Bergman Orthogonal Polynomials: Construction, Asymptotics, Zeros and Shape Recovery

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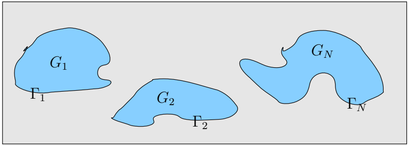
Definition: Bergman polynomials $\{p_n\}$



Γ_j, $j = 1, \ldots, N$, a system of disjoint and mutually exterior Jordan curves in $\mathbb{C}$, $\boxed{G_j := \text{int}(\Gamma_j)}$, $\boxed{\Gamma := \cup_{j=1}^N \Gamma_j}$, $\boxed{G := \cup_{j=1}^N G_j}$.

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

Construction of p_n's

Algorithm: Conventional Gram-Schmidt (GS)

Apply the Gram-Schmidt process to the monomials

$$1, z, z^2, z^3, \ldots$$

Main ingredient: the moments

$$\mu_{m,k} := \langle z^m, z^k \rangle = \int_G z^m \overline{z}^k dA(z), \quad m, k = 0, 1, \ldots .$$

The above algorithm has been been suggested by pioneers of Numerical Conformal Mapping (like P. Davis and D. Gaier) as the standard procedure for constructing Bergman polynomials. It was subsequently used by researchers in this area (e.g. Kokkinos, Papamichael, Sideridis and Warby). It has been even employed in the numerical conformal FORTRAN package `BKMPACK` of Warby.

Instability Indicator

The GS method is notorious for its instability. For measuring it, when orthonormalizing a system $S_n := \{u_1, u_2, \ldots, u_n\}$ of functions, the following instability indicator has been proposed by J.M. Taylor, (Proc. R.S. Edin., 1978):

$$\boxed{I_n := \frac{\|u_n\|^2_{L^2(G)}}{\min_{u \in \mathrm{span}(S_{n-1})} \|u_n - u\|^2_{L^2(G)}}, \quad n \in \mathbb{N}.}$$

Note that, when S_n is an orthonormal system, then $\boxed{I_n = 1}$. When S_n is linearly dependent then $\boxed{I_n = \infty}$. Also, if $G_n := [\langle u_m, u_k \rangle]^n_{m,k=1}$, denotes the Gram matrix associated with S_n then,

$$\boxed{\kappa_2(G_n) \geq I_n},$$

where $\kappa_2(G_n)$ is the spectral condition number of G_n.

Instability of the Conventional GS

In the single-component case $N = 1$, consider the monomials $u_j := z^{j-1}$, $j = 1, 2, \ldots, n$. Then, for the conventional GS we have the following result:

Theorem (N. Papamichael and M. Warby, Numer. Math., 1986.)

Assume that the curve Γ is piecewise-analytic without cusps and let

$$\boxed{L := d(\Gamma)/\mathrm{cap}(\Gamma)} \quad (\geq 1),$$

where $d(\Gamma) := \max\{|z| : z \in \Gamma\}$ and $\mathrm{cap}(\Gamma)$ denotes the capacity of Γ. Then,

$$\boxed{c_1(\Gamma)\, L^{2n} \leq I_n \leq c_2(\Gamma)\, L^{2n}}.$$

Note that $L = 1$, iff $G \equiv \mathbb{D}$. Also, when G is the 8×2 rectangle centered at the origin, then $L = 3/\sqrt{2} \approx 2.12$. In this case, $I_{25} \asymp 10^{16}$ and the method breaks down in MATLAB, for $\boxed{n = 25}$.

The Arnoldi algorithm in NLA

Let $A \in \mathbb{C}^{m,m}$, $b \in \mathbb{C}^m$ and consider the Krylov subspace

$$K_k := \text{span}\{b, Ab, A^2b \dots, A^{k-1}b\}.$$

The Arnoldi algorithm produces an orthonormal set of vectors $\{v_1, v_2, \dots, v_k\}$ as follows:

W. Arnoldi (Quart. Appl. Math., 1951)

At the n-th step, apply GS to orthonormalize the vector Av_{n-1} (instead of $A^{n-1}b$) against the (already computed) orthonormal vectors $\{v_1, v_2, \dots, v_{n-1}\}$.

