Bergman polynomials and Bergman operators

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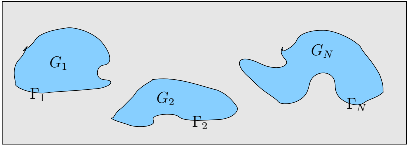
Bergman polynomials $\{p_n\}$ on an archipelago G



Γ_j, $j = 1, \ldots, N$, a system of disjoint and mutually exterior Jordan curves in $\mathbb{C}$, $\boxed{G_j := \operatorname{int}(\Gamma_j)}$, $\boxed{\Gamma := \cup_{j=1}^N \Gamma_j}$, $\boxed{G := \cup_{j=1}^N G_j}$.

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}.$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

Construction of p_n's

Algorithm: Conventional Gram-Schmidt (GS)

Apply the Gram-Schmidt process to the monomials

$$1, z, z^2, z^3, \ldots$$

Main ingredient: the moments

$$\mu_{m,k} := \langle z^m, z^k \rangle = \int_G z^m \overline{z}^k dA(z), \quad m, k = 0, 1, \ldots.$$

The above algorithm has been been suggested by pioneers of Numerical Conformal Mapping (like P. Davis and D. Gaier and P. Henrici) in the 1960's as the standard procedure for constructing Bergman polynomials. It was subsequently used by researchers in this area in the 1980's. It has been even employed in the numerical conformal mapping FORTRAN package `BKMPACK` of Warby.

Instability Indicator

The GS method is notorious for its instability. For measuring it, when orthonormalizing a system $S_n := \{u_0, u_1, \ldots, u_n\}$ of functions, the following instability indicator has been proposed by J.M. Taylor, (Proc. R.S. Edin., 1978):

$$\boxed{I_n := \frac{\|u_n\|^2_{L^2(G)}}{\min_{u \in \mathrm{span}(S_{n-1})} \|u_n - u\|^2_{L^2(G)}}, \quad n \in \mathbb{N}.}$$

Note that, when S_n is an orthonormal system, then $\boxed{I_n = 1}$. When S_n is linearly dependent then $\boxed{I_n = \infty}$. Also, if $G_n := [\langle u_m, u_k \rangle]^n_{m,k=0}$, denotes the Gram matrix associated with S_n then,

$$\boxed{\kappa_2(G_n) \geq I_n},$$

where $\kappa_2(G_n) := \|G_n\|_2 \|G_n^{-1}\|_2$ is the spectral condition number of G_n.

Instability of the Conventional GS process

In the single-component case $N = 1$, consider the monomial basis $S_n = \{1, z, z^2, \ldots, z^n\}$. Then, for the conventional GS process we have the following result:

Theorem (Papamichael & Warby, Numer. Math., 1986)

Assume that the curve Γ is piecewise-analytic without cusps and let

$$\boxed{L := \|z\|_{L^\infty(\Gamma)}/\text{cap}(\Gamma)} \quad (\geq 1),$$

where $\text{cap}(\Gamma)$ *denotes the logarithmic capacity of* Γ. *Then,*

$$\boxed{c_1(\Gamma)\, L^{2n} \leq I_n \leq\ c_2(\Gamma)\, L^{2n}}.$$

Note that $L = 1$, iff $G \equiv \mathbb{D}_r$ and that I_n is sensitive to the relative position of G w.r.t. the origin. When G is the 8×2 rectangle centered at the origin, then $L = 3/\sqrt{2} \approx 2.12$. In this case, $I_{25} \asymp 10^{16}$ and the method breaks down in MATLAB or FORTRAN, for $n = 25$.

The Arnoldi algorithm in Numerical Linear Algebra

Let $A \in \mathbb{C}^{m,m}$, $b \in \mathbb{C}^m$ and consider the Krylov subspace

$$K_k := \text{span}\{b, Ab, A^2 b \ldots, A^{k-1} b\}.$$

The Arnoldi algorithm produces an orthonormal basis $\{v_1, v_2, \ldots, v_k\}$ of K_k as follows:

W. Arnoldi (Quart. Appl. Math., 1951)

At the n-th step, apply GS to orthonormalize the vector Av_{n-1} (instead of $A^{n-1}b$) against the (already computed) orthonormal vectors $\{v_1, v_2, \ldots, v_{n-1}\}$.

