Bergman shift operators for Jordan domains

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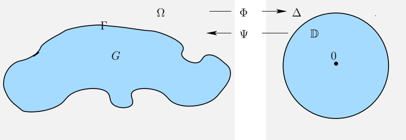
Bergman polynomials $\{p_n\}$ on an Jordan domain G



$\Gamma := \partial G$ $\quad$ $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}.$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

Shift Operator

Let $L_a^2(G)$ denote the Bergman space of square integrable and analytic functions in G and consider the Bergman shift operator on $L_a^2(G)$. That is,

$$S_z : L_a^2(G) \to L_a^2(G) \quad \text{with} \quad S_z f = zf.$$

Properties of S_z

(i) S_z defines a subnormal operator on $L_a^2(G)$.

(ii) $\sigma(S_z) = \overline{G}$ and $\sigma_{ess}(S_z) = \partial G$ (Axler, Conway & McDonald, Can. J. Math.,1982).

(iii) $S_z^*(f) = P_G(\overline{z}f)$, where P_G denotes the orthogonal projection from $L^2(G)$ to $L_a^2(G)$.

Proof of (iii): For any $f, g \in L_a^2(G)$ it holds that

$$\langle S_z^* f, g\rangle = \langle f, S_z g\rangle = \langle f, zg\rangle = \langle \overline{z}f, g\rangle = \langle P_G(\overline{z}f), g\rangle.$$

Short-term recurrences for Bergman polynomials $\{p_n\}$

In general it holds that

$$zp_n(z) = \sum_{k=0}^{n+1} b_{k,n} p_k(z), \quad \text{where} \quad b_{k,n} := \langle zp_n, p_k \rangle.$$

Definition

We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy an $m+1$-term recurrence relation, if for any $n \geq m-1$,

$$zp_n(z) = b_{n+1,n} p_{n+1}(z) + b_{n,n} p_n(z) + \ldots + b_{n-m+1,n} p_{n-m+1}(z).$$

Lemma (Putinar & St, CAOT, 2007)

If the Bergman polynomials $\{p_n\}$ satisfy an $m+1$-term recurrence relation, then for any $p \in \mathbb{P}_d$, it holds that $S_z^* p \in \mathbb{P}_{d+m-1}$.

Proof of Lemma

Using the short-term recurrence relation we have:

$$\langle S_z^* p, p_n\rangle = \langle p, S_z p_n\rangle = \langle p, z p_n\rangle = \langle p, \sum_{k=0}^{n-m+1} b_{k,n} p_k\rangle = \sum_{k=0}^{n-m+1} b_{k,n}\langle p, p_k\rangle.$$

Thus, $\langle S_z^* p, p_n\rangle = 0$, for any $d < n-m+1$, i.e., for $n > d+m-1$.

Only ellipses carry 3-term recurrence relations

Theorem (Putinar & St, CAOT, 2007)

If the Bergman polynomials $\{p_n\}$ satisfy a 3-term recurrence relation, then $\Gamma = \partial G$ is an ellipse.

Proof. We use Havin's Lemma: $\boxed{L^2(G) = L^2_a(G) \oplus \partial W^{1,2}_0(G)}$.

$$\overline{z} = P_G(\overline{z}) + \partial g = S^*_z(\overline{z}) + \partial g = p + \partial g,$$

where $g \in W^{1,2}_0(G)$ and $\deg(p) \leq 1$. Hence, by integration we obtain

$$z\overline{z} = Q(z) + g(z) + \overline{f(z)}, \quad Q' = p, \quad f \in L^2_a(G).$$

From uniqueness in $W^{1,2}_0(G)$ we further obtain $f = \overline{Q} + const$. Hence, $|z|^2 = Q(z) + \overline{Q(z)} + c + g(z)$, for $z \in G$, and from regularity of G:

$$|z|^2 = Q(z) + \overline{Q(z)} + c, \quad z \in \Gamma.$$

Only ellipses carry short-term recurrences for p_n

In fact, a great deal more can be obtained:

Theorem (St, C. R. Acad. Sci. Paris, 2010)

Assume that:

(i) $\Gamma = \partial G$ *is piecewise analytic without cusps.*

(ii) *The Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy an* $m+1$*-term recurrence relation, with some* $m \geq 2$.

Then $m = 2$ *and* Γ *is an ellipse.*

The above theorem refines some deep results of Putinar & St (CAOT, 2007) and Khavinson & St (Springer, 2010).