The Arnoldi algorithm for OP's

Let μ be an (non-trivial) finite Borel measure with compact support on $\mathbb{C}$ and consider the series of orthonormal polynomials

$$p_n(z,\mu) := \lambda_n(\mu) z^n + \cdots, \quad \lambda_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

generated by the inner product

$$\langle f, g \rangle_\mu = \int f(z)\overline{g(z)} d\mu(z).$$

Arnoldi GS for orthonormal polynomials

At the n-th step, apply GS to orthonormalize the polynomial zp_{n-1} (instead of z^n) against the (already computed) orthonormal polynomials $\{p_0, p_1, \ldots, p_{n-1}\}$.

Stability of Arnoldi GS

In the case of Arnoldi GS, the instability indicator is given by:

$$I_{n+1} := \frac{\|zp_{n-1}\|^2_{L^2(G)}}{\min_{p\in\mathbb{P}_{n-1}} \|zp_{n-1} - p\|^2_{L^2(G)}}, \quad n \in \mathbb{N}.$$

Theorem (I)

It holds,

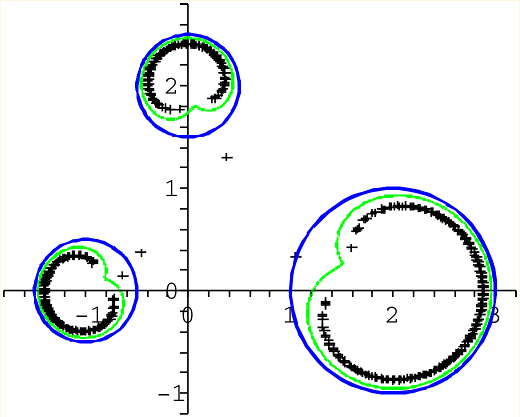
$$1 \leq I_{n+1} \leq \max_{z\in\operatorname{supp}(\mu)} |z| \frac{\lambda^2_{n-1}(\mu)}{\lambda^2_n(\mu)}, \quad n \in \mathbb{N}.$$

Typically: When $d\mu \equiv |dz|$ (Szegö polynomials), or $d\mu \equiv dA$ (Bergman polynomials), then

$$c_1(\Gamma) \leq \frac{\lambda_{n-1}(\mu)}{\lambda_n(\mu)} \leq c_2(\Gamma), \quad n \in \mathbb{N}.$$

Also, when $d\mu \equiv dx$ on $[a, b] \subset \mathbb{R}$, this ratio tends to $(b-a)/4$.

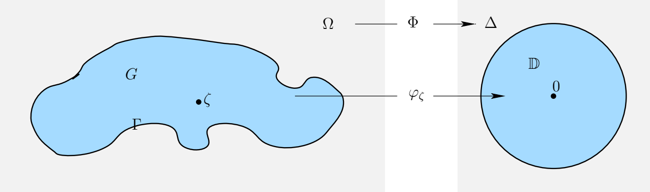
Zeros of Bergman polys: Three Disks



Zeros of the Bergman polynomials p_n, $n = 140$, 150 and 160.

Theory in: Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

Single-component case $N = 1$



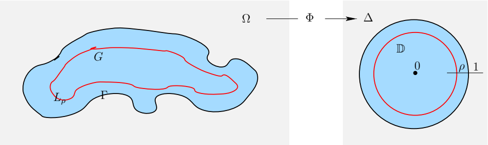
$$\boxed{\Omega := \overline{\mathbb{C}} \setminus \overline{G}}$$

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \boxed{\text{cap}(\Gamma) = 1/\gamma}$$

The Bergman polynomials of G:

$$p_n(z) = \lambda_n z^n + \cdots , \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

Strong asymptotics when Γ is analytic



T. Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\operatorname{ext}(L_\rho)$, then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\rho^{2n},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\},} \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\sqrt{n}\,\rho^n, \quad z \in \overline{\Omega}.$$

Strong asymptotics when Γ is smooth

We say that $\Gamma \in C(p,\alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if Γ is given by $z = g(s)$, where s is the arclength, with $g^{(p)} \in \mathrm{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \mathrm{Lip}\alpha$.

P.K. Suetin, Proc. Steklov Inst. Math. AMS (1974)

Assume that $\Gamma \in C(p+1,\alpha)$, with $p+\alpha > 1/2$. Then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\frac{1}{n^{2(p+\alpha)}},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\{1 + A_n(z)\},} \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$

Strong asymptotics for Γ non-smooth

Theorem (St, C. R. Acad. Sci. Paris, 2010)

Assume that Γ *is piecewise analytic without cusps. Then,*

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \textit{where} \quad 0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

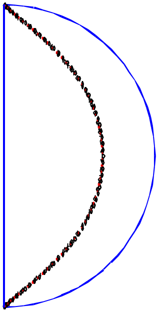
and for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c_1(\Gamma)}{\text{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

Numerical example: Half-disk



$$\gamma = \frac{1}{\text{cap}(\Gamma)} = \frac{3\sqrt{3}}{4}$$

We compute, by using the Arnoldi GS process (in finite precision), the Bergman polynomials $p_n(z)$ for the unit half-disk, for n up to 60 and test the hypothesis

$$\alpha_n := 1 - \frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} \approx C\,\frac{1}{n^s}.$$

Numerical example: Half-disk

n	α_n	s
51	0.003 263 458 678	-
52	0.003 200 769 764	0.998 887
53	0.003 140 444 435	0.998 899
54	0.003 082 351 464	0.998 911
55	0.003 026 369 160	0.998 923
56	0.002 972 384 524	0.998 934
57	0.002 920 292 482	0.998 946
58	0.002 869 952 027	0.998 957
59	0.002 821 401 485	0.998 968
60	0.002 774 426 207	0.998 979

The numbers indicate clearly that $\alpha_n \approx C\,\frac{1}{n}$. Accordingly, we have made conjectures regarding strong asymptotics in Oberwolfach Reports (2004) and ETNA (2006).

Ratio asymptotics for λ_n

Corollary (St, C. R. Acad. Sci. Paris, 2010)

$$\boxed{\sqrt{\frac{n+1}{n}}\,\frac{\lambda_{n-1}}{\lambda_n}=\operatorname{cap}(\Gamma)+\xi_n},\quad \textit{where}\quad |\xi_n|\le c(\Gamma)\,\frac{1}{n},\quad n\in\mathbb{N}.$$

The above relation provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the Bergman polynomials. In addition:

Corollary

$$c_1(\Gamma)\le I_n\le c_2(\Gamma),\quad n\in\mathbb{N}.$$

Hence, under the assumptions of the previous theorem, the Arnoldi GS for Bergman polynomials, in the single component case, is stable.

Ratio asymptotics for $p_n(z)$

Corollary (St, C. R. Acad. Sci. Paris, 2010)

For any $z \in \Omega$, and sufficiently large $n \in \mathbb{N}$,

$$\boxed{\sqrt{\frac{n+1}{n+2}}\,\frac{p_{n+1}(z)}{p_n(z)} = \Phi(z)\{1 + B_n(z)\}},$$

where

$$|B_n(z)| \leq \frac{c_1(\Gamma)}{\sqrt{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

The above relation provides the means for computing approximations to the conformal map Φ in Ω, by simply taking the ratio of two consequent Bergman polynomials. This leads to an efficient algorithm for recovering the shape of G, from a finite collection of its power moments $\langle z^m, z^k\rangle$, $m, k = 0, 1, \ldots, n$.

Only ellipses carry finite-term recurrences for p_n

Definition

We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a $(M+1)$-term recurrence relation, if for any $n \geq M-1$,

$$zp_n(z) = a_{n+1,n}p_{n+1}(z) + a_{n,n}p_n(z) + \ldots + a_{n-M+1,n}p_{n-M+1}(z).$$

Theorem (St, C. R. Acad. Sci. Paris, 2010)

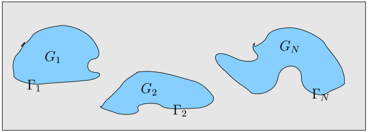
Assume that:

- $\Gamma = \partial G$ *is piecewise analytic without cusps;*
- *the Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy a* $(M+1)$*-term recurrence relation, with some* $M \geq 2$*.*

Then $M = 2$ *and* Γ *is an ellipse.*

The above theorem refines results of Putinar & St (CAOT, 2009) and Khavinson & St (Springer, 2010).

Leading coefficients in Archipelago



Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic, $j = 1, 2, \ldots, N$. Then

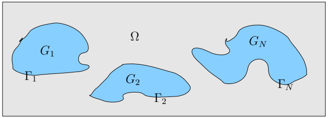
$$c_1(\Gamma)\sqrt{\frac{n+1}{\pi}}\frac{1}{\mathrm{cap}(\Gamma)^{n+1}} \leq \lambda_n \leq c_2(\Gamma)\sqrt{\frac{n+1}{\pi}}\frac{1}{\mathrm{cap}(\Gamma)^{n+1}}, \quad n \in \mathbb{N}.$$

Corollary

$$c_3(\Gamma) \leq I_n \leq c_4(\Gamma), \quad n \in \mathbb{N}.$$

Hence, the Arnoldi GS, for Bergman polynomials on an archipelago, is stable.

Bergman polynomials in Archipelago



Let $g_\Omega(z,\infty)$ denote the Green function of $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ with pole at ∞.

Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic. Then:

(i) *There exists a positive constant C, so that*

$$|p_n(z)| \le \frac{C}{\operatorname{dist}(z,\Gamma)}\sqrt{n}\exp\{n\, g_\Omega(z,\infty)\}, \quad z \notin \overline{G}. \tag{1}$$

(ii) *For every $\epsilon > 0$ there exist a constant $C_\epsilon > 0$, such that*

$$|p_n(z)| \ge C_\epsilon\sqrt{n}\exp\{n\, g_\Omega(z,\infty)\}, \quad \operatorname{dist}(z,\mathrm{Co}(\overline{G})) \ge \epsilon.$$

Recovery of Jordan domains (case $N = 1$)

Truncated Moments Problem

Given the finite $n + 1 \times n + 1$ section

$$[\mu_{m,k}]_{m,k=0}^{n}, \quad \mu_{m,k} := \int_G z^m \overline{z}^k dA(z),$$

of the infinite complex moment matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$ associated with a bounded Jordan domain G, compute a good approximation to its boundary Γ.

Theorem (Davis & Pollak, Trans. AMS, 1956)

The infinite matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$ defines uniquely Γ.

An algorithm based on ratio asymptotics

The Recovery Algorithm

(I) Use the Arnoldi GS to compute $p_0, p_1, \ldots, p_n$.

(II) Compute the coefficients of the Laurent series of the ratio

$$\sqrt{\frac{n}{n+1}}\,\frac{p_n(z)}{p_{n-1}(z)} = a^{(n)}z + a_0^{(n)} + \frac{a_1^{(n)}}{z} + \frac{a_2^{(n)}}{z^2} + \frac{a_3^{(n)}}{z^3} + \cdots, \quad (2)$$

(III) Revert (2) and truncate to obtain

$$\Psi_n(w) = b^{(n)}w + b_0^{(n)} + \frac{b_1^{(n)}}{w} + \frac{b_2^{(n)}}{w^2} + \frac{b_3^{(n)}}{w^3} + \cdots + \frac{b_n^{(n)}}{w^n}.$$

(IV) Approximate Γ by $\widetilde{\Gamma} := \{z :\ z = \Psi_n(e^{it}),\ t \in [0, 2\pi]\}$.

