 and Theorem 2.4 below show that a lot more can be said about the behavior of $\|H_n\|_{L^2(\Omega)}$ under additional assumptions on Γ.

2.1 Results for Rectifiable Boundary

We assume now that the boundary Γ is *rectifiable*. (Further assumptions on Γ will be imposed in various parts of the paper.) For rectifiable Γ, Cauchy's integral formula yields the following representation for the Faber polynomial $F_n(z)$ and its corresponding singular part $E_n(z)$:

$$F_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi^n(\zeta)}{\zeta - z}\, d\zeta, \quad z \in G, \tag{2.10}$$

$$E_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi^n(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Omega. \tag{2.11}$$

It is well known that the assumption on Γ implies the facts that Φ' belongs to the Smirnov class $E^1(\Omega)$, that both Φ' and Ψ' have nontangential limits almost everywhere on Γ and $\partial\mathbb{D}$, respectively, and that they are integrable with respect to the arclength measure, i.e.,

$$\int_\Gamma \left|\Phi'(\zeta)\right| |d\zeta| < \infty \quad \text{and} \quad \int_\mathbb{T} \left|\Psi'(t)\right| |dt| < \infty; \tag{2.12}$$

see, e.g., [15, Chap. 10], [28], and [39, §6.3]. Hence $H_n \in E^1(\Omega)$, and therefore (2.4) yields the following two Cauchy representations:

$$G_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi^n(\zeta)\Phi'(\zeta)}{\zeta - z}\, d\zeta, \quad z \in G, \tag{2.13}$$

and

$$H_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi^n(\zeta)\Phi'(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Omega; \tag{2.14}$$

cf. [15, Theorem 10.4]. We note the following estimate, which is a simple consequence of (2.12) and the representation (2.14):

$$\left|H_n(z)\right| \le \frac{c_1(\Gamma)}{\operatorname{dist}(z, \Gamma)}, \quad z \in \Omega. \tag{2.15}$$

Next, we single out three identities, which we are going to use below.

Lemma 2.2 *Assume that the boundary Γ is rectifiable. Then, for any $m, n \in \mathbb{N}$, it holds that*

$$\frac{1}{2\pi i} \int_\Gamma \Phi^m(z)\Phi'(z)\overline{\Phi^{n+1}(z)}\, dz = \delta_{m,n} \tag{2.16}$$

and

$$\int_\Gamma H_m(z)\overline{\Phi^{n+1}(z)}\, dz = 0 = \int_\Gamma \Phi^m(z)\Phi'(z)\overline{E_{n+1}(z)}\, dz, \tag{2.17}$$

where $\delta_{m,n}$ denotes Kronecker's delta function.

Proof Since $\Phi' \in E^1(\Omega)$, the application of Cauchy's theorem and the change of variables $w = \Phi(z)$ give, for any $R > 1$,

$$\frac{1}{2\pi i}\int_\Gamma \Phi^m(z)\Phi'(z)\overline{\Phi^{n+1}(z)}\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi^m(z)\Phi'(z)}{\Phi^{n+1}(z)}\,dz$$
$$= \frac{1}{2\pi i}\int_{L_R} \frac{\Phi^m(z)\Phi'(z)}{\Phi^{n+1}(z)}\,dz = \frac{1}{2\pi i}\int_{|w|=R} \frac{w^m\,dw}{w^{n+1}},$$

and the result (2.16) follows from the residue theorem.

Similarly, we have

$$\frac{1}{2\pi i}\int_\Gamma H_m(z)\overline{\Phi^{n+1}(z)}\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{H_m(z)}{\Phi^{n+1}(z)}\,dz,$$

and the first identity in (2.17) follows, again, from the residue theorem.

Since E_{n+1} and Φ are analytic on L_R, for $R > 1$, it follows from integration by parts and (2.7) that

$$\frac{1}{2\pi i}\int_{L_R} \Phi^m(z)\Phi'(z)\overline{E_{n+1}(z)}\,dz = \frac{1}{2(m+1)\pi i}\int_{L_R} \big(\Phi^{m+1}(z)\big)'\overline{E_{n+1}(z)}\,dz$$
$$= -\frac{1}{2(m+1)\pi i}\int_{L_R} \overline{\Phi^{m+1}(z)}\big(E_{n+1}(z)\big)'\,dz$$
$$= -\frac{n+1}{2(m+1)\pi i}\int_{L_R} \overline{\Phi^{m+1}(z)}H_n(z)\,dz,$$

and the second identity in (2.17) then follows by letting $R \to 1$ and using the first identity in (2.17). □

With the help of $G_n(z)$, we define an auxiliary polynomial that plays a crucial role in the course our study:

$$q_{n-1}(z) := G_n(z) - \frac{\gamma^{n+1}}{\lambda_n}p_n(z), \quad n \in \mathbb{N}. \tag{2.18}$$

Observe that $q_{n-1}(z)$ has degree at most $n-1$, but it can be identical to zero, as the special case $G \equiv \mathbb{D}$ shows.

By noting the relation

$$p_n(z) = \frac{\lambda_n}{\gamma^{n+1}}\Phi^n(z)\Phi'(z)\left\{1 + \frac{H_n(z)}{\Phi^n(z)\Phi'(z)} - \frac{q_{n-1}(z)}{\Phi^n(z)\Phi'(z)}\right\}, \tag{2.19}$$

which follows at once from (2.4) and (2.18) and is valid for any $z \in \Omega$ (since $\Phi'(z) \neq 0$), it is not surprising that we formulate our results in terms of the following two sequences of nonnegative numbers:

$$\beta_n := \frac{n+1}{\pi}\|q_{n-1}\|^2_{L^2(G)}, \quad n \in \mathbb{N}, \tag{2.20}$$

and

$$\varepsilon_n := \frac{n+1}{\pi}\|H_n\|^2_{L^2(\Omega)}, \quad n \in \mathbb{N}. \tag{2.21}$$

The proof of Theorems 1.1 and 1.2 amount, eventually, to establishing that the two sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\varepsilon_n\}_{n\in\mathbb{N}}$ decay to zero like $O(1/n)$. To this end, a representation of β_n and ε_n as line integrals will be useful:

Lemma 2.3 *Assume that the boundary Γ is rectifiable. Then, for any $n \in \mathbb{N}$, it holds that*

$$\beta_n = \frac{1}{2\pi i}\int_\Gamma q_{n-1}(z)\overline{E_{n+1}(z)}\,dz \tag{2.22}$$

and

$$\varepsilon_n = -\frac{1}{2\pi i}\int_\Gamma H_n(z)\overline{E_{n+1}(z)}\,dz. \tag{2.23}$$

Proof To derive (2.22), we use the orthogonality of p_n (2.7), and Green's formula to conclude, in steps, that

$$\begin{aligned}\|q_{n-1}\|^2_{L^2(G)} &= \left\langle q_{n-1}, G_n - \frac{\gamma^{n+1}}{\lambda_n}p_n\right\rangle_G = \langle q_{n-1}, G_n\rangle_G \\ &= \int_G q_{n-1}(z)\overline{G_n(z)}\,dA(z) = \frac{1}{n+1}\int_G q_{n-1}(z)\overline{F'_{n+1}(z)}\,dA(z) \\ &= \frac{\pi}{n+1}\frac{1}{2\pi i}\int_\Gamma q_{n-1}(z)\overline{F_{n+1}(z)}\,dz.\end{aligned}$$

Hence, from (2.1),

$$\frac{n+1}{\pi}\|q_{n-1}\|^2_{L^2(G)} = \frac{1}{2\pi i}\int_\Gamma q_{n-1}(z)\overline{E_{n+1}(z)}\,dz + \frac{1}{2\pi i}\int_\Gamma q_{n-1}(z)\overline{\Phi^{n+1}(z)}\,dz,$$

and the result (2.22) follows because the last integral vanishes, as can be readily seen after replacing $\overline{\Phi^{n+1}(z)}$ by $1/\Phi^{n+1}(z)$ and applying the residue theorem.

Next, we recall that E_{n+1} is analytic in Ω, including ∞, and continuous on $\overline{\Omega}$, and that $H_n \in L^2_a(\Omega) \cap E^1(\Omega)$. The result (2.23) follows from the application of Green's formula in the unbounded domain Ω and (2.7). That is,

$$\begin{aligned}-\frac{1}{2\pi i}\int_\Gamma H_n(z)\overline{E_{n+1}(z)}\,dz &= \frac{1}{\pi}\int_\Omega H_n(z)\overline{E'_{n+1}(z)}\,dA(z) \\ &= \frac{n+1}{\pi}\|H_n\|^2_{L^2(\Omega)}.\end{aligned}$$

□

It turns out that the strong asymptotic error α_n has a very simple connection with the quantities β_n and ε_n; namely,

$$\alpha_n = \beta_n + \varepsilon_n.$$

(This, actually, explains the presence of the fractional term $(n+1)/\pi$ in the definition of β_n and ε_n above.)

Lemma 2.4 *Assume that the boundary Γ is rectifiable. Then, for any $n \in \mathbb{N}$, it holds that*

$$\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - (\beta_n + \varepsilon_n). \tag{2.24}$$

Proof Green's formula, in conjunction with (2.7), yields:

$$\|G_n\|^2_{L^2(G)} = \frac{1}{n+1}\int_G G_n(z)\overline{F'_{n+1}(z)}\,dA(z) = \frac{\pi}{n+1}\frac{1}{2\pi i}\int_\Gamma G_n(z)\overline{F_{n+1}(z)}\,dz.$$

Next, replace in the last integral $G_n(z)$ and $F_{n+1}(z)$ by their counterparts given in the splittings (2.1) and (2.3), respectively, to obtain:

$$\begin{aligned}\int_\Gamma G_n(z)\overline{F_{n+1}(z)}\,dz &= \int_\Gamma \Phi^n(z)\Phi'(z)\overline{\Phi^{n+1}(z)}\,dz + \int_\Gamma \Phi^n(z)\Phi'(z)\overline{E_{n+1}(z)}\,dz \\ &\quad + \int_\Gamma H_n(z)\overline{\Phi^{n+1}(z)}\,dz + \int_\Gamma H_n(z)\overline{E_{n+1}(z)}\,dz.\end{aligned}$$

It therefore follows from Lemma 2.2 that

$$\|G_n\|^2_{L^2(G)} = \frac{\pi}{n+1}\left[1 + \frac{1}{2\pi i}\int_\Gamma H_n(z)\overline{E_{n+1}(z)}\,dz\right],$$

which, in view of Lemma 2.3, yields the relation

$$\|G_n\|^2_{L^2(G)} = \frac{\pi}{n+1}(1-\varepsilon_n), \quad n \in \mathbb{N}. \tag{2.25}$$

On the other hand, since $p_n \perp q_{n-1}$, we have from Pythagoras' theorem that

$$\|G_n\|^2_{L^2(G)} = \left\|\frac{\gamma^{n+1}}{\lambda_n}p_n + q_{n-1}\right\|^2_{L^2(G)} = \frac{\gamma^{2(n+1)}}{\lambda_n^2} + \|q_{n-1}\|^2_{L^2(G)}, \tag{2.26}$$

and (2.24) follows by comparing (2.25) with (2.26) and using the definition of β_n in (2.20). □

Remark 2.1 It follows immediately from (2.24) and (2.25) that

$$0 \le \beta_n + \varepsilon_n < 1, \quad 0 \le \varepsilon_n < 1 \quad \text{and} \quad 0 \le \beta_n < 1. \tag{2.27}$$

In particular, these inequalities lead to the following three estimates:

$$\|G_n\|_{L^2(G)} \le \sqrt{\frac{\pi}{n+1}}, \quad n \in \mathbb{N}, \tag{2.28}$$

and

$$\|q_{n-1}\|_{L^2(G)} < \sqrt{\frac{\pi}{n+1}}, \quad \|H_n\|_{L^2(\Omega)} < \sqrt{\frac{\pi}{n+1}}, \quad n \in \mathbb{N}, \tag{2.29}$$

provided that Γ is rectifiable. The inequality in (2.28) is sharp, as the case $G \equiv \mathbb{D}$ shows. Furthermore, Lemma 2.3 implies that β_n and ε_n vanish simultaneously if G is a disk.