Matrix representation for S_z

The Bergman operator S_z has the following upper Hessenberg matrix representation with respect to the Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G:

$$\mathcal{M} = \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} & b_{04} & b_{05} & \cdots \\ b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & \cdots \\ 0 & b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & \cdots \\ 0 & 0 & b_{32} & b_{33} & b_{34} & b_{35} & \cdots \\ 0 & 0 & 0 & b_{43} & b_{44} & b_{45} & \cdots \\ 0 & 0 & 0 & 0 & b_{54} & b_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

where $\boxed{b_{k,n} = \langle zp_n, p_k \rangle}$ are the Fourier coefficients of $S_z p_n = zp_n$.

Note

The eigenvalues of the $n \times n$ principal submatrix $\mathcal{M}_n$ of $\mathcal{M}$ coincide with the zeros of p_n.

Banded Hessenberg matrices for OP's are Jacobi

In the Numerical Linear Algebra jargon the short-term recurrence theorem reads as follows:

Theorem

If the upper Hessenberg matrix $\mathcal{M}$ is banded, with constant bandwidth ≥ 3, then it is tridiagonal, i.e., a Jacobi matrix.

This result should put an end to the long search in Numerical Linear Algebra, for practical semi-iterative methods (aka polynomial iteration methods) based on short-term recurrence relations of orthogonal polynomials.

Example: $G \equiv \mathbb{D}$

This example shows why modern text books on Functional Analysis or Operators Theory do not refer to matrices: Indeed, in this case we have:

$$p_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, \quad n = 0, 1, \ldots .$$

Therefore, in the matrix representation $\mathcal{M}$ of S_z the only non-zero diagonals are the main subdiagonal, and hence for any $n \in \mathbb{N}$, $\mathcal{M}_n$ is a nilpotent matrix. As a result, the Caley-Hamilton theorem implies:

$$\sigma(\mathcal{M}_n) = \{0\}.$$

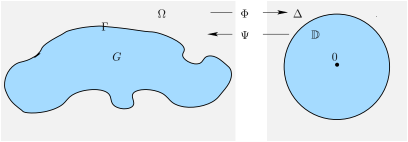
This is in sharp contrast to:

$$\sigma_{ess}(\mathcal{M}) = \sigma_{ess}(S_z) = \{w : |w| = 1\}$$

and

$$\sigma(\mathcal{M}) = \sigma(S_z) = \{w : |w| \leq 1\}.$$

The inverse conformal map Ψ



Recall that

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots ,$$

and let $\Psi := \Phi^{-1} : \{w : |w| > 1\} \to \Omega$, denote the inverse conformal map. Then,

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots , \quad |w| < 1,$$

where

$$b = \text{cap}(\Gamma) = 1/\gamma.$$

The Toeplitz matrix with (continuous) symbol Ψ

$$T_\psi = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\ b & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \cdots \\ 0 & b & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ 0 & 0 & b & b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & 0 & 0 & b & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & 0 & 0 & b & b_0 & b_1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & b & b_0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Spectral properties

Theorem (St, Constr, Approx., to appear)

If Γ is piecewise analytic without cusps, then

$$|b_n| \leq c_1(\Gamma)\frac{1}{n^{1+\omega}}, \quad n \in \mathbb{N}, \tag{1}$$

where $\omega\pi$ $(0 < \omega < 2)$ is the smallest exterior angle of Γ.

Therefore, in this case, the symbol Ψ of the Toeplitz matrix T_Ψ belongs to the Wiener algebra. Thus, T_Ψ defines a bounded linear operator on the Hilbert space $l^2(\mathbb{N})$ and

$$\boxed{\sigma_{ess}(T_\Psi) = \Gamma}; \tag{2}$$

see e.g., Bottcher & Grudsky, Toeplitz book, 2005.

Faber polynomials of G

The Faber polynomial of the 2nd kind $G_n(z)$, is the polynomial part of the expansion of the Laurent series expansion of $\Phi^n(z)\Phi'(z)$ at ∞:

$$G_n(z) = \Phi^n(z)\Phi'(z) + O\left(\frac{1}{z}\right), \quad z \to \infty.$$

These polynomials satisfy the recurrence relation:

$$\boxed{zG_n(z) = bG_{n+1}(z) + \sum_{k=0}^{n} b_k G_{n-k}(z)}, \quad n = 0, 1, \ldots,$$

Recall: $zp_n(z) = \sum_{k=0}^{n+1} b_{k,n} p_k(z)$.

Note

The eigenvalues of the $n \times n$ principal submatrix T_n of T_ψ coincide with the zeros of G_n.

Ratio asymptotics for λ_n

Theorem (St, C. R. Acad. Sci. Paris, 2010 & Constr. Approx.)

Assume that Γ *is piecewise analytic without cusps. Then, for* $n \in \mathbb{N}$,

$$\boxed{\sqrt{\frac{n+2}{n+1}}\,\frac{\lambda_n}{\lambda_{n+1}} = \operatorname{cap}(\Gamma) + \xi_n}, \quad \textit{where} \quad |\xi_n| \le c(\Gamma)\,\frac{1}{n}.$$

The above relation provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the Bergman polynomials.

Some coincidence

Recall

$$\sigma_{ess}(\mathcal{M}) = \Gamma = \sigma_{ess}(T_\Psi).$$

Consider the main subdiagonal $b_{n+1,n}$ of $\mathcal{M}$. Then:

$$b_{n+1,n} = \langle zp_n, p_{n+1}\rangle = \langle \lambda_n z^{n+1} + \cdots, p_{n+1}\rangle = \langle \lambda_n z^{n+1}, p_{n+1}\rangle = \frac{\lambda_n}{\lambda_{n+1}}.$$

Since $\mathrm{cap}(\Gamma) = b$, it follows from the ratio asymptotics for λ_n, that:

Corollary

$$\sqrt{\frac{n+2}{n+1}}\, b_{n+1,n} = b + O\left(\frac{1}{n}\right), \quad n \in \mathbb{N}.$$

That is, the main subdiagonal of the upper Hessenberg matrix $\mathcal{M}$ tends to the main subdiagonal of the Toeplitz matrix T_ψ.

$\mathcal{M} \to T_\psi$ diagonally

The next series of theorems show that the connection between the two matrices $\mathcal{M}$ and T_Ψ is much more substantial.

Theorem (Saff & St., CAOT, 2012)

Assume that Γ *is piecewise analytic without cusps. Then, it holds as* $n \to \infty$,

$$\sqrt{\frac{n+2}{n+1}} b_{n+1,n} = b + O\left(\frac{1}{n}\right), \tag{3}$$

and for $k \geq 0$,

$$\sqrt{\frac{n-k+1}{n+1}} b_{n-k,n} = b_k + O\left(\frac{1}{\sqrt{n}}\right), \tag{4}$$

where O *depends on* k *but not on* n.

$\mathcal{M} \to T_\psi$ diagonally: Smooth curve

Improvements in the order of convergence occur in cases when Γ is smooth.

Theorem (Saff & St., CAOT, 2012)

Assume that $\Gamma \in C(p+1,\alpha)$, *with* $p+\alpha > 1/2$. *Then, it holds as* $n \to \infty$,

$$\sqrt{\frac{n+2}{n+1}}\, b_{n+1,n} = b + O\left(\frac{1}{n^{2(p+\alpha)}}\right), \tag{5}$$

and for $k \geq 0$,

$$\sqrt{\frac{n-k+1}{n+1}}\, b_{n-k,n} = b_k + O\left(\frac{1}{n^{p+\alpha}}\right), \tag{6}$$

where O depends on k but not on n.

$\mathcal{M} \to T_\psi$ diagonally: Analytic curve

For the case of an analytic boundary Γ further improved asymptotic results can be obtained.

Theorem (Saff & St., CAOT, 2012)

Assume that the boundary Γ is analytic and let $\rho < 1$ be the smallest index for which Φ is conformal in the exterior of L_ρ. Then, it holds as $n \to \infty$,

$$\sqrt{\frac{n+2}{n+1}}\, b_{n+1,n} = b + O(\rho^{2n}), \tag{7}$$

and for $k \geq 0$,

$$\sqrt{\frac{n-k+1}{n+1}}\, b_{n-k,n} = b_k + O(\sqrt{n}\rho^n), \tag{8}$$

where O depends on k but not on n.

Is $\mathcal{M} - T_\psi$ compact?

Corollary

If the upper Hessenberg matrix $\mathcal{M}$ is banded, with constant bandwidth, then $\mathcal{M} - T_\psi$ defines a compact operator on $l^2(\mathbb{N})$.

Is the following true?

The adjoint operator $S_z^* : L_a^2(G) \to L_a^2(G)$ maps $\mathbb{P}$ into itself if and only if $\Gamma := \partial G$ is an ellipse.

The above is equivalent to:

Conjecture, Khavinson & Shapiro, 1992

The Dirichlet problem for G has a polynomial solution for any polynomial data on Γ if an only if $\Gamma = \partial G$ is an ellipse.