Convergence of the recovery algorithm

$\Psi := \Phi^{[-1]} : \{w : |w| > 1\} \to \Omega$, the inverse conformal map, where

$$\Psi(z) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots . \qquad \boxed{b = \text{cap}(\Gamma)}$$

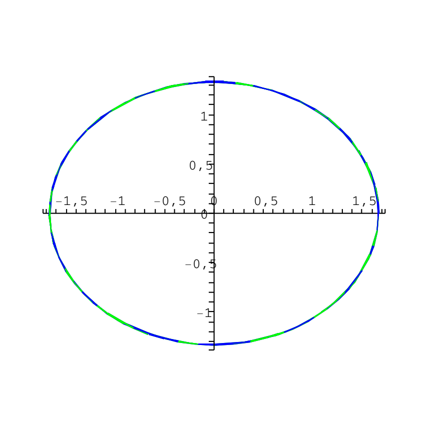
Theorem

Let $B_n(z)$ denote the error function in the ratio asymptotics of the Bergman polynomials. Then,

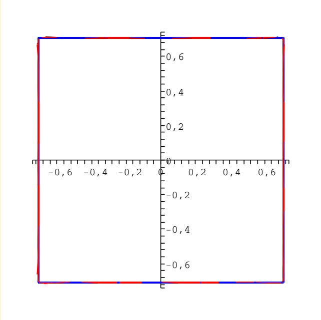
$$\max_{|w|=1} |\Psi(w) - \Psi_n(w)| \leq C(\Gamma) \max_{z \in \Gamma} |B_n(z)|, \quad n \in \mathbb{N}.$$

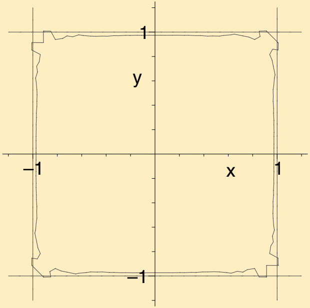
Numerical Examples

Recovery of the canonical ellipse, with $n = 3$



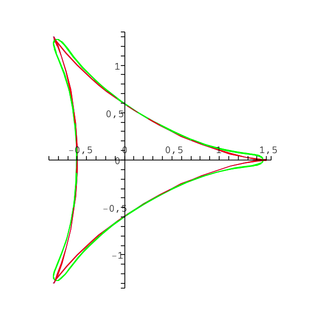
Recovery of the square, with $n = 16$

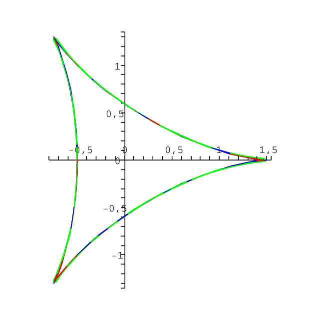




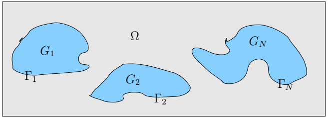
Comparison: The exponential transform algorithm of Gustafsson, He, Milanfar & Putinar, Inverse Problems (2000).

Recovery of the cusped hypocycloid, with $n = 10$ and $n = 20$.





Discovery of an Archipelago



$$\boxed{G_j := \text{int}(\Gamma_j)}, \quad \boxed{\Gamma := \cup_{j=1}^{N} \Gamma_j}, \quad \boxed{G := \cup_{j=1}^{N} G_j}.$$

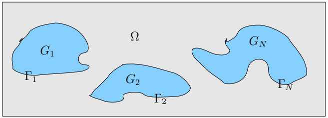
Truncated moments problem

Starting with the finite $n+1 \times n+1$ section

$$[\mu_{m,k}]_{m,k=0}^{n}, \quad \mu_{m,k} := \int_G z^m \overline{z}^k dA(z),$$

of the associated infinite complex moment matrix $[\mu_{m,k}]_{m,k=0}^{\infty}$, compute a good approximation to G.

Discovery of an Archipelago



Archipelago Recovery Algorithm

(I) Use the Arnoldi-type GS to compute $p_0, p_1, \ldots, p_n$.

