Christoffel, Bergman, Faber: Boundary behavior

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Elwin Christoffel
1829–1900

Georg Faber
1877 –1966

Stefan Bergman
1895 –1977

Orthonormal polynomials

Let μ be a finite positive Borel measure having compact and infinite support $\mathcal{S} := \text{supp}(\mu)$ in the complex plane $\mathbb{C}$. Then, the measure μ yields the Lebesgue spaces $L^2(\mu)$ with inner product

$$\langle f, g\rangle_\mu := \int f(z)\overline{g(z)}d\mu(z)$$

and norm

$$\|f\|_{L^2(\mu)} := \langle f, f\rangle_\mu^{1/2}.$$

Let $\{p_n(\mu, z)\}_{n=0}^{\infty}$ denote the sequence of orthonormal polynomials associated with μ. That is, the unique sequence of the form

$$p_n(\mu, z) = \kappa_n(\mu)z^n + \cdots, \quad \kappa_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

satisfying $\langle p_m(\mu, \cdot), p_n(\mu, \cdot)\rangle_\mu = \delta_{m,n}$.

Christoffel functions

The monic orthogonal polynomials $p_n(\mu, z)/\kappa_n(\mu)$, can be defined by the extremal property

$$\left\| \frac{1}{\kappa_n(\mu)} p_n(\mu, \cdot) \right\|_{L^2(\mu)} := \min_{z^n+\cdots} \|z^n + \cdots\|_{L^2(\mu)} = \frac{1}{\kappa_n(\mu)}.$$

A related extremal problem leads to the sequence $\{\lambda_n(\mu, z)\}_{n=0}^{\infty}$ of the Christoffel functions. These are defined, for any $z \in \mathbb{C}$, by

$$\lambda_n(\mu, z) := \inf\{\|P\|^2_{L^2(\mu)},\ P \in \mathbb{P}_n \text{ with } P(z) = 1\},$$

where $\mathbb{P}_n$ is the space of polynomials of degree $\leq n$.

Christoffel functions

The Cauchy-Schwarz inequality yields that

$$\frac{1}{\lambda_n(\mu, z)} = \sum_{j=0}^{n} |p_j(\mu, z)|^2, \quad z \in \mathbb{C}.$$

That is, $\lambda_n(\mu, z)$ is the inverse of the diagonal of the Christoffel-Darboux kernel of degree n, defined by

$$K_n^\mu(z, \zeta) := \sum_{j=0}^{n} \overline{p_j(\mu, \zeta)} p_j(\mu, z).$$

Note:

$$\langle p, K_n^\mu(\cdot, \zeta) \rangle_\mu = p(\zeta), \quad p \in \mathbb{P}_n.$$

In particular,

$$K_n^\mu(\zeta, \zeta) = \langle K_n^\mu(\cdot, \zeta), K_n^\mu(\cdot, \zeta) \rangle_\mu = \|K_n^\mu(\cdot, \zeta)\|^2_{L^2(\mu)}.$$

Asymptotics for general measures

Let Ω denote the unbounded component of $\overline{\mathbb{C}} \setminus \mathcal{S}$.

Theorem (Ambroladge, JAT 1995)

$$\lim_{n\to\infty} \lambda_n(\mu, z) = \frac{1}{\sum_{j=0}^{n} |p_j(\mu, z)|^2} = \mu(\{z\}), \quad z \in \partial\Omega.$$

Corollary

For every $\varepsilon > 0$,

$$\limsup_{n\to\infty} |p_n(\mu, z)| n^{\frac{1}{2}+\varepsilon} = \infty,$$

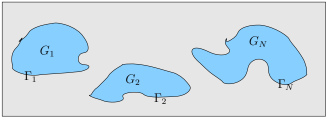
everywhere in $\partial\Omega$ outside the discrete spectrum of μ.