The Arnoldi algorithm for OP's

Let μ be a (non-trivial) finite Borel measure with compact support $\boxed{\text{supp}(\mu)}$ on $\mathbb{C}$ and consider the series of orthonormal polynomials

$$p_n(z,\mu) := \lambda_n(\mu)z^n + \cdots, \quad \lambda_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

generated by the inner product

$$\langle f, g\rangle_\mu = \int f(z)\overline{g(z)}d\mu(z).$$

Arnoldi GS for Orthonormal Polynomials

At the n-th step, apply GS to orthonormalize the polynomial zp_{n-1} (instead of z^n) against the (already computed) orthonormal polynomials $\{p_0, p_1, \ldots, p_{n-1}\}$.

Used by Gragg & Reichel, in Linear Algebra Appl. (1987), for the construction of Szegö polynomials.

Stability of the Arnoldi GS

In the case of the Arnoldi GS, the instability indicator is given by:

$$I_n = \frac{\|zp_{n-1}\|^2_{L^2(G)}}{\min_{p\in\mathbb{P}_{n-1}} \|zp_{n-1} - p\|^2_{L^2(G)}}, \quad n \in \mathbb{N}.$$

Theorem

It holds,

$$1 \le I_n \le \|z\|_{L^\infty(\mathrm{supp}(\mu))} \frac{\lambda^2_{n-1}(\mu)}{\lambda^2_n(\mu)}, \quad n \in \mathbb{N}.$$

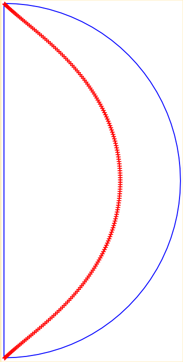
Typically: When $d\mu \equiv |dz|$ (Szegö polynomials), or $d\mu \equiv dA$ (Bergman polynomials), then

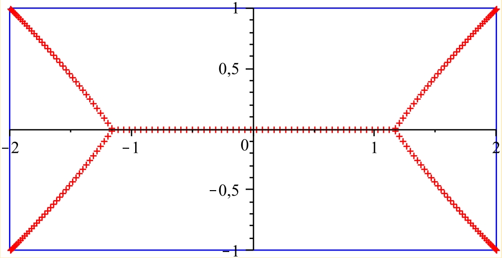
$$\boxed{c_1(\Gamma) \le \frac{\lambda_{n-1}(\mu)}{\lambda_n(\mu)} \le c_2(\Gamma)}, \quad n \in \mathbb{N}.$$

When $d\mu \equiv w(x)dx$ on $[a, b] \subset \mathbb{R}$, this ratio tends to a constant.

Half-disk and rectangle

Zeros of the Bergman polynomial p_{200}.

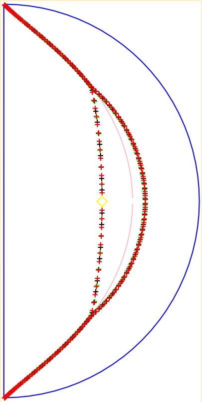


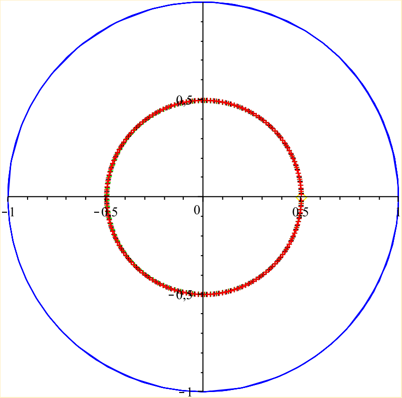


Theory (left) in: Levin, Saff & St, Constr. Approx., 2003.
Theory (right) in: Mina-Diaz, Saff & St, CMFT, 2005.