2.2 Results for Quasiconformal Boundary

For the next three results, we need additional assumptions on Γ. Their respective proofs are given in Sect. 4. The first of them is essential in the proof of Theorem 2.1 below and is of independent interest in the sense that it provides an estimate for the integral on Γ of the product of two functions (one of them defined on $\overline{G}$ and the other on $\overline{\Omega}$) in terms of associated L^2-norms in G and Ω.

Lemma 2.5 *Assume that Γ is quasiconformal and rectifiable. Then, for any f analytic in G, continuous on $\overline{G}$ and g analytic in Ω, continuous on $\overline{\Omega}$, with $g' \in L^2(\Omega)$, it holds that*

$$\left|\frac{1}{2i}\int_\Gamma f(z)\overline{g(z)}\,dz\right| \le \frac{k}{\sqrt{1-k^2}}\,\|f\|_{L^2(G)}\left\|g'\right\|_{L^2(\Omega)}, \tag{2.30}$$

where k is the reflection factor of Γ defined in (1.10).

It is readily verified that in the case when Γ is a circle, both sides of (2.30) vanish.

The second result shows that the sequence $\{\beta_n\}$ is dominated by the sequence $\{\varepsilon_n\}$. Note, in particular, that β_n, ε_n, and k vanish simultaneously if Γ is a circle.

Theorem 2.1 *Assume that Γ is quasiconformal and rectifiable. Then, for any $n \in \mathbb{N}$, it holds that*

$$0 \le \beta_n \le \frac{k^2}{1-k^2}\,\varepsilon_n, \tag{2.31}$$

where k is the reflection factor of Γ.

The third result relates the decay of $\{\varepsilon_n\}$ to that of the coefficients of the exterior conformal map Ψ and is essential to the proof of Theorem 1.3.

Theorem 2.2 *Assume that Γ is quasiconformal. Then, for any $n \in \mathbb{N}$, it holds that*

$$\varepsilon_n \ge \frac{\pi\,(1-k^2)}{A(G)}\,(n+1)\,|b_{n+1}|^2, \tag{2.32}$$

where $A(G)$ denotes the area of G and k is the reflection factor of Γ.

2.3 Results for Piecewise Analytic Boundary

The next two theorems are established for Γ *piecewise analytic without cusps*. This means that Γ consists of a finite number of analytic arcs, say N, that meet at corner points z_j, $j = 1, \dots, N$, where they form exterior angles $\omega_j \pi$, with $0 < \omega_j < 2$. The proofs of these theorems are given in Sect. 5.

The relation (2.19) reveals that in order to derive the strong asymptotics for $p_n(z)$ in Ω, we need suitable estimates for $q_{n-1}(z)$ and $H_n(z)$ there. For $q_{n-1}(z)$, this is provided by Corollary 3.1 below. Regarding $H_n(z)$, we can use the estimate (2.15), which is valid for Γ rectifiable. However, under the current assumption on Γ, more can be obtained.

Theorem 2.3 *Assume that Γ is piecewise analytic without cusps. Then, for any $n \in \mathbb{N}$, it holds that*

$$\left|H_n(z)\right| \le \frac{c_2(\Gamma)}{\operatorname{dist}(z, \Gamma)} \frac{1}{n}, \quad z \in \Omega, \tag{2.33}$$

where $c_2(\Gamma)$ depends on Γ only.

Regarding the L^2-norm of H_n, we have the following estimate; cf. (2.29).

Theorem 2.4 *Assume that Γ is piecewise analytic without cusps. Then, for any $n \in \mathbb{N}$, it holds that*

$$\|H_n\|_{L^2(\Omega)} \le c_3(\Gamma) \frac{1}{n}, \tag{2.34}$$

where $c_3(\Gamma)$ depends on Γ only.

It is interesting to note a uniformity aspect in both the estimates (2.33) and (2.34) in the sense that the geometry of Γ, as it is measured by the values of $\omega_j \pi$, does not influence the way that $H_n(z)$ and $\|H_n\|_{L^2(\Omega)}$ tend to zero. This is somewhat surprising when compared with similar results in approximation theory for domains with corners, and it can be attributed to the fact that the effect of ω_j's "cancels out" in the representation (2.14) of $H_n(z)$, see (5.3) and Remark 5.2 below.

We conclude this section with a simple consequence of Theorems 2.1 and 2.4.

Corollary 2.1 *Assume that Γ is piecewise analytic without cusps. Then, for any $n \in \mathbb{N}$, it holds that*

$$0 \le \varepsilon_n \le c_4(\Gamma) \frac{1}{n} \tag{2.35}$$

and

$$\|q_{n-1}\|_{L^2(G)} \le c_5(\Gamma) \frac{1}{n}, \tag{2.36}$$

where $c_4(\Gamma)$ and $c_5(\Gamma)$ depend on Γ only.

A comparison between (2.36) and (2.29) reveals the gain in the rate of decay of $\|q_{n-1}\|_{L^2(G)}$ under the additional assumption on Γ.

3 A Polynomial Lemma

In the proof of Theorem 1.2, we require an estimate for the growth of the polynomial $q_{n-1}(z)$ in Ω in terms of its L^2-norm in G. This is the purpose of the next lemma, which is of independent interest. Its own proof is given in Sect. 4.4 below. We use $\mathbb{P}_n$ to denote the space of the polynomials of degree up to n.

Lemma 3.1 *Assume that Γ is quasiconformal and rectifiable. Then, for any $P \in \mathbb{P}_n$, it holds that*

$$\left|P(z)\right| \le \frac{1}{\operatorname{dist}(z,\Gamma)\sqrt{1-k^2}} \sqrt{\frac{n+1}{\pi}}\, \|P\|_{L^2(G)} \left|\Phi(z)\right|^{n+1}, \quad z \in \Omega, \tag{3.1}$$

where k is the reflection factor of Γ.

Regarding sharpness of the inequality (3.1), we note that the order $1/2$ of n cannot be improved in general, as the choice $P \equiv p_n$ and the strong asymptotics for smooth Γ of Sect. 1 show. Furthermore, the constant term is asymptotically optimal for $z \to \infty$, as the choice $P(z) = z^n$, with $G = \mathbb{D}$ (hence $k = 0$) shows.

Lemma 3.1 should be compared with the following well-known result, which gives the growth of a polynomial in terms of its uniform norm on $\overline{G}$. Hereafter, we use $\|\cdot\|_K$ to denote the uniform norm on the set K.

Lemma (Bernstein–Walsh) *For any $P \in \mathbb{P}_n$, it holds that*

$$\left|P(z)\right| \le \|P\|_{\overline{G}} \left|\Phi(z)\right|^n, \quad z \in \Omega. \tag{3.2}$$

We note that (3.2) is valid under more general assumptions for $\overline{G}$; see, e.g., [44, p. 153]. We also note the following norm-comparison result, which was quoted by Suetin in [50, p. 38] under the assumption that Γ is smooth:

$$\|P\|_{\overline{G}} \le c(\Gamma)\, n\, \|P\|_{L^2(G)}, \quad P \in \mathbb{P}_n. \tag{3.3}$$

To underline the importance of Lemma 3.1 for our work here, we observe that the combination of (3.2) with (3.3) gives the estimate

$$\left|P(z)\right| \le c(\Gamma)\, n\, \|P\|_{L^2(G)} \left|\Phi(z)\right|^n, \quad z \in \Omega, \tag{3.4}$$

and this for $P \equiv q_{n-1}$, together with Corollary 2.1, yields

$$\left|q_{n-1}(z)\right| \le c(\Gamma) \left|\Phi(z)\right|^n, \quad z \in \Omega, \tag{3.5}$$

provided Γ is smooth. Unfortunately, (3.5) is not adequate for delivering the strong asymptotics for $p_n(z)$, even for smooth Γ; see the proof of Theorem 1.2 in Sect. 6.

On the other hand, the combination of Lemma 3.1 with Corollary 2.1 yields the following finer estimate, which suffices to convey that $A_n(z) = O(1/\sqrt{n})$ in (1.6):

Corollary 3.1 *Assume that Γ is piecewise analytic without cusps. Then, for any $n \in \mathbb{N}$, it holds that*

$$\left|q_{n-1}(z)\right| \le \frac{c_1(\Gamma)}{\operatorname{dist}(z,\Gamma)} \frac{1}{\sqrt{n}} \left|\Phi(z)\right|^n, \quad z \in \Omega, \tag{3.6}$$

where $c_1(\Gamma)$ depends on Γ only.

4 Proofs for Quasiconformal Boundary

Assume now that Γ is a quasiconformal curve. Our arguments in this section are based on the use of a *K-quasiconformal reflection* $y : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined, for some $K \ge 1$, by Γ and a fixed point a in G. Below, we collect together some well-known properties of $y(z)$ which are important for our work here, and we refer to the four monographs [2, 4, 32] and [6] for a concise account of results in quasiconformal mapping theory; see also [9, §6].

Remark 4.1 (Properties of quasiconformal reflection) With the above notation, it holds that:

(A1) $\overline{y}$ is a K-quasiconformal mapping $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$;
(A2) $y(G) = \Omega$, $y(\Omega) = G$, with $y(a) = \infty$ and $y(\infty) = a$;
(A3) $y(z) = z$, for every $z \in \Gamma$ and $y(y(z)) = z$, for all $z \in \mathbb{C}$.

For a function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, we use the notation f_z and $f_{\overline{z}}$ to denote its formal complex derivatives

$$f_z := \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial x}\right) \quad \text{and} \quad f_{\overline{z}} := \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial x}\right).$$

We recall that $f_z = f'$ and $f_{\overline{z}} = 0$ whenever f is analytic and

$$\overline{f_z} = \overline{f}_{\overline{z}} \quad \text{and} \quad \overline{f_{\overline{z}}} = \overline{f}_z. \tag{4.1}$$

We also recall the chain rule for formal derivatives, that is, if $\zeta = g(z)$, then

$$(f \circ g)_{\overline{z}} = f_\zeta\big(g(z)\big)\, g_{\overline{z}}(z) + f_{\overline{\zeta}}\big(g(z)\big)\, \overline{g_z(z)}. \tag{4.2}$$

The property (A1) implies that y is a sense-reversing homeomorphism of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ satisfying, almost everywhere in $\mathbb{C}$,

$$\left|\frac{y_z}{y_{\overline{z}}}\right| = \left|\frac{\overline{y}_{\overline{z}}}{\overline{y}_z}\right| \le k := \frac{K-1}{K+1} < 1. \tag{4.3}$$

It further implies that y belongs to the Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Recall that we refer to k as the reflection factor of Γ associated with y.

Let

$$J\bigl(y(z)\bigr) := |y_z|^2 - |y_{\overline{z}}|^2$$

denote the Jacobian of the transformation $y : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, and note that $J(y(z)) < 0$ because $y(z)$ is sense-reversing. It follows easily from (4.3) that

$$|y_{\overline{z}}|^2 \le \frac{-1}{1-k^2} J\bigl(y(z)\bigr) \quad \text{and} \quad |y_z|^2 \le \frac{-k^2}{1-k^2} J\bigl(y(z)\bigr) \tag{4.4}$$

almost everywhere in $\mathbb{C}$. Thus, the change of variables $\zeta = y(z)$ and the property (A2) yield immediately the two estimates

$$\int_\Omega |y_{\overline{z}}|^2 \, dA(z) \le \frac{1}{1-k^2} A(G) \quad \text{and} \quad \int_\Omega |y_z|^2 \, dA(z) \le \frac{k^2}{1-k^2} A(G), \tag{4.5}$$

where $A(G)$ stands for the area of G.

4.1 Proof of Lemma 2.5

Since the function $g(z)$ is analytic in Ω, it follows from (4.1) and the chain rule (4.2) that

$$\bigl[(\overline{g} \circ y)(z)\bigr]_{\overline{z}} = \overline{g'\bigl(y(z)\bigr)}\, \overline{y_z}, \quad z \in G.$$

Hence, it is easy to verify that the function $[(\overline{g} \circ y)(z)]_{\overline{z}}$ is square integrable in G. This is a consequence of the assumption $g' \in L^2(\Omega)$ and the second inequality in (4.4). Indeed, using the change of variables $\zeta = y(z)$, we have:

$$\begin{aligned}\int_G \bigl|\bigl[(\overline{g} \circ y)(z)\bigr]_{\overline{z}}\bigr|^2 \, dA(z) &= \int_G \bigl|g'\bigl(y(z)\bigr)\bigr|^2 |y_z|^2 \, dA(z) \\ &\le \frac{-k^2}{1-k^2} \int_G \bigl|g'\bigl(y(z)\bigr)\bigr|^2 J\bigl(y(z)\bigr) \, dA(z) \\ &= \frac{k^2}{1-k^2} \int_\Omega \bigl|g'(\zeta)\bigr|^2 \, dA(\zeta).\end{aligned} \tag{4.6}$$

Next, we set

$$I := \frac{1}{2i} \int_\Gamma f(z) \overline{g(z)} \, dz$$

and observe that since $y(z) = z$, for $z \in \Gamma$, I can be written as

$$I = \frac{1}{2i} \int_\Gamma f(z) \, \overline{g\bigl(y(z)\bigr)} \, dz.$$

Finally, we note that the function $g(y(z))$ defines a quasiconformal extension of $g(z)$ into G, which is continuous on $\overline{G}$. Therefore, from the assumptions on f, g, and Γ, we conclude by means of Green's formula that

$$I = \int_G \left[f(z)\,(\overline{g} \circ y)(z) \right]_{\overline{z}}\, dA(z) = \int_G f(z) \left[(\overline{g} \circ y)(z) \right]_{\overline{z}}\, dA(z),$$

and the estimate (2.30) then follows by applying the Cauchy–Schwarz inequality to the last integral and using (4.6). □

4.2 Proof of Theorem 2.1

In view of Lemma 2.1 and (2.7), we note that $E'_{n+1} \in L^2(\Omega)$ and apply the result of Lemma 2.5 with $f \equiv q_{n-1}$ and $g \equiv E_{n+1}$ in the expression of β_n given by (2.22) to obtain:

$$\beta_n \le \frac{k}{\sqrt{1-k^2}} \frac{1}{\pi} \|q_{n-1}\|_{L^2(G)} \left\| E'_{n+1} \right\|_{L^2(\Omega)}.$$

Therefore, using the definition of β_n and ε_n in (2.20) and (2.21), we conclude that

$$\begin{aligned} \beta_n^2 &\le \frac{k^2}{1-k^2} \frac{(n+1)^2}{\pi^2} \|q_{n-1}\|^2_{L^2(G)} \|H_n\|^2_{L^2(\Omega)} \\ &= \frac{k^2}{1-k^2} \beta_n\, \varepsilon_n, \end{aligned}$$

which yields at once the required estimate (2.31). □

4.3 Proof of Theorem 2.2

Assume that $R > 1$ is large enough so that the expansion (2.3) is valid for all $z \in L_R$. Then, from the residue theorem and the splitting (2.1), we have

$$c_1^{(n+1)} = \frac{1}{2\pi i} \int_{L_R} E_{n+1}(z)\, dz = -\frac{1}{2\pi i} \int_{L_R} \Phi^{n+1}(z)\, dz.$$

Next, by differentiating the expansion (1.3) of $\Psi(w)$ and applying again the residue theorem, we see that

$$-(n+1) b_{n+1} = \frac{1}{2\pi i} \int_{|w|=R} w^{n+1} \Psi'(w)\, dw = \frac{1}{2\pi i} \int_{L_R} \Phi^{n+1}(z)\, dz. \tag{4.7}$$

Therefore, for any $n \in \mathbb{N}$,

$$c_1^{(n+1)} = (n+1) b_{n+1}.$$

This, in view of (2.3) and (2.7), shows that $a_2^{(n)} = -b_{n+1}$, where $a_2^{(n)}$ is the coefficient of $1/z^2$ in the expansion (2.6) of $H_n(z)$. Hence, another application of the residue theorem yields that

$$-b_{n+1}=\frac{1}{2\pi i}\int_{L_R}H_n(z)z\,dz=\frac{1}{2\pi i}\int_{\Gamma}H_n(z)z\,dz.$$

Furthermore, by using the fact that $y(z)=z$, for $z\in\Gamma$, and the properties of $H_n(z)$ and $y(z)$ in Ω, we obtain with the help of Green's formula in the unbounded domain Ω:

$$b_{n+1}=-\frac{1}{2\pi i}\int_{\Gamma}H_n(z)y(z)\,dz=\frac{1}{\pi}\int_{\Omega}H_n(z)y_{\overline{z}}\,dA(z). \tag{4.8}$$

The last integral can be estimated by means of the Cauchy–Schwarz inequality and the first inequality in (4.5). Indeed,

$$\left|\int_{\Omega}H_n(z)y_{\overline{z}}\,dA(z)\right|\le\|H_n\|_{L_2(\Omega)}\left[\frac{1}{1-k^2}A(G)\right]^{1/2},$$

and the required result emerges from (4.8) and the definition of ε_n. □

4.4 Proof of Lemma 3.1

Let $P\in\mathbb{P}_n$ and fix $z\in\Omega$. Then, the function $P(z)/\Phi^{n+1}(z)$ is analytic in Ω, continuous on Γ, and vanishes at ∞. Hence, from Cauchy's formula and the property $y(\zeta)=\zeta$, for $\zeta\in\Gamma$, we have

$$\frac{P(z)}{\Phi^{n+1}(z)}=-\frac{1}{2\pi i}\int_{\Gamma}\frac{g(\zeta)\,d\zeta}{\Phi^{n+1}(\zeta)}=-\frac{1}{2\pi i}\int_{\Gamma}\frac{g(\zeta)\,d\zeta}{(\Phi^{n+1}\circ y)(\zeta)},$$

where $g(\zeta):=P(\zeta)/(\zeta-z)$. Now, the function $1/\Phi^{n+1}\circ y$ is continuous on $\overline{G}$, and its $\partial/\partial_{\overline{z}}$ derivative belongs to $L^2(G)$; see (4.10) below. Hence, from Green's formula, we have that

$$\begin{aligned}\frac{P(z)}{\Phi^{n+1}(z)}&=-\frac{1}{\pi}\int_{G}\left[\frac{g(\zeta)}{(\Phi^{n+1}\circ y)(\zeta)}\right]_{\overline{\zeta}}dA(\zeta)\\&=\frac{n+1}{\pi}\int_{G}g(\zeta)\,\frac{\Phi'(y(\zeta))\,y_{\overline{\zeta}}}{(\Phi^{n+2}\circ y)(\zeta)}\,dA(\zeta),\end{aligned} \tag{4.9}$$

where we made use of the fact that g is analytic on $\overline{G}$. Next, using (4.4), it is readily seen that

$$\begin{aligned}\int_{G}\frac{|\Phi'(y(\zeta))|^2\,|y_{\overline{\zeta}}|^2}{|(\Phi^{n+2}\circ y)(\zeta)|^2}\,dA(\zeta)&\le\frac{-1}{1-k^2}\int_{G}\frac{|\Phi'(y(\zeta))|^2\,J(y(\zeta))}{|(\Phi^{n+2}\circ y)(\zeta)|^2}\,dA(\zeta)\\&=\frac{1}{1-k^2}\int_{\Omega}\frac{|\Phi'(t)|^2\,dA(t)}{|\Phi^{n+2}(t)|^2}\\&=\frac{1}{1-k^2}\int_{\Delta}\frac{dA(w)}{|w^{n+2}|^2}=\frac{1}{1-k^2}\,\frac{\pi}{(n+1)}.\end{aligned} \tag{4.10}$$

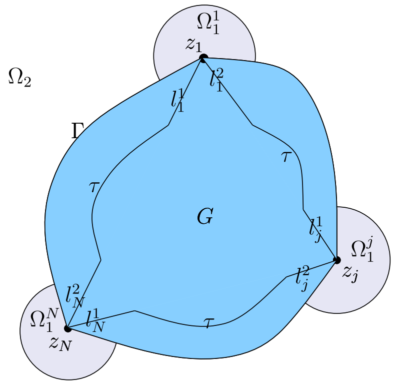

Fig. 1 The two decompositions: $\Gamma' = l \cup \tau$ and $\Omega = \Omega_1 \cup \Omega_2$

Obviously,

$$\int_G \left|g(\zeta)\right|^2 dA(\zeta) \le \frac{\|P\|^2_{L^2(G)}}{(\mathrm{dist}(z, \Gamma))^2},$$

and the result (3.1) follows from (4.10) and the application of the Cauchy–Schwarz inequality to the integral in (4.9). □

5 Proofs for Piecewise Analytic Boundary

We recall our assumption that Γ consists of N analytic arcs, which meet at corner points z_j, $j = 1, \dots, N$, forming there exterior angles $\omega_j \pi$, with $0 < \omega_j < 2$.

The basic idea underlying the work in this section is simple. Extend, using Schwarz reflection, Φ across each arc of Γ inside G so that this extension is conformal in the exterior of a piecewise analytic Jordan curve Γ', which shares with Γ the same corners z_j and otherwise lies in G. Γ' can be chosen so that Φ is analytic on Γ', apart from z_j. Hence, the four representations (2.10), (2.11), (2.13), and (2.14) remain valid if Γ is deformed to Γ'. Next, divide Γ' into two parts: a part l containing arcs emanating from the corners z_j, and a part τ constituting the complement $\Gamma' \setminus l$, so that there exists a compact set $B := B(\Gamma)$ of G which contains τ. When $\zeta \in \tau$, $\Phi(\zeta)^n$ decays geometrically to zero, i.e., $|\Phi(\zeta)|^n = O(\rho^n)$, for some $\rho := \rho(\Gamma) < 1$, and therefore its own contribution is negligible when compared with the contribution of $\Phi(\zeta)^n$, for $\zeta \in l$. To make things more precise, we assume (as we may) that l is formed by linear segments, and we number these two segments meeting at z_j by l_j^i, $i = 1, 2$; see Fig. 1.

Subsequently, we make extensive use of the following four inequalities.

Remark 5.1 (Behavior of Φ near an analytic corner) For any $\zeta \in l_j^i$, the following hold:

(i) $|\Phi(\zeta) - \Phi(z_j)| \ge c\,|\zeta - z_j|^{1/\omega_j}$;
(ii) $|\Phi'(\zeta)| \le c\,|\zeta - z_j|^{1/\omega_j - 1}$;
(iii) $|\Phi(\zeta)| \le 1 - c\,|\zeta - z_j|^{1/\omega_j}$;
(iv) $\mathrm{dist}(\zeta, \Gamma) \ge c\,|\zeta - z_j|$.

(In Remark 5.1 and below, we use the symbol c generically in order to denote positive constants, possibly different ones, that depend on Γ only.)

The inequalities (i) and (ii) emerge from Lehman's asymptotic expansions for conformal mappings near an analytic corner [31]. The third inequality follows easily from (i) because reflection preserves angles. Finally, (iv) is a simple fact of conformal mapping geometry.

5.1 Proof of Theorem 2.3

The proof follows similar lines as those taken in [22] for deriving an estimate for $F_n(z)$ in G, with one significant difference, though. Here z lies in Ω rather than G, and thus z is allowed to tend to Γ without having to alter the curve Γ'. As a consequence, the set B defined above does not depend on z, and thus $\operatorname{dist}(z,\tau) \geq \operatorname{dist}(z, B) > \operatorname{dist}(\Gamma, B) = c(\Gamma)$.

The details are as follows: From the discussion above, it is easy to see that, for $z \in \Omega$,

$$\begin{aligned} H_n(z) &= \frac{1}{2\pi i}\int_{\Gamma'} \frac{\Phi^n(\zeta)\Phi'(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\sum_j \int_{l_j^1\cup l_j^2} \frac{\Phi^n(\zeta)\Phi'(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\tau} \frac{\Phi^n(\zeta)\Phi'(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\sum_j \int_{l_j^1\cup l_j^2} \frac{\Phi^n(\zeta)\Phi'(\zeta)}{\zeta - z}\,d\zeta + O\bigl(\rho^n\bigr) \end{aligned} \tag{5.1}$$

for some $\rho := \rho(\Gamma) < 1$, independent of z. Hence, we only need to estimate the integral

$$I_j^i := \int_{l_j^i} \frac{\Phi^n(\zeta)\Phi'(\zeta)}{\zeta - z}\,d\zeta.$$

Let s denote the arclength on l_j^i measured from z_j. Then, Remark 5.1 yields the following two inequalities, which hold for any $\zeta \in l_j^i$:

$$\bigl|\Phi(\zeta)\bigr| \leq 1 - cs^{1/\omega_j} < \exp\bigl(-cs^{1/\omega_j}\bigr) \quad \text{and} \quad \bigl|\Phi'(\zeta)\bigr| \leq cs^{1/\omega_j - 1}. \tag{5.2}$$

Since $1/\omega_j > 1/2$, these imply

$$\bigl|I_j^i\bigr| \leq \frac{c}{\operatorname{dist}(z,\Gamma')}\int_0^\infty \mathrm{e}^{-cns^{1/\omega_j}} s^{1/\omega_j - 1}\,ds = \frac{c\,\omega_j}{\operatorname{dist}(z,\Gamma')}\,\frac{1}{n}, \tag{5.3}$$

and the required estimate (2.33) follows from (5.1).

The next result is needed in establishing Theorem 2.4.

Lemma 5.1 *With* $\omega \in (0, 2]$ *and* $k \in \mathbb{N}$, *set* $\delta := k^{-\omega}$, *and let*

$$I(\omega, k) := \int_0^\delta \left[\int_r^\infty e^{-ks^{1/\omega}} s^{1/\omega - 2}\, ds \right]^2 r\, dr. \tag{5.4}$$

Then,

$$I(\omega, k) \le \frac{c}{k^2}. \tag{5.5}$$

(In the statement and proof of Lemma 5.1, the positive constants c depend on ω only.)

Proof We consider separately the four complementary cases: (I) $\omega = 1$, (II) $0 < \omega < 1$, (III) $\omega = 2$, and (IV) $1 < \omega < 2$.

Case (I): $\omega = 1$. Note that

$$I(1, k) = \int_0^\delta \left[\int_r^\infty \frac{e^{-ks}}{s}\, ds \right]^2 r\, dr = \int_0^\delta E_1^2(kr)\, r\, dr,$$

where $E_1(x)$ denotes the exponential integral $E_1(x) := \int_x^\infty t^{-1} e^{-t}\, dt$, with $x > 0$. Using the formula $\int_0^\infty E_1^2(x)\, dx = 2\log 2$, we thus have

$$I(1, k) \le \delta \int_0^\delta E_1^2(kr)\, dr < \frac{\delta}{k} \int_0^\infty E_1^2(x)\, dx = \frac{c}{k^2}.$$

Case (II): $0 < \omega < 1$. Now $1/\omega > 1$. Consequently, for $r > 0$,

$$\int_r^\infty e^{-ks^{1/\omega}} s^{1/\omega - 2}\, ds \le \int_0^\infty e^{-ks^{1/\omega}} s^{1/\omega - 2}\, ds = \omega\, \Gamma(1 - \omega)\, k^{\omega - 1},$$

where $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$ denotes the Gamma function with argument $x > 0$. This yields

$$I(\omega, k) \le c \frac{\delta^2}{k^{2(1-\omega)}} = \frac{c}{k^2}. \tag{5.6}$$

Case (III): $\omega = 2$. We note first the formula, valid for $r > 0$,

$$\int_r^\infty e^{-ks^{1/2}} s^{-3/2}\, ds = 2\big(e^{-k r^{1/2}} r^{-1/2} - k\, E_1\big(k\, r^{1/2}\big)\big).$$

Therefore,

$$I(2, k) < c \int_0^\infty e^{-2k\, r^{1/2}}\, dr + c\, k^2 \int_0^\infty E_1^2\big(k\, r^{1/2}\big)\, r\, dr = \frac{c}{k^2} + k^2 \frac{c}{k^4} = \frac{c}{k^2}.$$

Case (IV): $1 < \omega < 2$. The result for $1 < \omega < 2$ can be established as a special case of $\omega = 1$ and $\omega = 2$. To see this, set $h(\omega, s) := e^{-ks^{1/\omega}} s^{1/\omega - 2}$ and split the integral from r to ∞ in (5.4) into three parts:

$$\int_r^\infty h(\omega,s)\,ds = \int_r^\delta h(\omega,s)\,ds + \int_\delta^1 h(\omega,s)\,ds + \int_1^\infty h(\omega,s)\,ds.$$

Next, observe that if $s \in (0,\delta) \cup (1,\infty)$, then $h(\omega,s)$ is an increasing function of ω, hence $h(\omega,s) \le h(2,s)$. On the other hand, when $s \in (\delta,1)$, then $h(\omega,s)$ is a decreasing function of ω, thus $h(\omega,s) \le h(1,s)$.

Summing up, we therefore have

$$\int_r^\infty h(\omega,s)\,ds \le \int_r^\infty \mathrm{e}^{-ks^{1/2}} s^{-3/2}\,ds + \int_r^\infty \frac{\mathrm{e}^{-ks}}{s}\,ds,$$

and the result (5.5) follows easily using the estimates given in Cases (I) and (III). □

5.2 Proof of Theorem 2.4

We choose positive quantities

$$\delta_j = \delta_{n,j} := c\,n^{-\omega_j}, \quad j = 1,\dots,N, \tag{5.7}$$

where c is small enough so that any two of the N domains $\Omega_1^j := \{z \in \Omega : |z - z_j| < \delta_j\}$ are disjoint from each other. Next, we set $\Omega_1 := \bigcup_j^N \Omega_1^j$ and split Ω into two parts Ω_1 and Ω_2; see Fig. 1.

Using this partition of Ω, we express $\|H_n\|^2_{L^2(\Omega)}$ as the sum of two integrals over Ω_1 and Ω_2. This gives

$$\begin{aligned}\|H_n\|^2_{L^2(\Omega)} &= \int_{\Omega_1} \left|H_n(z)\right|^2 dA(z) + \frac{1}{(n+1)^2} \int_{\Omega_2} \left|E'_{n+1}(z)\right|^2 dA(z) \\ &=: J_1(n) + J_2(n),\end{aligned} \tag{5.8}$$

where we made use of (2.7). Hence, deriving the estimate (2.34), it now amounts to showing that: (a) $J_1(n) = O(1/n^2)$, and (b) $J_2(n) = O(1/n^2)$.

(a) Let

$$T_j(n) := \int_{\Omega_1^j} \left|H_n(z)\right|^2 dA(z), \quad j = 1,\dots,N,$$

so that

$$J_1(n) = \sum_{j=1}^N T_j(n). \tag{5.9}$$

With $z \in \Omega_1^j$, set $r := |z - z_j|$, and observe that, in view of Remark 5.1(iv), $|\zeta - z| \approx s + r$ if $\zeta \in l_j^1 \cup l_j^2$, while $|\zeta - z| \ge c$ if $\zeta \in l_k^1 \cup l_k^2$, with $k \ne j$, where

s denotes the arclength on l_j^i measured from z_j. Consequently, since $\Omega_1^j \subset \{z : |z - z_j| < \delta_j\}$, we obtain, using (5.1) and (5.2):

$$\begin{aligned}
T_j(n) &\le c \int_0^{\delta_j} \left[\int_0^\infty \frac{\mathrm{e}^{-cns^{1/\omega_j}} s^{1/\omega_j - 1}}{r+s}\, ds + \sum_{k \neq j} \int_0^\infty \mathrm{e}^{-cns^{1/\omega_k}} s^{1/\omega_k - 1}\, ds \right]^2 r\, dr \\
&\le c \int_0^{\delta_j} \left[\int_0^\infty \frac{\mathrm{e}^{-cns^{1/\omega_j}} s^{1/\omega_j - 1}}{r+s}\, ds \right]^2 r\, dr \\
&\quad + c \int_0^{\delta_j} \left[\sum_{k \neq j} \int_0^\infty \mathrm{e}^{-cns^{1/\omega_k}} s^{1/\omega_k - 1}\, ds \right]^2 r\, dr. \qquad (5.10)
\end{aligned}$$

Since

$$\int_0^\infty \mathrm{e}^{-cns^{1/\omega_k}} s^{1/\omega_k - 1}\, ds = \frac{\omega_k}{cn},$$

it follows from (5.7) that

$$\int_0^{\delta_j} \left[\sum_{k \neq j} \int_0^\infty \mathrm{e}^{-cns^{1/\omega_k}} s^{1/\omega_k - 1}\, ds \right]^2 r\, dr \le \frac{c}{n^2} \int_0^{\delta_j} r\, dr \le \frac{c}{n^{2(1+\omega_j)}}. \qquad (5.11)$$

Next by splitting the integral on $(0, \infty)$ into the two parts $(0, r)$ and (r, ∞), we get that

$$\begin{aligned}
\int_0^{\delta_j} \left[\int_0^\infty \frac{\mathrm{e}^{-cns^{1/\omega_j}} s^{1/\omega_j - 1}}{r+s}\, ds \right]^2 r\, dr &\le c \int_0^{\delta_j} \left[\int_0^r \frac{\mathrm{e}^{-cns^{1/\omega_j}} s^{1/\omega_j - 1}}{r}\, ds \right]^2 r\, dr \\
&\quad + c \int_0^{\delta_j} \left[\int_r^\infty \mathrm{e}^{-cns^{1/\omega_j}} s^{1/\omega_j - 2}\, ds \right]^2 r\, dr.
\end{aligned}$$

Now we use the estimate

$$\int_0^r \mathrm{e}^{-cns^{1/\omega_j}} s^{1/\omega_j - 1}\, ds = \frac{c}{n}\left(1 - \mathrm{e}^{-cnr^{1/\omega_j}}\right) < c\, r^{1/\omega_j}$$

and the result of Lemma 5.1 to deduce, respectively,

$$\int_0^{\delta_j} \left[\int_0^r \frac{\mathrm{e}^{-cns^{1/\omega_j}} s^{1/\omega_j - 1}}{r}\, ds \right]^2 r\, dr < c\, \delta_j^{2/\omega_j} = c\, \frac{1}{n^2} \qquad (5.12)$$

and

$$\int_0^{\delta_j} \left[\int_r^\infty \mathrm{e}^{-cns^{1/\omega_j}} s^{1/\omega_j - 2}\, ds \right]^2 r\, dr \le c\, \frac{1}{n^2}.$$

Summing up, we conclude from (5.10) that

$$T_j(n) \le c\,\frac{1}{n^2} \quad \text{for } j = 1, \dots, N,$$

which, in view of (5.9), leads to the required estimate

$$J_1(n) \le \frac{c}{n^2}. \tag{5.13}$$

(b) By using Cauchy's integral formula for the derivative in (2.11) and arguing as in Sect. 5.1, we obtain, for $z \in \Omega$, that

$$\begin{aligned} E'_{n+1}(z) &= \frac{1}{2\pi i} \int_{\Gamma'} \frac{\Phi^{n+1}(\zeta)}{(\zeta - z)^2}\, d\zeta \\ &= \frac{1}{2\pi i} \sum_j \int_{l_j^1 \cup l_j^2} \frac{\Phi^{n+1}(\zeta)}{(\zeta - z)^2}\, d\zeta + O\big(\rho^n\big), \end{aligned} \tag{5.14}$$

with $\rho := \rho(\Gamma) < 1$ independent of z.

Assume now that $z \in \Omega_2$ and $\zeta \in l_j^1 \cup l_j^2$, $j = 1, \dots, N$. Then, the triangle inequality and Remark 5.1(iv) imply that $|\zeta - z| \ge c\,|z - z_j|$. Thus, by using (5.2), we get

$$\left| \int_{l_j^1 \cup l_j^2} \frac{\Phi^{n+1}(\zeta)}{(\zeta - z)^2}\, d\zeta \right| \le \frac{c}{|z - z_j|^2} \int_0^\infty \mathrm{e}^{-cns^{1/\omega_j}}\, ds = \frac{c\,\Gamma(\omega_j)}{|z - z_j|^2} \frac{1}{n^{\omega_j}}.$$

This, in conjunction with (5.14), leads to the estimate

$$\int_{\Omega_2} \big|E'_{n+1}(z)\big|^2\, dA(z) \le c \sum_j \frac{1}{n^{2\omega_j}} \int_{\Omega_2} \frac{dA(z)}{|z - z_j|^4}.$$

Finally, since $\Omega_2 \subset \{z : |z - z_j| \ge \delta_j\}$, we have from (5.7) that

$$\begin{aligned} \int_{\Omega_2} \big|E'_{n+1}(z)\big|^2\, dA(z) &\le c \sum_j \frac{1}{n^{2\omega_j}} \int_{|z - z_j| > \delta_j} \frac{dA(z)}{|z - z_j|^4} \\ &= c \sum_j \frac{1}{n^{2\omega_j}\, \delta_j^2} = c, \end{aligned} \tag{5.15}$$

and this, in view of the definition of $J_2(n)$ in (5.8), yields the required estimate

$$J_2(n) \le \frac{c}{n^2}. \tag{5.16}$$

□

Remark 5.2 It is interesting to note that the choice for δ_j given by (5.7) keeps the estimates (5.13) and (5.16) in balance in the sense that any other choice for δ_j will result in a weaker estimate for the decay of $\|H_n\|_{L^2(\Omega)}$, as a comparison of (5.6) and (5.12) with (5.15) shows.

6 Proof of the Main Theorems

Proof of Theorem 1.1 The assumption of the theorem implies that Γ is quasiconformal and rectifiable. Hence, by comparing (1.4) with (2.24), we see that

$$\alpha_n = \beta_n + \varepsilon_n, \tag{6.1}$$

and the result (1.5) emerges immediately in view of Theorem 2.1 and Corollary 2.1. □

Proof of Theorem 1.2 Theorem 1.1 implies that

$$\frac{\lambda_n}{\gamma^{n+1}} = \sqrt{\frac{n+1}{\pi}}\{1+\xi_n\}, \quad n \in \mathbb{N}, \tag{6.2}$$

where

$$0 \le \xi_n \le c_1(\Gamma)\,\frac{1}{n}. \tag{6.3}$$

Therefore, from (2.19), we have, for $z \in \Omega$, that

$$p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\{1+\xi_n\}\left\{1 + \frac{H_n(z)}{\Phi^n(z)\Phi'(z)} - \frac{q_{n-1}(z)}{\Phi^n(z)\Phi'(z)}\right\},$$

which, in comparison with (1.6), gives the following explicit expression for the error $A_n(z)$:

$$A_n(z) = \xi_n + \{1+\xi_n\}\frac{1}{\Phi'(z)}\left\{\frac{H_n(z)}{\Phi^n(z)} - \frac{q_{n-1}(z)}{\Phi^n(z)}\right\}. \tag{6.4}$$

The required result (1.7) then emerges by using the estimates (6.3), (2.33), and (3.6), for ξ_n, $H_n(z)$ and $q_{n-1}(z)$, respectively. □

Proof of Theorem 1.3 The proof follows immediately from (6.1) and Theorem 2.2, since $\beta_n \ge 0$. □

7 Applications

Strong asymptotics for orthogonal polynomials with respect to measures supported on the real line have played a crucial role in the development of the theory of orthogonal polynomials in $\mathbb{R}$. In order to argue that this would be the case for Bergman polynomials as well, we present briefly a number of applications based on the strong asymptotics of Sect. 1 and the associated theory developed in Sects. 2–5.

7.1 Zeros of the Bergman Polynomials

A well-known result of Fejer asserts that *all the zeros of* $\{p_n(z)\}_{n\in\mathbb{N}}$ *are contained on the convex hull* $\mathrm{Co}(\overline{G})$ of $\overline{G}$. This was refined by Saff [42] to the interior of $\mathrm{Co}(\overline{G})$. To these should be added a result of Widom [53] to the effect that, *on any closed subset* B *of* $\Omega \cap \mathrm{Co}(\overline{G})$ *and for any* $n \in \mathbb{N}$, *the number of zeros of* $p_n(z)$ *on* B *is bounded independently of* n. This of course doesn't preclude the possibility that if $B \neq \emptyset$, then $p_n(z)$ has a zero on B for every $n \in \mathbb{N}$. The next theorem, which is a simple consequence of Theorem 1.2, shows that, under an additional assumption on Γ, the zeros of the sequence $\{p_n(z)\}_{n\in\mathbb{N}}$ cannot be accumulated in Ω.

Theorem 7.1 *Assume that Γ is piecewise analytic without cusps. Then, for any closed set $B \subset \Omega$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $p_n(z)$ has no zeros on B.*

7.2 Weak Asymptotics

The important class **Reg** of measures of orthogonality was introduced by Stahl and Totik in [47, Def. 3.1.2]. Since the area measure dA on G belongs to **Reg**, it follows that

$$\lim_{n\to\infty} \left|p_n(z)\right|^{1/n} = \left|\Phi(z)\right| \tag{7.1}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \mathrm{Co}(\overline{G})$; see [47, Theorem 3.1.1(ii)]. The next theorem shows how this result can be made more precise under an additional assumption on the boundary.

Theorem 7.2 *Assume that Γ is piecewise analytic without cusps. Then,*

$$\lim_{n\to\infty} \left|p_n(z)\right|^{1/n} = \left|\Phi(z)\right|$$

locally uniformly in Ω.

Proof The proof follows at once after utilizing Theorem 7.1 in [44, Theorem III.4.7]. □

For an account on weak asymptotics for Bergman polynomials defined by a system of disjoint Jordan curves, we refer to [25, Prop. 3.1].

7.3 Ratio Asymptotics

The following two corollaries are simple consequences of Theorems 1.1 and 1.2.

Corollary 7.1 *Assume that Γ is piecewise analytic without cusps. Then, for any $n \in \mathbb{N}$, it holds that*

$$\sqrt{\frac{n+1}{n+2}}\,\frac{\lambda_{n+1}}{\lambda_n} = \gamma + \varsigma_n, \tag{7.2}$$

where

$$|\varsigma_n| \leq c_1(\Gamma)\,\frac{1}{n}. \tag{7.3}$$

Corollary 7.2 *Under the assumptions of Corollary 7.1, for any $z \in \Omega$ and sufficiently large $n \in \mathbb{N}$, it holds that*

$$\sqrt{\frac{n+1}{n+2}}\,\frac{p_{n+1}(z)}{p_n(z)} = \Phi(z)\bigl\{1 + B_n(z)\bigr\}, \tag{7.4}$$

where

$$\left|B_n(z)\right| \leq \frac{c_2(\Gamma)}{\operatorname{dist}(z,\Gamma)|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_3(\Gamma)\,\frac{1}{n}. \tag{7.5}$$

Remark 7.1 The ratio asymptotics above are derived as a consequence of Theorems 1.1 and 1.2. Thus, we are obliged to assume that Γ is piecewise analytic without cusps. Based, however, on substantial numerical evidence (an instance is shown in Table 3 below), we believe that the ratio asymptotics hold, in the sense that $\varsigma_n = o(1)$ and $B_n(z) = o(1)$, under weaker assumptions on Γ.

7.4 Stability of the Arnoldi GS for Polynomials

Let μ be a (nontrivial) finite Borel measure supported on a compact (and infinite) subset K of the complex plane, and let $\{p_n(z,\mu)\}_{n=0}^{\infty}$ denote the associated sequence of orthonormal polynomials

$$p_n(z,\mu) := \lambda_n(\mu)z^n + \cdots, \quad \lambda_n(\mu) > 0,\ n = 0,1,2,\ldots,$$

generated by the inner product

$$\langle f, g\rangle_\mu := \int f(z)\overline{g(z)}\,d\mu(z).$$

A standard way to construct the sequence $\{p_n(z,\mu)\}_{n=0}^{\infty}$, even to prove its existence and uniqueness, is by using the Gram–Schmidt (GS) process. This process is designed to turn, in iterative fashion, any polynomial sequence $\{P_n\}_{n=0}^{\infty}$ into an orthonormal sequence. The main ingredients in the computation are the complex moments $\langle z^m, z^k\rangle_\mu$. The conventional way to apply the GS process is by choosing the monomials as the starting up sequence, that is, by setting $P_n(z) = z^n$. Indeed, this was suggested (see, e.g., [26, §18.3–18.4]) and was eventually used (see, e.g., [37]) by people working in numerical conformal mapping, where the need for constructing orthonormal polynomials arises from the application of the Bergman kernel method and its variants.

By the *Arnoldi* GS, we mean the application of the GS process in the following way: At the k-step, where the orthonormal polynomial p_k is to be constructed, use the polynomials $\{p_0, p_1, \ldots, p_{k-1}, zp_{k-1}\}$ rather than the monomials as the starting up sequence.

Regarding the stability properties of the Arnoldi GS, we note that it is not difficult to show that

$$1 \leq I_n \leq \|z\|_K \frac{\lambda_n^2(\mu)}{\lambda_{n-1}^2(\mu)} \tag{7.6}$$

for the instability indicator

$$I_n := \frac{\|P_n\|_{L^2(\mu)}^2}{\min_{P\in\operatorname{span}(S_{n-1})}\|P_n - P\|_{L^2(\mu)}^2}, \quad n \in \mathbb{N}, \tag{7.7}$$

introduced by Taylor in [52] for the purpose of measuring the instability of the application of the GS process in orthonormalizing the set of polynomials $S_n := \{P_0, P_1, \dots, P_n\}$. Note that $I_n = 1$ if S_n is already an orthonormal set, while $I_n = \infty$ if S_n is linearly dependent.

In view of Corollary 7.1, the estimate (7.6) implies that the Arnoldi GS process for computing the Bergman polynomials of G is stable in the sense that the instability indicator I_n does not increase (in fact remains uniformly bounded) with n. This is in sharp contrast to the conventional GS, where I_n *increases geometrically fast* with n. More specifically, the following estimate for the conventional GS was derived in [37, Theorem 3.1]:

$$c_4(\Gamma)L^{2n} \le I_n \le c_5(\Gamma)L^{2n}, \tag{7.8}$$

where $L := \|z\|_\Gamma/\text{cap}(\Gamma)$. Note that $L > 1$ unless G is a disk centered at the origin, where $L = 1$. For a comprehensive account of the damaging effects of the conventional GS process to the computation of Bergman polynomials, we refer to [37].

It is interesting to note that although Arnoldi's original paper [5] appeared in 1951, and the Arnoldi implementation of the GS process has been used in numerical linear algebra since then, we first encountered its implementation in connection with the computation of orthogonal polynomials much later in [23], where it was proposed for the computation of Szegő polynomials without reference, however, to its stability properties.

7.5 Computation of $\Phi(z)$ and cap(Γ)

Since cap$(\Gamma) = b = 1/\gamma$, Corollary 7.1 provides the means for computing approximations to the capacity of Γ by using only the leading coefficients of the Bergman polynomials. Similarly, Corollary 7.2 suggests a simple numerical method for computing approximations to the conformal map $\Phi(z)$. This is quite appealing in the sense that the Bergman polynomials, alone, suffice to provide approximations to both interior conformal map $G \to \mathbb{D}$ (via the well-known Bergman kernel method) and exterior conformal map $\Omega \to \Delta$ associated with the same Jordan curve. We refer to [34] for the current state of the convergence theory of the Bergman kernel method. Regarding the exterior map, we propose here the following approximation algorithm.

`Approximation of Capacities and Exterior Conformal Maps`

1. Compute the complex moments

$$\mu_{m,k} := \langle z^m, z^k \rangle_G = \int_G z^m \overline{z}^k \, dA(z), \quad m, k = 0, 1, \dots, n. \tag{7.9}$$

2. Employ the Arnoldi GS process to construct the Bergman polynomials $\{p_k\}_{k=0}^n$ using the moments $\mu_{m,k}$.

Table 2 Square: The approximation $b^{(n)}$ of $\mathrm{cap}(\Pi_4) = 0.834\,626\,841\cdots$

n	$b^{(n)}$	t_n	s
100	0.834 640 612	1.37e-05	–
110	0.834 638 233	1.14e-05	1.9902
120	0.834 636 420	9.58e-06	1.9911
130	0.834 635 009	8.16e-06	1.9918
140	0.834 633 888	7.04e-06	1.9924
150	0.834 632 982	6.14e-06	1.9930
160	0.834 632 341	5.39e-06	1.9934
170	0.834 631 626	4.78e-06	1.9938
180	0.834 631 111	4.26e-06	1.9942
190	0.834 630 674	3.83e-06	1.9945
200	0.834 630 301	3.46e-06	1.9949

3. Set

$$b^{(n)} := \sqrt{\frac{n+1}{n}}\frac{\lambda_{n-1}}{\lambda_n} \quad \text{and} \quad \Phi_n(z) := \sqrt{\frac{n}{n+1}}\frac{p_n(z)}{p_{n-1}(z)}. \tag{7.10}$$

4. Approximate $\mathrm{cap}(\Gamma)$ by $b^{(n)}$ and $\Phi(z)$ by $\Phi_n(z)$.

We demonstrate the performance of the above algorithm in the computation of capacities only. We do so by presenting numerical results for two examples: (a) the canonical square with boundary Π_4, discussed in Sect. 7.8 below, and (b) the 3-cusped hypocycloid with boundary H_3, defined by (1.14) with $m = 3$. We note that H_3 does not satisfy the requirements of Corollary 7.1. The capacity of Π_4 is given explicitly in (7.23), while clearly, $\mathrm{cap}(H_3) = 1$. In both cases, the complex moments are known explicitly. The details of the presentation are as follows:

Let t_n denote the error in approximating the capacity, i.e.,

$$t_n := b^{(n)} - \mathrm{cap}(\Gamma). \tag{7.11}$$

Since $\mathrm{cap}(\Gamma) = b$, it follows from Corollary 7.1 that

$$|t_n| \le c(\Gamma)\frac{1}{n}, \quad n \in \mathbb{N}. \tag{7.12}$$

In Tables 2 and 3, we report the computed values of $b^{(n)}$ and t_n, with n varying from 100 to 400. We also report the values of the parameter s, which is designed to test the hypothesis that $|t_n| \approx 1/n^s$. All computations presented in this paper were carried out on a desktop PC using the computing environment MAPLE in high precision. Thus, in view of the stability properties of the Arnoldi GS process discussed in Sect. 7.4, we expect all the figures quoted in the tables to be correct.

The numbers listed on the tables show that the proposed algorithm constitutes a valid method for computing capacities. It is interesting to note that in both cases

Table 3 Hypocycloid: The approximation $b^{(n)}$ of $\operatorname{cap}(\Pi_4) = 1$

n	$b^{(n)}$	t_n	s
300	1.000 117 809	1.17e-04	–
310	1.000 112 347	1.12e-04	1.447
320	1.000 107 296	1.07e-04	1.448
330	1.000 102 615	1.02e-04	1.449
340	1.000 098 267	9.82e-04	1.449
350	1.000 094 219	9.42e-05	1.450
360	1.000 090 443	9.04e-05	1.451
370	1.000 086 914	8.69e-05	1.452
380	1.000 083 610	8.36e-05	1.453
390	1.000 080 511	8.05e-05	1.454
400	1.000 077 600	7.76e-05	1.455

the presented values of $b^{(n)}$ decay monotonically to the capacity. Also, the values of the parameter s indicate clearly that for the case of the square, $|t_n| \approx 1/n^2$. This behavior can be explained if $\alpha_n \approx 1/n$ for the strong asymptotic error of the leading coefficient. For the case of the cusped hypocycloid however, no safe conclusions can be drawn for the behavior of t_n from the reported values on Table 3.

Based on the important applications of the ratio asymptotics outlined above (see also Sect. 7.6), we think that the solution of the following problem will be of significance in developing further the theory of orthogonal polynomials in the complex plane.

Problem 7.1 *Characterize all the measures of orthogonality μ, with* $\operatorname{supp}(\mu) = K$, *for which it holds that*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}(\mu)}{\lambda_n(\mu)} = \frac{1}{\operatorname{cap}(K)}. \tag{7.13}$$

Since the property $\mu \in \mathbf{Reg}$ is equivalent to

$$\lim_{n\to\infty} \lambda_n^{1/n}(\mu) = \frac{1}{\operatorname{cap}(K)} \tag{7.14}$$

see [47, Theorem 3.1.1], it follows that the measures satisfying (7.13) form a subclass of **Reg**. We note, however, that there are known instances where the limit points of the sequence $\{\lambda_{n+1}(\mu)/\lambda_n(\mu)\}_{n\in\mathbb{N}}$ constitute a finite set, as in the case of Bergman polynomials defined on a system of disjoint symmetric lemniscates (see [25, §7]), or where they fill up a whole interval, as in the case of Szegő polynomials defined on a system of disjoint smooth Jordan curves (see [54, Theorem 9.2]).

7.6 Finite Recurrence Relations and Dirichlet Problems

Definition 7.1 We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy an $(M+1)$-*term recurrence relation* if, for any $n \geq M-1$,

$$zp_n(z) = a_{n+1,n}p_{n+1}(z) + a_{n,n}p_n(z) + \cdots + a_{n-M+1,n}p_{n-M+1}(z).$$

A direct application of the ratio asymptotics for $\{p_n\}_{n\in\mathbb{N}}$, given by Corollary 7.2, leads to the next two theorems. These refine, respectively, Theorems 2.2 and 2.1 of [30] in the sense that they weaken the C^2-smoothness assumption on Γ. For their proof, it is sufficient to note that: (a) the two theorems are equivalent to each other, and (b) the reason for assuming that Γ is C^2-smooth in Theorem 2.2 of [30] was to ensure the ratio asymptotics of the Bergman polynomials; see [30, §4 Rem. (i)].

Theorem 7.3 *Assume that Γ is piecewise analytic without cusps. If the Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy an $(M+1)$-term recurrence relation, with some $M \geq 2$, then $M = 2$ and Γ is an ellipse.*

Theorem 7.4 *Let G be a bounded simply-connected domain with Jordan boundary Γ, which is piecewise analytic without cusps. Assume that there exists a positive integer $M := M(G)$ with the property that the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } G, \\ u = \overline{z}^m z^n & \text{on } \Gamma, \end{cases} \tag{7.15}$$

has a polynomial solution of degree $\leq m(M-1)+n$ in z and of degree $\leq n(M-1)+m$ in $\overline{z}$, for all positive integers m and n. Then Γ is an ellipse and $M = 2$.

Theorem 7.4 confirms a special case of the so-called Khavinson and Shapiro conjecture; see [29] for results reporting on the recent progress in this direction. We note that the equivalence between the two properties "the Bergman polynomials of G satisfy a finite-term recurrence relation" and "any Dirichlet problem in G, with polynomial data, possesses a polynomial solution" was first established in [40].

7.7 Shape Recovery from Partial Measurements

Given a finite $n+1 \times n+1$ section

$$[\mu_{m,k}]_{m,k=0}^{n}, \quad \mu_{m,k} := \int_G z^m \overline{z}^k \, dA(z) \tag{7.16}$$

of the infinite complex moment matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$, associated with a bounded Jordan domain G, the *truncated moments problem* consists of computing an approximation Γ_n to its boundary Γ by using only the data provided by (7.16). Regarding existence and uniqueness, we note a result of Davis and Pollak [13] stating that the infinite matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$ defines uniquely the curve Γ. Corollary 7.2 and the discussion in Sect. 7.4 regarding the stability of the Arnoldi GS process suggest the following algorithm:

`Reconstruction from Moments Algorithm`

1. Use the Arnoldi GS process to construct the Bergman polynomials $\{p_k\}_{k=0}^{n}$ from the given complex moments μ_{mk}, $m,k=0,1,\dots,n$.
2. Compute the coefficients of the Laurent series expansion of the ratio

$$\Phi_n(z) := \sqrt{\frac{n}{n+1}}\frac{p_n(z)}{p_{n-1}(z)} = \gamma^{(n)}z + \gamma_0^{(n)} + \frac{\gamma_1^{(n)}}{z} + \frac{\gamma_2^{(n)}}{z^2} + \cdots. \tag{7.17}$$

3. Revert the series (7.17) using the explicit method described in [17, p. 764]. This leads to:

$$b^{(n)} := 1/\gamma^{(n)} = \sqrt{\frac{n+1}{n}}\frac{\lambda_{n-1}}{\lambda_n}, \qquad b_0^{(n)} := -b^{(n)}\gamma_0^{(n)}/\gamma^{(n)}$$

   and

$$\Psi_n(w) := b^{(n)}w + b_0^{(n)} + \frac{b_1^{(n)}}{w} + \frac{b_2^{(n)}}{w^2} + \frac{b_3^{(n)}}{w^3} + \cdots + \frac{b_n^{(n)}}{w^n},$$

   where $-k\,b_k^{(n)}/b^{(n)}$, $k=1,2,\dots,n$, is the coefficient of $1/z$ in the Laurent series expansion of $[\Phi_n(z)/\gamma^{(n)}]^k$ about infinity.
4. Approximate Γ by $\Gamma_n := \{z : z = \Psi_n(e^{it}),\ t \in [0, 2\pi]\}$.

For applications to the 2D image reconstruction arising from tomographic data, we refer to [25]. Here we highlight the performance of the reconstruction algorithm by applying it to the recovery of three shapes, where the defining curves come from different classes: one analytic, one with corners and one with cusps, for which the theory of Sect. 7.3 does not apply. In each case, we start by computing a finite set of complex moments and then follow the four steps of the algorithm. We note that in all three examples the complex moments are known explicitly.

In Figs. 2–4, we depict the computed approximation Γ_n against the original curve Γ. Note that in the first two plots, the fitting of the two curves is not far from being perfect. Even in the cusped case, pictured in Fig. 4, the fitting is remarkably close despite the low degree of the moment matrix used.

In Fig. 2, we illustrate the reconstruction of an ellipse by using only the first 16 moments in (7.16), i.e., by taking $n=3$.

In Fig. 3, we reconstruct a square by using the complex moments up to the degree 16. We have chosen $n=16$ so that the result can be compared with the recovery of a square, as shown on page 1067 of [24] obtained using the *exponential transform algorithm* of the opus cited. This is another reconstruction algorithm based on moments. Of course, for concluding results regarding the comparison of the two algorithms, more experiments need to be conducted.

In order to show that our reconstruction algorithm works equally well for domains where the theory above does not apply, we use it for the recovery of the boundary

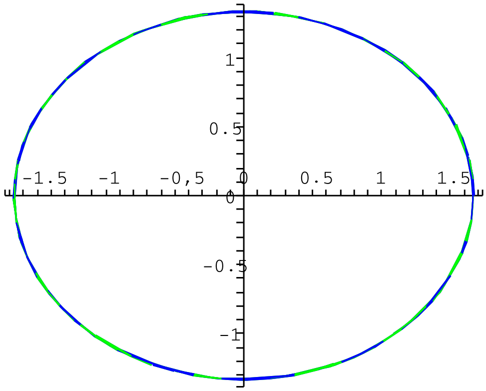

Fig. 2 Recovery of an ellipse, with $n = 3$

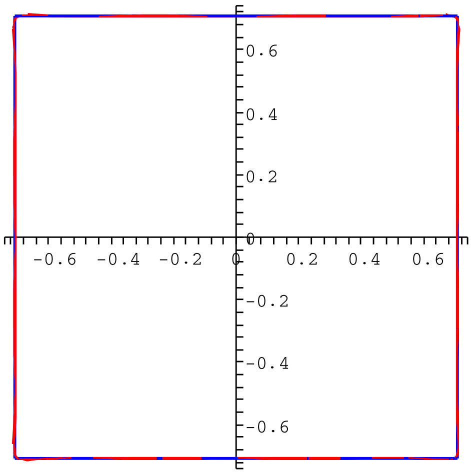

Fig. 3 Recovery of a square, with $n = 16$

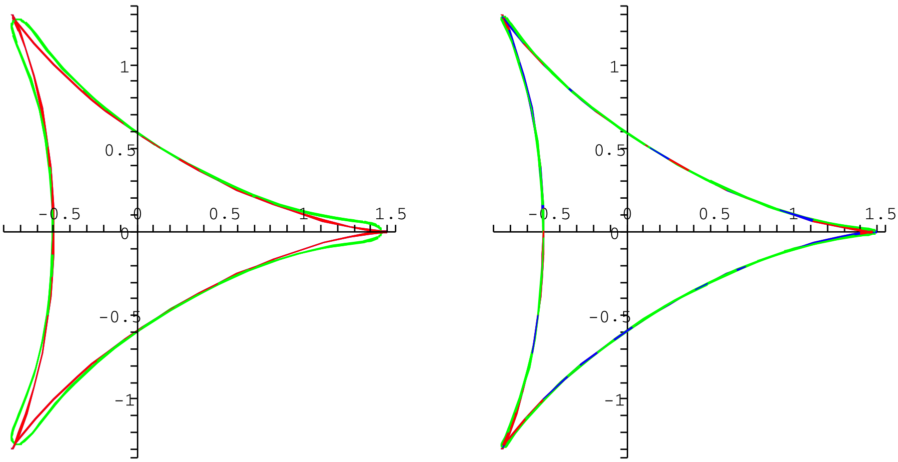

Fig. 4 Recovery of a 3-cusped hypocycloid, with $n = 10$ (*left*) and $n = 20$ (*right*)

H_3 of the 3-cusped hypocycloid defined by (1.14) with $m = 3$. The application of the algorithm with $n = 10$ and $n = 20$ is depicted in Fig. 4.

Concluding, we note that the above algorithm is not suited for reconstructing unions of disjoint Jordan domains, in contrast to the *archipelagos reconstruction algorithm* of [25]. On the other hand, the simplicity of the construction and the proximity of the two curves Γ_n and Γ shown in the figures support that the proposed algorithm is more efficient when it comes to recovering single Jordan domains.

7.8 Coefficient Estimates

We recall the expansion (1.3) of the inverse conformal mapping $\Psi : \Delta \to \Omega$ and note that $\Psi(w)/b$ belongs to the well-known class Σ of univalent functions; see, e.g., [38] and [16].

The following result settles, in a certain sense, the associated coefficient problem for an important subclass of Σ. We refer to [16, §4.9] for a comprehensive discussion of the coefficient problem for other subclasses of Σ.