Proposition (Saff & St, Mat. Sb. 2018)

$$\lim_{n\to\infty} \lambda_n(\mu, z) = 0,$$

locally uniformly in Ω.

Bergman polynomials on an archipelago: $d\mu = dA|_G$



$G := \cup_{j=1}^N G_j.$

G_j, $j = 1, \ldots, N$, a system of disjoint and mutually exterior Jordan domains at positive distance.

$$\langle f, g \rangle_G := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle_G^{1/2}$$

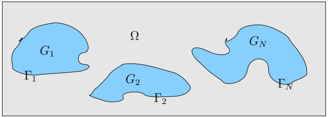
The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure A on G:

$$\langle p_m, p_n \rangle_G = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \ldots .$$

Christoffel on archipelagoes $d\mu = dA|_G$



$\Omega := \overline{\mathbb{C}} \setminus \overline{G}$

Let $g_\Omega(z, \infty)$ denote the Green function of Ω with pole at ∞.

Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

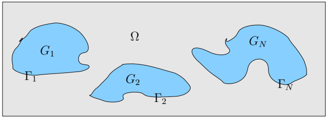
$$\lambda_n(z) > \pi \operatorname{dist}(z, \partial G), \quad z \in G.$$

If every ∂G_j is analytic then, as $n \to \infty$:

$$\begin{aligned} n^2 \lambda_n(z) &\asymp 1, \quad z \in \partial\Omega. \\ n\lambda_n(z) &\asymp \exp\{-2n\, g_\Omega(z, \infty)\}, \quad z \in \Omega. \end{aligned}$$

Note: $g_\Omega(z, \infty)$ increases from 0 on ∂G to $+\infty$ at the point of infinity.
This theorem gave rise to an reconstruction algorithm from moments.

Christoffel on archipelagoes $d\mu = WdA|_G$



Theorem (Totik, Trans. AMS, 2010)

Suppose $d\mu = WdA|_G$, with continuous W such that,

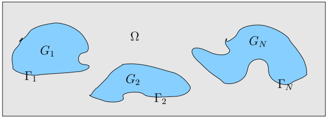
$$\operatorname{cap}\left(\{z : W(z) > 0\} \cap G\right) = \operatorname{cap}(\overline{G}).$$

If z_0 is the center of some C^2 Jordan subarc of $\partial\Omega$, then,

$$\lim_{n\to\infty} n^2 \lambda_n(z_0) = 2\pi W(z_0) \left(\frac{\partial g_{\Omega(z_0,\infty)}}{\partial \mathbf{n}} \right)^{-2}.$$

*The same holds, for $\mu \in$ **Reg**, with $\operatorname{supp}(\mu) = \overline{G}$, such that $d\mu = WdA$, in some open disk around z_0, with W continuous at z_0.*

Christoffel on archipelagoes $d\mu = dA|_G$

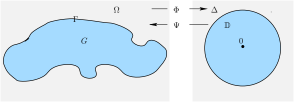


Theorem (Totik & Varga, London Math. Soc., 2015)

Assume that $z_0 \in \partial\Omega$ is formed by two C^{1+} arcs, of exterior angle $\omega\pi$, with $1 \leq \omega < 2$. Then,

$$\lambda_n(z_0) \asymp \frac{1}{n^{2\omega}}$$

Christoffel on island $d\mu = dA|_G$



$\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$ $\boxed{\Gamma = \partial G = \partial\Omega}$

Theorem (Beckermann, Putinar, Saff & St, Found. C. Math., 2021)

Assume that ∂G is piece-wise analytic without cusps. Then,

$$(n+1)\lambda_n(z) = \pi \frac{|\Phi(z)|^2 - 1}{|\Phi'(z)|^2 |\Phi(z)|^{2(n+1)}} \left(1 + O(\frac{1}{n})\right),$$

locally uniformly in Ω.