Weighted Bergman polys: $d\mu(z) = |z - \frac{1}{2}|^2 dA(z)$

Zeros of the weighted Bergman polynomials p_{100}, p_{150} and p_{200}.

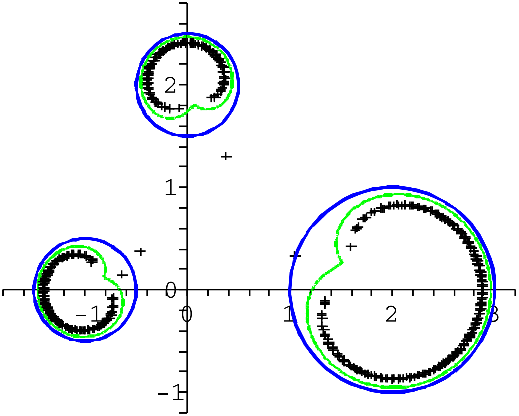




Theory in: Mina-Diaz, Saff & St, CMFT, 2005.

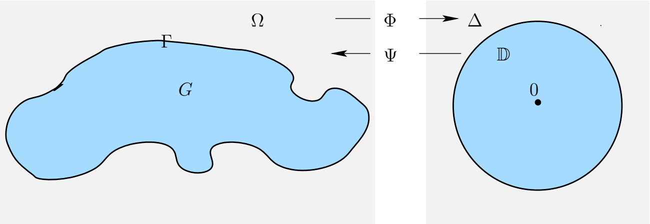
Three-disk archipelago

Zeros of the Bergman polynomials p_{140}, p_{150} p_{160}.



Theory in: Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

Asymptotics: Single-component case $N = 1$



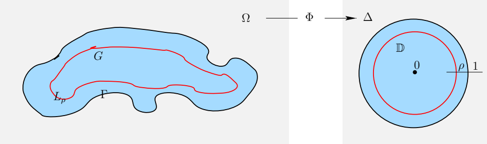
$$\boxed{\Omega := \overline{\mathbb{C}} \setminus \overline{G}}$$

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \boxed{\text{cap}(\Gamma) = 1/\gamma}$$

The Bergman polynomials of G:

$$p_n(z) = \lambda_n z^n + \cdots , \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

Strong asymptotics when Γ is analytic



T. Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\text{ext}(L_\rho)$, then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\rho^{2n},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}}, \qquad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\sqrt{n}\,\rho^n, \qquad z \in \overline{\Omega}.$$

Strong asymptotics when Γ is smooth

We say that $\Gamma \in C(p,\alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if Γ is given by $z = g(s)$, where s is the arclength, with $g^{(p)} \in \mathrm{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \mathrm{Lip}\alpha$.

P.K. Suetin, Proc. Steklov Inst. Math. AMS (1974)

Assume that $\Gamma \in C(p+1,\alpha)$, with $p+\alpha > 1/2$. Then, for $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\frac{1}{n^{2(p+\alpha)}},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}},$$

where

$$|A_n(z)| \le c_2(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$

Strong asymptotics for Γ non-smooth

Theorem (St, C. R. Acad. Sci. Paris, 2010)

Assume that Γ is piecewise analytic without cusps. Then, for $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \textit{where} \quad 0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n},$$

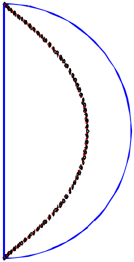
and for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c_1(\Gamma)}{\mathrm{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

Numerical example: Half-disk



$$\gamma = \frac{1}{\text{cap}(\Gamma)} = \frac{3\sqrt{3}}{4}$$

We compute, by using the Arnoldi GS process (in finite precision), the Bergman polynomials $p_n(z)$ for the unit half-disk, for n up to 60 and test the hypothesis

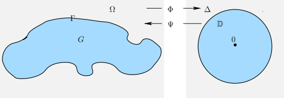
$$\alpha_n := 1 - \frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} \approx C\,\frac{1}{n^s}.$$

Numerical example: Half-disk

n	α_n	s
51	0.003 263 458 678	-
52	0.003 200 769 764	0.998 887
53	0.003 140 444 435	0.998 899
54	0.003 082 351 464	0.998 911
55	0.003 026 369 160	0.998 923
56	0.002 972 384 524	0.998 934
57	0.002 920 292 482	0.998 946
58	0.002 869 952 027	0.998 957
59	0.002 821 401 485	0.998 968
60	0.002 774 426 207	0.998 979

The numbers indicate clearly that $\alpha_n \approx C\,\frac{1}{n}$. Accordingly, we have made conjectures regarding strong asymptotics in Oberwolfach Reports (2004) and ETNA (2006).