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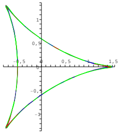
Example: G is a 3-cusped hypocycloid



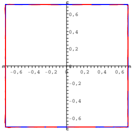
Assume that $\nu(P)$ denotes the *normalized counting measure of zeros* of the polynomial P. Also let μ_Γ denote the *equilibrium measure* on Γ, note that $\text{supp}(\mu_\Gamma) = \Gamma$ and recall $\sigma_{ess}(\mathcal{M}) = \Gamma = \sigma_{ess}(T_\Psi)$. Then:
Levin, Saff & St., Constr. Approx. (2003):

$$\boxed{\nu(p_n) \xrightarrow{*} \mu_\Gamma, \quad n \to \infty, \; n \in \mathcal{N}}, \quad \mathcal{N} \subset \mathbb{N}.$$

He & Saff, JAT (1994):

$$\boxed{\sigma(T_n) \subset [0, 1.5] \cup [0, 1.5e^{i2\pi/3}] \cup [0, 1.5e^{i4\pi/3}]}.$$

Example: G is the square



Recall: $\sigma_{ess}(\mathcal{M}) = \Gamma = \sigma_{ess}(T_\Psi)$.

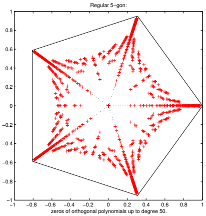
Maymeskul & Saff, JAT (2003):

$$\boxed{\sigma(M_n) \subset \text{the two diagonals}}.$$

Kuijlaars & Saff, Math. Proc. Cambrigde Phil. Soc. (1995):

$$\boxed{\nu(G_n) \xrightarrow{*} \mu_\Gamma, \quad n \to \infty, \ n \in \mathcal{N}}, \quad \mathcal{N} \subset \mathbb{N}$$

Example: G is the canonical pentagon



Recall: $\sigma_{ess}(\mathcal{M}) = \Gamma = \sigma_{ess}(T_\Psi)$.

Levin, Saff & St., Constr. Approx. (2003):

$$\boxed{\nu(p_n) \stackrel{*}{\longrightarrow} \mu_\Gamma, \quad n \to \infty,\ n \in \mathcal{N}}, \quad \mathcal{N} \subset \mathbb{N}$$

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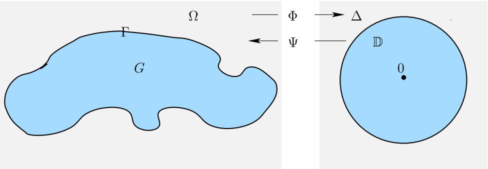
$$\boxed{\nu(G_n) \stackrel{*}{\longrightarrow} \mu_\Gamma, \quad n \to \infty,\ n \in \mathcal{N}}, \quad \mathcal{N} \subset \mathbb{N}$$

Asymptotics



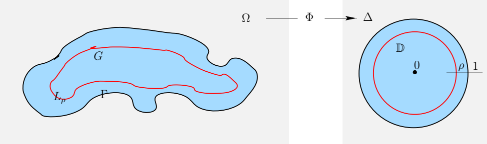
$$\boxed{\Omega := \overline{\mathbb{C}} \setminus \overline{G}}$$

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \boxed{\mathrm{cap}(\Gamma) = 1/\gamma}$$

The Bergman polynomials of G:

$$p_n(z) = \lambda_n z^n + \cdots , \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

Strong asymptotics when Γ is analytic



T. Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\mathrm{ext}(L_\rho)$, then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\rho^{2n},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}}, \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\sqrt{n}\,\rho^n, \quad z \in \overline{\Omega}.$$

Strong asymptotics when Γ is smooth

We say that $\Gamma \in C(p,\alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if Γ is given by $z = g(s)$, where s is the arclength, with $g^{(p)} \in \mathrm{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \mathrm{Lip}\alpha$.

P.K. Suetin, Proc. Steklov Inst. Math. AMS (1974)

Assume that $\Gamma \in C(p+1,\alpha)$, with $p+\alpha > 1/2$. Then, for $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\frac{1}{n^{2(p+\alpha)}},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}},$$

where

$$|A_n(z)| \le c_2(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$

Strong asymptotics for Γ non-smooth

Theorem (St, C. R. Acad. Sci. Paris, 2010)

Assume that Γ *is piecewise analytic without cusps. Then, for* $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \text{where} \quad 0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n},$$

and for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c_1(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

Ratio asymptotics for $p_n(z)$

Corollary (St, C. R. Acad. Sci. Paris, 2010)

For any $z \in \Omega$, and sufficiently large $n \in \mathbb{N}$,

$$\boxed{\sqrt{\frac{n}{n+1}}\,\frac{p_n(z)}{p_{n-1}(z)} = \Phi(z)\{1 + B_n(z)\}},$$

where

$$|B_n(z)| \leq \frac{c_1(\Gamma)}{\sqrt{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

The above relation provides the means for computing approximations to the conformal map Φ. This leads to an efficient algorithm for recovering the shape of G, from a finite collection of its power moments $\langle z^m, z^k \rangle_{m,k=0}^n$. This method was actually commented as unsuitable by P. Henrici, in *Computational Complex Analysis, Vol. III (1986)*, because of the instability of the Conventional GS.