Note:

- Based on the strong asymptotics for Bergman polynomials in Ω.
- $O(\cdot)$, has $\mathrm{dist}(z, \partial\Omega)$ in the denominator.
- Here, $g_\Omega(z, \infty) = \log |\Phi(z)|$.

Christoffel on island $d\mu = WdA|_G$

Let $z_j \in \partial G$ and $\alpha_j > -2$, $j = 1, 2, \cdots, N$, and consider the weight

$$W(z) = h(z) \prod_{j=1}^{N} |z - z_j|^{a_j},$$

where h is bounded above and below in G by positive constants.

Theorem (Andrievskii, Constr. Approx., 2017)

Assume that ∂G is bounded by a finite number of Dini-smooth arcs that form at z_0 an exterior angle of opening $\omega\pi$, $0 < \omega < 2$. Then,

$$\lambda_n(z_0) \asymp \frac{1}{n^{2\omega}}$$

The paper contains sharp estimates for quasidisks.

Bergman estimate on $\Gamma := \partial G$ $\quad \mu = dA|_G$

Theorem (St, Contemp. Math., 2015)

Assume that Γ is piecewise analytic without cusps, and let z_0 be a corner with exterior angle $\omega\pi$, $0 < \omega_j < 2$. Then,

$$|p_n(z_0)| \leq c(\Gamma, z_0) n^{\omega - \frac{1}{2}}, \quad n \in \mathbb{N}.$$

- This yields for $0 < \omega < 1/2$: $\boxed{\lim_{n\to\infty} p_n(z_0) = 0.}$
- Recall Ambroladge: $\boxed{\limsup_{n\to\infty} |p_n(z_0)| n^{\frac{1}{2}+\varepsilon} = \infty}$, for every $\varepsilon > 0$.
- For z_0 inside an analytic subarc of Γ (whence $\omega = 1$), the proof relies on the result $\boxed{\lim_{n\to\infty} n^2 \lambda_n(z_0) = \frac{2\pi}{|\Phi'(z_0)|^2}}$ of Totik.

A Conjecture for Bergman on Γ

Conjecture (St, Contemp. Math., 2015)

Assume that Γ is piecewise analytic without cusps. Then, at any point z_o on Γ with exterior angle $\omega\pi$, $0 < \omega < 2$, it holds that

$$p_n(z_0) = \frac{\omega(n+1)^{\omega-1/2} {a_1}^{\omega} \Phi^{n+1-\omega}(z_0)}{\sqrt{\pi}\,\Gamma(\omega+1)}\{1+\beta_n(z_0)\},$$

with $\lim_{n\to\infty} \beta_n(z_0) = 0$.

The two intersecting circles

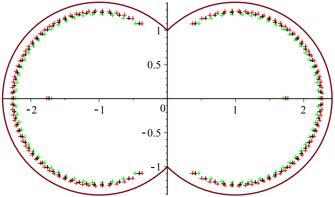
Consider the case where G is defined by the two intersecting circles $|z-1|=\sqrt{2}$ and $|z+1|=\sqrt{2}$. Then,

$$\Phi(z)=\frac{1}{2}\left(z-\frac{1}{z}\right).$$

We test the conjecture numerically for $z=i$ and $z=1+\sqrt{2}$.

The computations below were carried out in Maple 16 with 128 significant figures on a MacBook Pro. The construction of the Bergman polynomials was made by using the Arnoldi variation of the Gram-Schmidt algorithm as suggested in [St, Constr. Approx., 2013], where it is shown that the Taylor Indicator measuring instability does not increase with the number of basis functions used.
See also Vandermode with Arnoldi, in Brubeck, Nakatsukasa & Trefethen, SIAM Rev., 2021.

The two intersecting circles



Zeros of the Bergman polynomials $p_n(z)$, with $n = 80, 100, 120$.

Theorem (Saff & St, JAT 2015)

Let ν_n denote the normalised counting measure of zeros of p_n. Then

$$\nu_n \xrightarrow{*} \mu_\Gamma, \quad n \to \infty, \quad n \in \mathbb{N},$$

where μ_Γ denotes the equilibrium measure on Γ.

The reluctance of the zeros to approach the points $\pm i$, is due to the fact that $\boxed{d\mu_\Gamma(z) = |\Phi'(z)|ds}$, where s denotes the arclength on Γ.

A result of Lehman, Pac. J. Math., 1957

For the statement of a conjecture regarding the behaviour of $p_n(z)$ on Γ, we need a result of Lehman, for the asymptotics of both Φ and Φ'.

Theorem

Assume that $\omega\pi$, $0 < \omega \leq 2$, is the opening of the exterior angle at a point $z_0 \in \Gamma$, formed by two analytic arcs. Then, for any z near z_0:

$$\Phi(z) = \Phi(z) + a_1 (z - z_0)^{1/\omega} + o(|z - z_0|^{1/\omega}),$$

and

$$\Phi'(z) = \frac{1}{\omega} a_1 (z - z_0)^{1/\omega - 1} + o(|z - z_0|^{1/\omega - 1}),$$

with $a_1 \neq 0$.

This is an over-simplification. Logarithmic terms may appear, if ω is rational. However, they never appear when z_0 is formed by two straight line segments.

The two intersecting circles: $z = i$

Here: $\omega = 1/2\ \Phi(i) = i$, $\Gamma(3/2) = \sqrt{\pi}/2$ and $a_1 = 1/(2i)$.
The conjecture takes the form $p_n(i) = (i^n/\sqrt{2\pi})\{1 + \beta_n\}$, $\beta_n = o(1)$.

n	$\lvert\beta_n\rvert$	n	$\lvert\beta_n\rvert$
100	0.057 121	101	0.037 299
102	0.056 990	103	0.037 428
104	0.056 864	105	0.037 554
106	0.056 741	107	0.037 675
108	0.056 623	109	0.037 793
110	0.056 508	111	0.037 907
112	0.056 396	113	0.038 017
114	0.056 288	115	0.038 125
116	0.056 183	117	0.038 229
118	0.056 081	119	0.038 312
120	0.055 981		

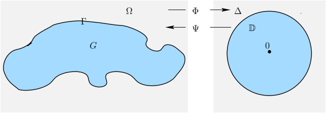
The two intersecting circles: $z = 1 + \sqrt{2}$

Here: $\omega = 1$, $\Phi(z) = 1$, $\beta_n \in \mathbb{R}$, and the conjecture takes the form

$$p_n(1+\sqrt{2}) = \sqrt{\frac{n+1}{\pi}}\,\frac{2+\sqrt{2}}{(1+\sqrt{2})^2}\{1+\beta_n\}, \quad \beta_n = o(1).$$

n	β_n	n	β_n
100	0.000 596	111	0.000 986
101	0.001 095	112	0.000 784
102	0.000 930	113	0.000 557
103	0.001 410	114	0.000 466
104	0.001 163	115	0.000 184
105	0.001 557	116	0.000 261
106	0.001 246	117	0.000 429
107	0.001 525	118	0.000 447
108	0.001 224	119	0.000 822
109	0.001 325	120	0.000 722
110	0.001 054		

Normalized Faber polynomials



$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots. \qquad \boxed{\mathrm{cap}(\Gamma) = 1/\gamma}$$

We consider the polynomial part of $\Phi^n(z)\Phi'(z)$ and denote the resulting series by $\{G_n\}_{n=0}^{\infty}$. Thus,

$$\Phi^n(z)\Phi'(z) = G_n(z) - H_n(z), \quad z \in \Omega,$$

with

$$G_n(z) = \gamma^{n+1} z^n + \cdots \quad \text{and} \quad H_n(z) = O\left(1/|z|^2\right), \quad z \to \infty.$$