A lower bound for α_n - Coefficient estimates



Let Ψ denote the inverse map $\Psi := \Phi^{[-1]} : \{w : |w| > 1\} \to \Omega$, i.e.,

$$\Psi(z) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots . \qquad \boxed{b = \mathrm{cap}(\Gamma)}$$

Theorem (St, arXiv, Sep 2009)

Assume that Γ is quasiconformal and rectifiable. Then, for any $n \in \mathbb{N}$,

$$\alpha_n \geq \frac{\pi(1-k^2)}{A(G)} (n+1) |b_{n+1}|^2 .$$

This provides a connection with the problem of estimating coefficients in Univalent Functions Theory. In particular, it implies that if $\{\alpha_n\}$ decays geometrically, then the curve Γ is analytic.

Ratio asymptotics for λ_n

Corollary (St, C. R. Acad. Sci. Paris, 2010)

$$\boxed{\sqrt{\frac{n+1}{n}}\frac{\lambda_{n-1}}{\lambda_n} = \mathrm{cap}(\Gamma) + \xi_n}, \quad \textit{where} \quad |\xi_n| \le c(\Gamma)\frac{1}{n}, \quad n \in \mathbb{N}.$$

The above relation provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the Bergman polynomials. In addition, it implies:

Corollary

$$\boxed{c_1(\Gamma) \le I_n \le c_2(\Gamma)}, \quad n \in \mathbb{N}.$$

Hence, the Arnoldi GS for Bergman polynomials, in the single component case, is stable.

Ratio asymptotics for $p_n(z)$

Corollary (St, C. R. Acad. Sci. Paris, 2010)

For any $z \in \Omega$, and sufficiently large $n \in \mathbb{N}$,

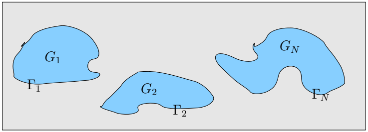
$$\boxed{\sqrt{\frac{n}{n+1}}\,\frac{p_n(z)}{p_{n-1}(z)} = \Phi(z)\{1 + B_n(z)\}},$$

where

$$|B_n(z)| \le \frac{c_1(\Gamma)}{\sqrt{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

The above relation provides the means for computing approximations to the conformal map Φ. This leads to an efficient algorithm for recovering the shape of G, from a finite collection of its power moments $\langle z^m, z^k\rangle_{m,k=0}^{n}$. This method was actually commented as unsuitable by P. Henrici, in *Computational Complex Analysis, Vol. III (1986)*, because of the instability of the Conventional GS.

Leading coefficients in archipelago



Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic, $j = 1, 2, \ldots, N$. Then, for $n \in \mathbb{N}$,

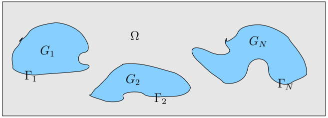
$$c_1(\Gamma)\sqrt{\frac{n+1}{\pi}}\,\frac{1}{\mathrm{cap}(\Gamma)^{n+1}} \leq \lambda_n \leq c_2(\Gamma)\sqrt{\frac{n+1}{\pi}}\,\frac{1}{\mathrm{cap}(\Gamma)^{n+1}}.$$

Corollary

$$\boxed{c_3(\Gamma) \leq I_n \leq c_4(\Gamma)}, \quad n \in \mathbb{N}.$$

Hence, the Arnoldi GS, for Bergman polynomials on an archipelago, is stable.

Bergman polynomials in archipelago



Let $g_\Omega(z,\infty)$ denote the Green function of $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ with pole at ∞.

Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic. Then, for $n \in \mathbb{N}$:

(i) *There exists a positive constant C, so that*

$$|p_n(z)| \leq \frac{C}{\text{dist}(z,\Gamma)}\sqrt{n}\exp\{n\, g_\Omega(z,\infty)\}, \quad z \notin \overline{G}.$$

(ii) *For every $\epsilon > 0$ there exist a constant $C_\epsilon > 0$, such that*

$$|p_n(z)| \geq C_\epsilon\sqrt{n}\exp\{n\, g_\Omega(z,\infty)\}, \quad \text{dist}(z, \text{Co}(\overline{G})) \geq \epsilon.$$

Discovery of a single island (case $N = 1$)

Truncated Moments Problem

Given the finite $n+1 \times n+1$ section

$$[\mu_{m,k}]_{m,k=0}^{n}, \quad \mu_{m,k} := \int_G z^m \overline{z}^k dA(z),$$

of the infinite complex moment matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$ associated with a bounded Jordan domain G, compute a good approximation to its boundary Γ.

Theorem (Davis & Pollak, Trans. AMS, 1956)

The infinite matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$ *defines uniquely* Γ*.*

Discovery of a single island

Island Recovery Algorithm

(I) Use the Arnoldi GS to compute $p_0, p_1, \dots, p_n$.

(II) Compute the coefficients of the Laurent series of the ratio

$$\sqrt{\frac{n}{n+1}}\frac{p_n(z)}{p_{n-1}(z)} = \gamma^{(n)}z + \gamma_0^{(n)} + \frac{\gamma_1^{(n)}}{z} + \frac{\gamma_2^{(n)}}{z^2} + \frac{\gamma_3^{(n)}}{z^3} + \cdots . \quad (1)$$

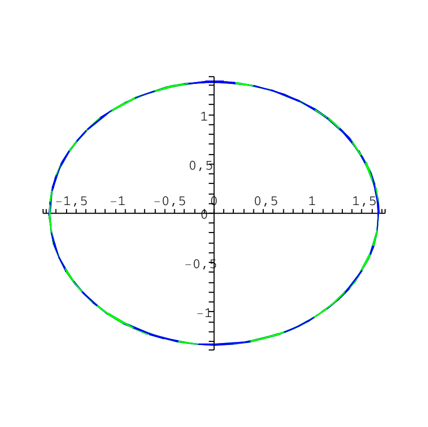
(III) Revert (1) and truncate to obtain

$$\Psi_n(w) := b^{(n)}w + b_0^{(n)} + \frac{b_1^{(n)}}{w} + \frac{b_2^{(n)}}{w^2} + \frac{b_3^{(n)}}{w^3} + \cdots + \frac{b_n^{(n)}}{w^n}.$$

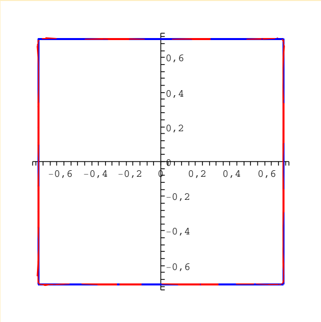
(IV) Approximate Γ by $\widetilde{\Gamma} := \{z : z = \Psi_n(e^{it}),\ t \in [0, 2\pi]\}$.

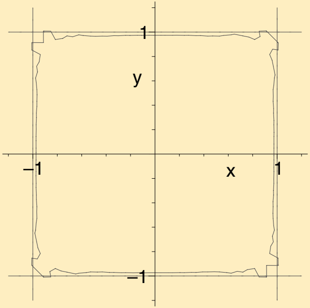
Numerical Examples

Recovery of the canonical ellipse, with $n = 3$.



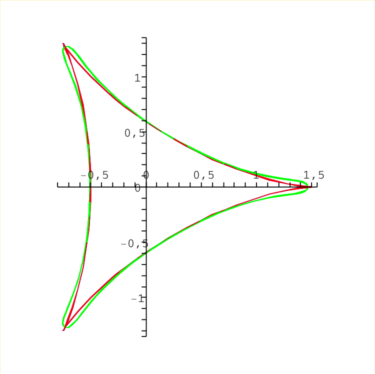
Recovery of the square, with $n = 16$.

