An Arnoldi Gram-Schmidt process and Hessenberg matrices for Orthonormal Polynomials

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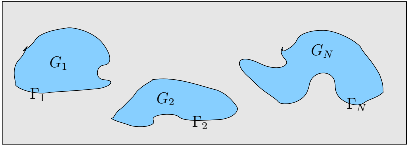
Bergman polynomials $\{p_n\}$ on an archipelago G



Γ_j, $j = 1, \ldots, N$, a system of disjoint and mutually exterior Jordan curves in $\mathbb{C}$, $\boxed{G_j := \operatorname{int}(\Gamma_j)}$, $\boxed{\Gamma := \cup_{j=1}^N \Gamma_j}$, $\boxed{G := \cup_{j=1}^N G_j}$.

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

Construction of p_n's

Algorithm: Conventional Gram-Schmidt (GS)

Apply the Gram-Schmidt process to the monomials

$$1, z, z^2, z^3, \ldots$$

Main ingredient: the moments

$$\mu_{m,k} := \langle z^m, z^k \rangle = \int_G z^m \overline{z}^k dA(z), \quad m, k = 0, 1, \ldots.$$

The above algorithm has been been suggested by pioneers of Numerical Conformal Mapping (like P. