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ON A DOMAIN DECOMPOSITION METHOD FOR THE COMPUTATION OF CONFORMAL MODULES

N. PAPAMICHAEL and S. N. STYLIANOPOULOS

Department of Mathematics and Statistics, Brunel University

Abstract. We consider a domain decomposition method for approximating the conformal modules of a certain class of long quadrilaterals.

1. INTRODUCTION

Let Ω be a Jordan domain in the complex z-plane ($z = x + iy$), and consider a system consisting of Ω and four points z_j; $j = 1, 2, 3, 4$, in counter-clockwise order on its boundary $\partial\Omega$. Such a system is said to be a quadrilateral

$$Q := \{\Omega; z_1, z_2, z_3, z_4\}. \tag{1.1}$$

the conformal module $m(Q)$ of Q is defined as follows:
Let R_H denote a rectangle of the form

$$R_H = \{(\xi, \eta) : 0 < \xi < 1,\ 0 < \eta < H\}, \tag{1.2}$$

in the w-plane ($w = \xi + i\eta$). Then, $m(Q)$ is the unique value of H for which Q is conformally equivalent to R_H. That is, for $H = m(Q)$ and for this value only there exists a unique conformal map

$$f : \Omega \to R_H = R_{m(Q)}, \tag{1.2a}$$

which takes the four points z_j; $j = 1, 2, 3, 4$, rexpectively onto the four corners of the rectangle, i.e. f is such that

$$f(z_1) = 0, f(z_2) = 1, f(z_3) = 1 + im(Q) \textit{ and } f(z_4) = im(Q). \tag{1.2b}$$

The conformal map (1.2) has many practical applications, and in these the value of $m(Q)$ is often of special significance; see e.g. [2], [4:§ 16.11] and [7].
Now let Ω be a domain bounded by the two straight lines $x = 0$ and $x = 1$, and two Jordan arcs with cartesian equations $y = -\tau_1(x)$ and $y = \tau_2(x)$, where τ_j; $j = 1, 2$, are positive in [0,1], and let z_j; $1, 2, 3, 4$, be the four corners of Ω. That is, let

$$Q = \{\Omega;\ z_1, z_2, z_3, z_4\}, \tag{1.3a}$$

where

$$\Omega = \{(x, y) : 0 < x < 1,\ -\tau_1(x) < y < \tau_2(x)\}, \tag{1.3b}$$

and

$$z_1 = -i\tau_1(0),\ z_2 = 1 - i\tau_1(1),\ z_3 = 1 + i\tau_2(1),\ z_4 = i\tau_2(0). \tag{1.3c}$$

Also, let

$$\Omega_1 = \{(x, y) : 0 < x < 1,\ -\tau_1(x) < y < 0\}, \tag{1.4a}$$

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and

$$\Omega_2 = \{(x,y) : 0 < x < 1,\ 0 < y < \tau_2(x)\}, \tag{1.4b}$$

so that $\bar\Omega = \bar\Omega_1, \cup \bar\Omega_2$, and let

$$Q_1 = \{\Omega_1; z_1, z_2, 1, 0\} \textit{ and } Q_2 = \{\Omega_2; 0, 1, z_3, z_4\} \tag{1.4c}$$

Then, as is shown in [4: p. 437], $m(Q) \geq m(Q_1) + m(Q_2)$, and equality occurs only in the two trivial cases whre: (a) Ω is a rectangle, and (b) $\tau_1(x) = \tau_2(x), x \in [0,1]$.
The purpose of this note is to consider the problem of approximating $m(Q)$ by the sum $m(Q_1) + m(Q_2)$ and, in particular, to give an estimate of $m(Q) - (m(Q_1) + m(Q_2))$. In other words, we are concerned with a method for approximating $m(Q)$, based on decomposing the original quadrilateral (1.3) into the two quadrilaterals given by (1.4). Such a method is of practical interest, mainly, because it can reduce considerably the "crowding" difficulties associated with the computation of the conformal modules of "long" quadrilaterals of the form (1.3); see [5].