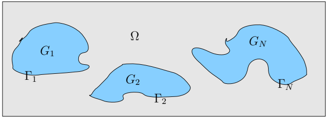
(II) Form the square root of the Christoffel function

$$\Lambda_n(z) := \frac{1}{\sqrt{\sum_{k=0}^{n} |p_k(z)|^2}}.$$

(III) Plot the zeros of p_j, $j = 1, 2, \ldots, n$.

(IV) Plot the level curves of the function $\Lambda_n(x + iy)$, on a suitable rectangular frame for (x, y) that surrounds the plotted zero set.

Theoretical Support of the Recovery Algorithm



Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

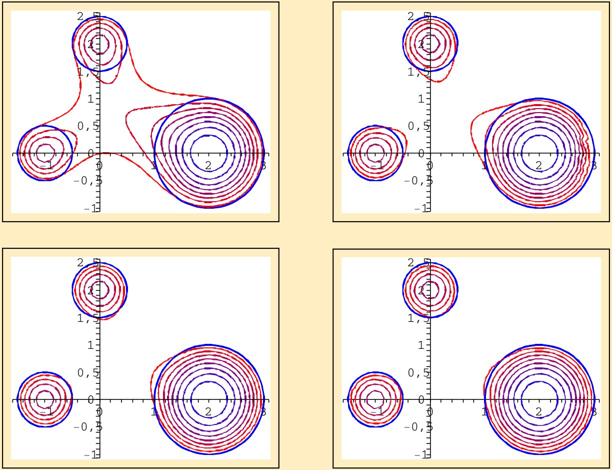
Assume that every Γ_j *is analytic and let* $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$. *Then,*

$$\begin{aligned}
\Lambda_n(z) &\asymp \operatorname{dist}(z,\Gamma), \quad z \in G, \quad n \to \infty \\
\Lambda_n(z) &\asymp \frac{1}{n}, \quad z \in \Gamma, \quad n \to \infty \\
\Lambda_n(z) &\asymp \exp\{-n\, g_\Omega(z,\infty)\}, \quad z \in \Omega, \quad n \to \infty.
\end{aligned}$$

where $g_\Omega(z,\infty)$ *denotes the Green function of* Ω *with pole at infinity.*

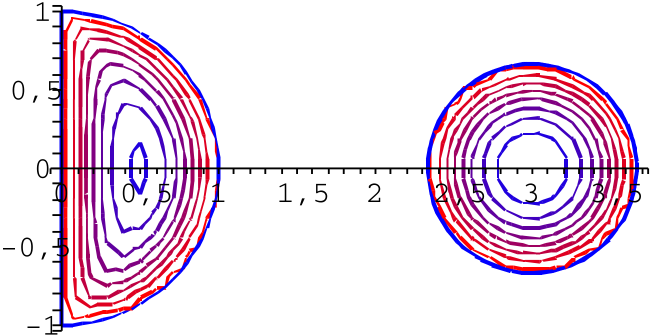
Note: $\boxed{g_\Omega(z,\infty) > 0, \quad z \in \Omega}$.

Recovery of three disks



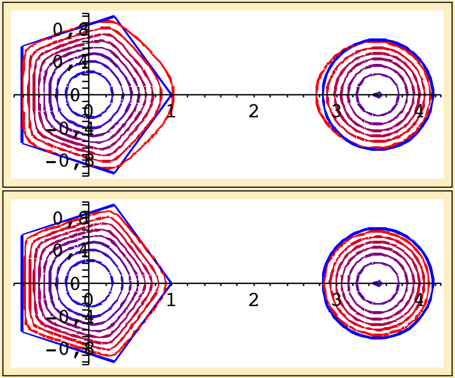
Level lines of $\Lambda_n(x+iy)$ on $\{(x,y) : -1 \leq x \leq 4, -2 \leq y \leq 2\}$, for $n = 25, 50, 75, 100$.

Recovery of half-disk and disk



Level lines of $\Lambda_{100}(x+iy)$ on $\{(x,y) : -1 \leq x \leq 6, -2 \leq y \leq 2\}$

Recovery of pentagon and disk



Level lines of $\Lambda_n(x+iy)$ on $\{(x,y) : -2 \le x \le 5, -2 \le y \le 2\}$, for $n = 25, 50$.