We define the normalized Faber polynomials by

$$f_n(z) := \sqrt{\frac{n+1}{\pi}} G_n(z) = \sqrt{\frac{n+1}{\pi}} \gamma^{n+1} z^n + \cdots.$$

Theorem (Pritsker, JAT 2002)

Assume that Γ *is rectifiable and let* $z_0 \in \Gamma$ *be formed by two analytic arcs meeting with exterior angle* $\omega\pi$, $0 < \omega \leq 2$. *Then,*

$$f_n(z_0) = \frac{\omega(n+1)^{\omega-1/2} a_1^{\omega} \Phi^{n+1-\omega}(z_0)}{\sqrt{\pi}\,\Gamma(\omega+1)} \{1 + o(1)\},$$

For polygonal Γ the above result was established by G. Szegő in his famous paper "Uber einen Satz von A. Markoff", Math. Z., 1925.

Theorem (P.K. Suetin, "Series of Faber Polynomials", 1998)

Assume that Γ *is the outer boundary of a continuum. Then, for z in the lever curve* $|\Phi(z)| = R > 1$ *it holds:*

$$f_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi^n(z)\Phi'(z) \left\{1 + O\left(\frac{r^n}{R^n}\right)\right\}, \quad 1 < r < R.$$

This, in particular, shows that moving level lines to the boundary, in the presence of corners, it does not work.

Theoretical motivation

The theoretical motivation for the conjecture came from:

Theorem (St, Constr. Approx., 2013)

Assume that Γ *is piece-wise analytic without cusps. Then,*

$$\|f_n - p_n\|^2_{L^2(G)} = O(\frac{1}{n}) \quad \textit{and} \quad f_n(z) = p_n(z)\left(1 + O(\frac{1}{\sqrt{n}})\right),$$

locally uniformly in Ω*.*

In [Beckermann & St, Constr., 2018], $O(\frac{1}{\sqrt{n}})$ was improved to $O(\frac{1}{n})$.

Theorem (St, Contemporary Math., 2016)

Assume that Γ *is piece-wise analytic without cusps. Then, for* $z_0 \in \Gamma$*,*

$$|f_n(z_0) - p_n(z_0)| \leq C(\Gamma, z_0), \quad n \in \mathbb{N}.$$

However, the Conjecture calls for $\boxed{f_n(z_0) - p_n(z_0) = o(1)}$.

Suggestion

Replace orthonormal in Christoffel by normalized Faber.

Set $L_n(z,z_0) := \sum_{j=0}^{n} \overline{f_j(z_0)} f_j(z)$ and note that $L_n(z_0,z_0) > 0$

Theorem

$$\|L_n(\cdot,z_0)\|^2_{L^2(G)} \le L_n(z_0,z_0), \quad z_0 \in \mathbb{C}.$$

Then, from the minimal property of the Christoffel functions,

$$\lambda_n(z_0) \le \frac{\|L_n(\cdot,z_0)\|^2_{L^2(G)}}{(L_n(z_0,z_0))^2} \le \frac{1}{L_n(z_0,z_0)}, \quad z_0 \in \mathbb{C}.$$

This leads to computable upper estimates for the Christoffel functions.

Lemma (I)

Assume that $z_0 \in \Gamma$ is formed by two analytic arcs meeting at angle π (whence $\omega = 1$). Then,

$$\lim_{n\to\infty} \frac{L_n(z_0, z_0)}{n^2} = \frac{|\Phi'(z_0)|^2}{2\pi}.$$

This is in agreement with Totik's result: $\boxed{\lim_{n\to\infty} n^2 \lambda_n(z_0) = \frac{2\pi}{|\Phi'(z_0)|^2}}$

Lemma (II)

If $z_0 \in \Gamma$ is an outward pointing cusp point (whence $\omega = 2$) then,

$$\lim_{n\to\infty} \frac{L_n(z_0, z_0)}{n^4} = \frac{|a_1|^2}{2\pi}.$$