Theorem 7.5 *Assume that Γ is piecewise analytic without cusps, and let $\omega\pi$, $0 < \omega < 2$, denote its smallest exterior angle. Then, it holds that*

$$|b_n| \le c_1(\Gamma)\frac{1}{n^{1+\omega}}, \quad n \in \mathbb{N}, \tag{7.18}$$

and the order $1 + \omega$ of $1/n$ is sharp in the sense that for certain ω, there exists a Jordan curve Γ of the same class such that

$$|b_n| \ge c_2(\Gamma)\frac{1}{n^{1+\omega}} \quad \textit{for infinitely many } n. \tag{7.19}$$

Proof The estimate (7.18) can be established by means of the tools developed in Sect. 5. More precisely, the following array of equations can be readily verified by using (4.7) and arguing as in Sect. 5.1:

$$\begin{aligned} -nb_n &= \frac{1}{2\pi i}\int_{L_R}\Phi^n(z)\,dz = \frac{1}{2\pi i}\int_{\Gamma'}\Phi^n(z)\,dz \\ &= \frac{1}{2\pi i}\sum_j\int_{l_j^1\cup l_j^2}\Phi^n(\zeta)\,d\zeta + \frac{1}{2\pi i}\int_\tau\Phi^n(\zeta)\,d\zeta \\ &= \frac{1}{2\pi i}\sum_j\int_{l_j^1\cup l_j^2}\Phi^n(\zeta)\,d\zeta + O\big(\rho^n\big) \end{aligned} \tag{7.20}$$

for some $\rho := \rho(\Gamma) < 1$.

Hence, we only need to estimate the integral

$$I_j^i := \int_{l_j^i}\Phi^n(\zeta)\,d\zeta.$$

This can be done by working as in deriving (5.3). Indeed, by using the estimate

$$\left|\Phi(\zeta)\right| \le 1 - cs^{1/\omega_j} < \exp\left(-cs^{1/\omega_j}\right), \quad \zeta \in l_j^i,$$

we obtain

$$\left|I_j^i\right| \le c \int_0^\infty \mathrm{e}^{-cns^{1/\omega_j}}\, ds = c\,\omega_j \Gamma(\omega_j)\, \frac{1}{n^{\omega_j}},$$

and the required result (7.18) follows from (7.20), with $\omega := \min_j\{\omega_j\}$.

An extremal domain, where (7.19) holds true, is provided by the case where Γ is the canonical square Π_4, with vertices at $1, i, -1$, and $-i$. In this case, $\omega = 3/2$, and by making use of the rotational symmetry of Π_4, it is easily seen that the Schwarz–Christoffel formula for the normalized conformal map $\Psi : \Delta \to \Omega$ takes the following expression:

$$\begin{aligned}\Psi(w) &= \operatorname{cap}(\Pi_4) \int \left(1 - \frac{1}{w}\right)^{\omega-1} \left(1 - \frac{i}{w}\right)^{\omega-1} \left(1 + \frac{1}{w}\right)^{\omega-1} \left(1 + \frac{i}{w}\right)^{\omega-1} dw \\ &= \operatorname{cap}(\Pi_4) \int \left(1 - \frac{1}{w^4}\right)^{\omega-1} dw,\end{aligned}$$

or, more explicitly,

$$\Psi(w) = \operatorname{cap}(\Pi_4) \left\{ w + \sum_{k=1}^{\infty} (-1)^{k+1} \binom{a}{k} \frac{1}{4k-1} \frac{1}{w^{4k-1}} \right\},$$

where $a := \omega - 1 = 1/2$ and $\binom{a}{k}$ denotes the binomial coefficient. Hence, for $n = 4k - l$, $k \in \mathbb{N}$, and $l \in \{0, 1, 2, 3\}$, we have

$$b_n = \begin{cases} \operatorname{cap}(\Pi_4)(-1)^{k+1}\binom{a}{k}\frac{1}{n}, & \text{if } l = 1, \\ 0, & \text{if } l \ne 1. \end{cases} \tag{7.21}$$

Now, using the properties of the Gamma function $\Gamma(z)$, it is easy to verify that

$$\binom{a}{k} = \frac{(-1)^k}{\Gamma(-a)} \frac{\Gamma(k-a)}{\Gamma(k+1)} = \frac{(-1)^k}{\Gamma(-a)} \left\{ \frac{1}{k^{1+a}} + O\left(\frac{1}{k^{2+a}}\right) \right\},$$

and this, in conjunction with (7.21), provides the required behavior

$$|b_n| \asymp \frac{1}{n^{1+\omega}} \tag{7.22}$$

for $n = 3, 7, 11, \ldots$.

Clearly, the above argument applies to any canonical polygon Π_m with m-sides. In particular, (7.22) holds true for any Π_m, $m \ge 3$, with $\omega = (m+2)/m$ and $n = km - 1$, $k \in \mathbb{N}$. Thus, any Π_m can serve as an extremal curve for the estimate (7.18). □

We note that, since $\Psi(1)=1$, it is not difficult to obtain the following expression for the capacity of Π_4 using the properties of hypergeometric functions:

$$\operatorname{cap}(\Pi_4)=\frac{\Gamma^2(1/4)}{4\pi^{3/2}}=0.834\,626\,841\,674\,072\cdots. \tag{7.23}$$

Remark 7.2 In the case where Γ is allowed to have cusps, we recall, from Sect. 1, the following estimate of Gaier [21, §4.1]:

$$|b_n|\le c(\Gamma)\frac{1}{n},\quad n\in\mathbb{N}.$$

This shows that the arguments on the first part of the proof of the theorem can be amended to cover the case of zero exterior angles but not of angles of opening 2π.

7.9 A Connection with Operator Theory

In a different reading, Theorem 7.5 brings in a connection with operator theory. To show this, we consider the Toeplitz matrix T_Ψ defined by the continuous extension of $\Psi(w)$ to the unit circle $\mathbb{T}:=\{w:|w|=1\}$. By this we mean the matrix

$$T_\Psi:=\begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & \cdots\\ b & b_0 & b_1 & b_2 & b_3 & \cdots\\ 0 & b & b_0 & b_1 & b_2 & \cdots\\ 0 & 0 & b & b_0 & b_1 & \cdots\\ 0 & 0 & 0 & b & b_0 & \cdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \tag{7.24}$$

defined by the coefficients of $\Psi(w)$ in its Laurent series expansion (1.3). If the boundary Γ is piecewise analytic without cusps, then Theorem 7.5 implies that $\sum_{n=0}^{\infty}|b_n|<\infty$, and hence that the symbol Ψ of the Toeplitz matrix T_Ψ belongs to the Wiener algebra; see, e.g., [10, §1.2–1.5]. This property leads to very interesting conclusions, for instance, to the conclusion that T_Ψ defines a bounded linear operator on the Hilbert space l^2 and that

$$\sigma_{\text{ess}}(T_\Psi)=\Gamma, \tag{7.25}$$

where we use $\sigma_{\text{ess}}(L)$ to denote the *essential spectrum* of a bounded linear operator L.

Consider next the multiplication by z operator $\mathcal{M}:f\to zf$ (also known as the *Bergman shift operator*) defined on the Hilbert space $L^2_a(G)$. We note that $\mathcal{M}$ is a bounded linear operator on $L^2_a(G)$ such that

$$\sigma_{\text{ess}}(\mathcal{M})=\Gamma;$$

see [7]. Hence, from (7.25), it follows that

$$\sigma_{\text{ess}}(\mathcal{M})=\sigma_{\text{ess}}(T_\Psi). \tag{7.26}$$

In a forthcoming paper [43], we employ results and tools from the present work to show that the connection between the two operators $\mathcal{M}$ and T_Ψ is much more substantial.

In order to emphasize the importance of the Bergman shift operator $\mathcal{M}$ in the theory of orthogonal polynomials, we note that the proof of Theorem 7.3 relies heavily on the properties of $\mathcal{M}$; see [40] and [30]. Furthermore, we note that the stable Arnoldi GS process is based on the use of the polynomial zp_{n-1}, i.e., on the application of $\mathcal{M}$ to p_{n-1}.

7.10 The Decay of the Bergman Polynomials in G

Here we refine the following estimate, which was derived in [35, p. 530] under the assumption that Γ is piecewise analytic without cusps: For any compact subset B of G and for any $n \in \mathbb{N}$, it holds that

$$\left|p_n(z)\right| \leq c_1(\Gamma, B)\, \frac{1}{n^s}, \quad z \in B, \tag{7.27}$$

where

$$s := \min_{1 \leq j \leq N} \left\{\omega_j/(2-\omega_j)\right\}. \tag{7.28}$$

(We use $c_j(\Gamma, B)$ to denote positive constants that depend only on Γ and B.) Note that $s \to 0$ if $\omega_j \to 0$, for some j, and hence for such cases, (7.27) predicts a very slow decay for $p_n(z)$. The next theorem, however, shows that this decay cannot be slower than $O(1/\sqrt{n})$.

Theorem 7.6 *Assume that Γ is piecewise analytic without cusps. Then, for any $n \in \mathbb{N}$, it holds that*

$$\left|p_n(z)\right| \leq c_2(\Gamma, B)\, \frac{1}{n^\sigma}, \quad z \in B, \tag{7.29}$$

where $\sigma := \max\{1/2, s\}$.

Proof By using Cauchy's formula for the derivative in (2.10) and by working as in the proof of Theorem 7.5, it is readily seen that

$$\left|F'_{n+1}(z)\right| \leq c_3(\Gamma, B)\, \frac{1}{n^\omega}, \quad z \in B, \tag{7.30}$$

where $\omega\pi$ $(0 < \omega < 2)$ is the smallest exterior angle of Γ. This, in view of (2.7), gives immediately

$$\left|G_n(z)\right| \leq c_4(\Gamma, B)\, \frac{1}{n^{1+\omega}}, \quad z \in B. \tag{7.31}$$

Next, since

$$\left|q_{n-1}(z)\right| \leq \frac{\|q_{n-1}\|_{L^2(G)}}{\sqrt{\pi}\,\mathrm{dist}(z, \Gamma)}, \quad z \in G,$$

see, e.g., [20, p. 4], we obtain from Corollary 2.1 the estimate

$$\left|q_{n-1}(z)\right| \le c_5(\Gamma, B)\,\frac{1}{n}, \quad z \in B. \tag{7.32}$$

Finally, from (2.18), we have that

$$\left|p_n(z)\right| \le \frac{\lambda_n}{\gamma^{n+1}}\left\{\left|G_n(z)\right| + \left|q_{n-1}(z)\right|\right\},$$

and this in view of (6.2) and (6.3) and (7.31) and (7.32) yields

$$\left|p_n(z)\right| \le c_6(\Gamma, B)\,\frac{1}{n^{1/2}}, \quad z \in B. \tag{7.33}$$

The result of the theorem follows by combining (7.27) with (7.33). □

Remark 7.3 Regarding sharpness of the exponent σ of n in (7.29), we recall the following result of [35, p. 531]: "If not all interior angles of Γ are of the form π/m, $m \in \mathbb{N}$, and if we disregard in the definition of s in (7.28) angles of this form, should they exist then for any $\varepsilon > 0$, there is a subsequence $\mathcal{N}_\varepsilon \subset \mathbb{N}$, such that for any $n \in \mathcal{N}_\varepsilon$:

$$\left|p_n(z)\right| \ge c_7(\Gamma, B)\,\frac{1}{n^{s+1/2+\varepsilon}}, \quad z \in B."$$

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