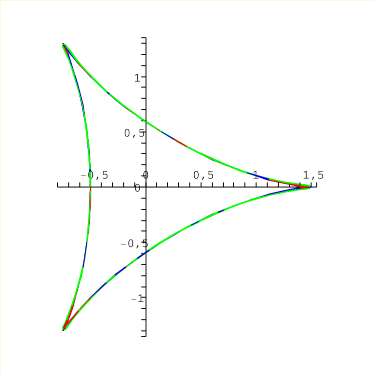




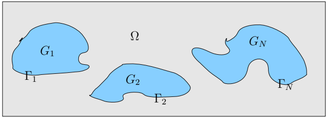
Comparison: The exponential transform algorithm of Gustafsson, He, Milanfar & Putinar, Inverse Problems (2000).

Recovery of the 3-cusped hypocycloid, with $n = 10$ and $n = 20$.





Discovery of an archipelago



$$\boxed{G_j := \text{int}(\Gamma_j)}, \quad \boxed{\Gamma := \cup_{j=1}^{N} \Gamma_j}, \quad \boxed{G := \cup_{j=1}^{N} G_j}.$$

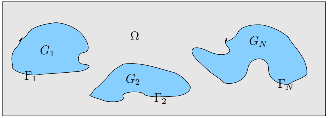
Truncated moments problem

Starting with the finite $n+1 \times n+1$ section

$$[\mu_{m,k}]_{m,k=0}^{n}, \quad \mu_{m,k} := \int_G z^m \overline{z}^k \, dA(z),$$

of the associated infinite complex moment matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$, compute a good approximation to G.

Discovery of an archipelago



Archipelago Recovery Algorithm
Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

(I) Use the Arnoldi GS to compute $p_0, p_1, \dots, p_n$.

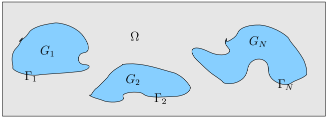
(II) Form the square root of the Christoffel function

$$\Lambda_n(z) := \frac{1}{\sqrt{\sum_{k=0}^{n} |p_k(z)|^2}}.$$

(III) Plot the zeros of p_j, $j = 1, 2, \dots, n$.

(IV) Plot the level curves of the function $\Lambda_n(x + iy)$, on a suitable rectangular frame for (x, y) that surrounds the plotted zero set.

Theoretical support of the recovery algorithm



Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic and let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$. Then,

$$\Lambda_n(z) \asymp \operatorname{dist}(z,\Gamma), \quad z \in G, \quad n \to \infty$$

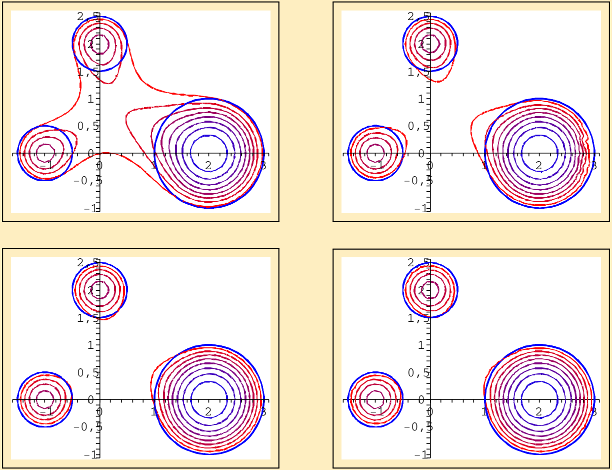
$$\Lambda_n(z) \asymp \frac{1}{n}, \quad z \in \Gamma, \quad n \to \infty$$

$$\Lambda_n(z) \asymp \frac{1}{\sqrt{n}} \exp\{-n\, g_\Omega(z,\infty)\}, \quad z \in \Omega, \quad n \to \infty.$$

where $g_\Omega(z,\infty)$ denotes the Green function of Ω with pole at infinity.

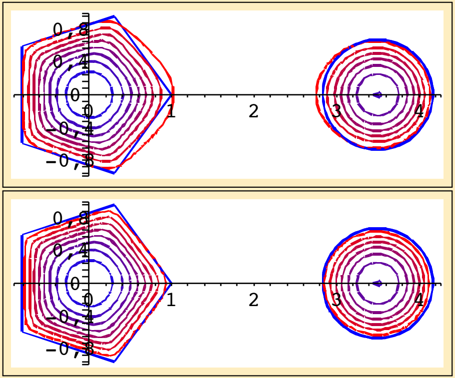
Note: $\boxed{g_\Omega(z,\infty) > 0, \quad z \in \Omega}$.

Recovery of three disks



Level lines of $\Lambda_n(x+iy)$ on $\{(x,y) : -1 \leq x \leq 4, -2 \leq y \leq 2\}$, for $n = 25, 50, 75, 100$.

Recovery of pentagon and disk



Level lines of $\Lambda_n(x+iy)$ on $\{(x,y) : -2 \leq x \leq 5, -2 \leq y \leq 2\}$, for $n = 25, 50$.

Only ellipses carry finite-term recurrences for p_n

Definition

We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy an $m+1$-term recurrence relation, if for any $n \geq m-1$,

$$zp_n(z) = a_{n+1,n}p_{n+1}(z) + a_{n,n}p_n(z) + \ldots + a_{n-m+1,n}p_{n-m+1}(z).$$

Theorem (St, C. R. Acad. Sci. Paris, 2010)

Assume that:

(i) $\Gamma = \partial G$ *is piecewise analytic without cusps.*

(ii) *The Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy an* $m+1$*-term recurrence relation, with some* $m \geq 2$*.*

Then $m = 2$ *and* Γ *is an ellipse.*

The above theorem refines some deep results of Putinar & St (CAOT, 2007) and Khavinson & St (Springer, 2010).

Connection with Operator Theory

For the rest we assume now that G is a bounded Jordan domain with $\Gamma := \partial G$.

$L^2_a(G)$: the Bergman space of

square integrable and analytic functions in G.

The Bergman (Shift) Operator $M_z : L^2_a(G) \to L^2_a(G)$

$$M_z f = zf.$$

Quiz

How many times did the Bergman Operator appear above?

The upper Hessenberg matrix $\mathcal{M}$

The Bergman operator M_z has the following upper Hessenberg matrix representation with respect to the Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G:

$$\mathcal{M} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & \cdots \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ 0 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\ 0 & 0 & a_{32} & a_{33} & a_{34} & a_{35} & \cdots \\ 0 & 0 & 0 & a_{43} & a_{44} & a_{45} & \cdots \\ 0 & 0 & 0 & 0 & a_{54} & a_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

where $\boxed{a_{k,n} = \langle zp_n, p_k \rangle}$ are the Fourier coefficients of $M_z p_n = zp_n$.

Note

The eigenvalues of the $n \times n$ principal submatrix $\mathcal{M}_n$ of $\mathcal{M}$ coincide with the zeros of p_n.

Banded Hessenberg matrices for OP's are Jacobi

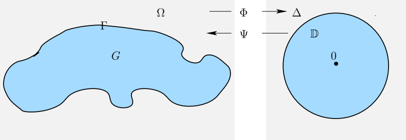
In the Numerical Linear Algebra jargon the finite-term recurrence theorem reads as follows:

Theorem

If the upper Hessenberg matrix $\mathcal{M}$ is banded, with constant bandwidth ≥ 3, then it is tridiagonal, i.e., a Jacobi matrix.

This result should put an end to the long search in Numerical Linear Algebra, for practical semi-iterative methods (aka polynomial iteration methods) based on short-term recurrence relations of orthogonal polynomials.

The inverse conformal map Ψ



Recall that

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots ,$$

and let $\Psi := \Phi^{-1} : \{w : |w| > 1\} \to \Omega$, denote the inverse conformal map. Then,

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots , \quad |w| \geq 1,$$

where

$$b = \text{cap}(\Gamma) = 1/\gamma.$$

The Toeplitz matrix with (continuous) symbol ψ

$$
T(\psi) = \begin{bmatrix}
b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\
b & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \cdots \\
0 & b & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\
0 & 0 & b & b_0 & b_1 & b_2 & b_3 & \cdots \\
0 & 0 & 0 & b & b_0 & b_1 & b_2 & \cdots \\
0 & 0 & 0 & 0 & b & b_0 & b_1 & \cdots \\
0 & 0 & 0 & 0 & 0 & b & b_0 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix}.
$$

Note

The eigenvalues of the $n \times n$ principal submatrix T_n of $T(\psi)$ coincide with the zeros of G_n, the 2nd kind Faber polynomial of degree n of G.

Comparison of spectra

For the upper Hessenberg matrix $\mathcal{M}$ we have $\boxed{\sigma_{ess}(\mathcal{M}) = \sigma_{ess}(M_z)}$.
Furthermore:

Theorem (Axler, Conway & McDonald, 1982)

$$\sigma_{ess}(M_z) = \Gamma.$$

Regarding the Toeplitz matrix $T(\psi)$ we have:

Theorem (Bottcher & Grudsky, Toeplitz book, 2005)

$$\sigma_{ess}(T(\psi)) = \psi(\mathbb{T}) \quad (= \Gamma).$$

Hence,

$$\boxed{\sigma_{ess}(T(\psi)) = \sigma_{ess}(\mathcal{M}).}$$

More coincidence: Main subdiagonal

Consider the main subdiagonal $a_{n+1,n}$ of $\mathcal{M}$. Then:

$$a_{n+1,n} = \langle zp_n, p_{n+1} \rangle = \langle \lambda_n z^{n+1} + \cdots , p_{n+1} \rangle = \langle \lambda_n z^{n+1}, p_{n+1} \rangle = \frac{\lambda_n}{\lambda_{n+1}}.$$

Since $\mathrm{cap}(\Gamma) = b$, it follows from the ratio asymptotics for λ_n, that:

Lemma

$$\sqrt{\frac{n+2}{n+1}}\, a_{n+1,n} = b + O\left(\frac{1}{n}\right), \quad n \in \mathbb{N}.$$

That is, the main subdiagonal of the upper Hessenberg matrix $\mathcal{M}$ tends to the main subdiagonal of the Toeplitz matrix $T(\psi)$.

Eventually: $\mathcal{M} \to T(\psi)$, diagonally!

Using the theory on strong asymptotics for non-smooth curves we have:

Theorem (Saff & St)

Assume that Γ is piecewise analytic without cusps. Then for any fixed $k \in \mathbb{N} \cup \{0\}$,

$$\sqrt{\frac{n+1}{n+k+1}}\, a_{n,n+k} = b_k + O\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty.$$

That is, the k-th diagonal of the upper Hessenberg matrix $\mathcal{M}$ tends to the k-th diagonal of the Toeplitz matrix $T(\psi)$.

Faber polynomials of G

The Faber polynomial $F_n(z)$ ($n \in \mathbb{N}$) of G, is the polynomial part of the Laurent series expansion of $\Phi^n(z)$ at ∞:

$$F_n(z) = \Phi^n(z) + O\left(\frac{1}{z}\right), \quad z \to \infty.$$

The Faber polynomial of the 2nd kind $G_n(z)$, is the polynomial part of the expansion of the Laurent series expansion of $\Phi^n(z)\Phi'(z)$ at ∞:

$$G_n(z) = \Phi^n(z)\Phi'(z) + O\left(\frac{1}{z}\right), \quad z \to \infty.$$

Note:

$$\boxed{G_n(z) = \frac{F'_{n+1}(z)}{n+1}}.$$

Recurrence relation for G_n

The Faber polynomials of the 2nd kind satisfy the recurrence relation,

$$\boxed{zG_n(z) = bG_{n+1}(z) + \sum_{k=0}^{n} b_k G_{n-k}(z)}, \quad n = 0, 1, \ldots,$$