2. ESTIMATE OF $m(Q) - (m(Q_1) + m(Q_2))$

Let Q and Q_j; $j = 1, 2$, denote the quadrilaterals defined by (1.3)- (1.4), and assume that the functions τ_j; $j = 1, 2$, in (1.3b) are absolutely continuous in [0,1] and

$$d_j := \underset{0\leq x\leq 1}{\text{ess}} \sup |\tau_j^1(x)| < \infty;\ j = 1, 2. \tag{2.1}$$

Also, let

$$m_j := \max_{0\leq x\leq 1} \{\exp(-\pi\tau_j(x))\}\,;\ j = 1, 2, \tag{2.2}$$

and

$$d = \max(d_1, d_2),\ m = \max(m_1, m_2),\ h = \min\{m(Q_1), m((Q_2)\}\,. \tag{2.3}$$

Then, our main result can be stated as follows.

THEOREM 2.1. *if*

$$\epsilon := d\{(1 + m^2)/(1 - m^2)\} < 1, \tag{2.4}$$

then

$$m(Q) - (m(Q_1) + m(Q_2)) < A(\epsilon)e^{-2\pi h}(2 + e^{-2\pi h}), \tag{2.5a}$$

where

$$A(\epsilon) = (\frac{8}{3})^{1/2}\epsilon^2 / \{(1 - \epsilon)(1 - \epsilon^2)^{1/2}\}\,. \tag{2.5b}$$

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REMARKS

2.1. Our method of proof makes extensive use of the theory of the Theodersen-Garrick method, and involves expressing the problem $f : \Omega \rightarrow R_{m(Q)}$ as an equivalent problem for the conformal map of a doubly-connected domain onto a circular annulus; see [1: pp. 194-206] and [3]. The detailed proof will be given in a subsequent paper [6], together with the analysis of the decomposition method for determining the full conformal map f.

2.2. The theorem shows that for "large" h,

$$m(Q) - (m(Q_1) + m(Q_2)) \sim e^{-2\pi h}. \tag{2.6}$$

That, is if Q is a "long" quadrilateral then $m(Q)$ can be approximated closely by $m(Q_1) + m(Q_2)$.

2.3. Although the bound (2.5) is in terms of $h = \min\{m(Q_1),\ m(Q_2)\}$, it is in general very easy to obtain crude estimates of $m(Q_j)$; $j = 1, 2$. For example,

$$\min_{0 \leq x \leq 1} \tau_j(x) < m(Q_j) < \max_{0 \leq x \leq 1} \tau_j(x); j = 1, 2.$$

2.4. The condition (2.4) is needed for our method of proof. However, the results of several numerical experiments suggest that (2.6) holds even when the condition (2.4) is violated; see [6].

2.5. Considerable simplifications occur in the case where $\tau_1(x) = c > 0, x \in [0,1]$, i.e. where Ω_1 is a rectangle of height c. In this special case corresponding to Theorem 2.1 we have the following:

THEOREM 2.2. *Let $\tau_1(x) = c > 0,\ x \in [0,1]$, so that $m(Q_1) = c$, and let d_2 and m_2 be defined by (2.1) and (2.2). If*

$$\epsilon := d_2 \left\{(1 + m_2^2)/(1 - m_2^2)\right\} < 1, \tag{2.7}$$

then

$$(MQ) - (c + m(Q_2)) < A(\epsilon) e^{-2\pi m(Q_2)}(1 + e^{-2\pi c}), \tag{2.8a}$$

where

$$A(\epsilon) = (\frac{2}{3})^{1/2} \epsilon^2 / \left\{(1 - \epsilon)(1 - \epsilon^2)^{1/2}\right\}. \ \blacksquare \tag{2.8b}$$

3. EXAMPLE

Let Q and Q_j; $j = 1, 2$, be defined by (1.3)-(1.4) with

$$\tau_1(x) = 1 + 0.2\, sech^2\,(2.5x) \text{ and } \tau_2(x) = 1 + 0.5x.$$

Then, by using the algorithms of [3] we find that

$$m(Q) = 2.251\,94, m(Q_1) = 1.065\,49 \text{ and } m(Q_2) = 1.186\,28.$$

Thus,

$$m(Q) - (m(Q_1) + m(Q_2)) = 1.7 \times 10^{-4},$$

whilst (2.5) with $h = m(Q_1)$ gives

$$m(Q) - (m(Q_1) + m(Q_2)) < 2.4 \times 10^{-3}.$$

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Department of Mathematics and Statistics, Brunel University, Uxbridge, Middx UB8 3PH, U. K.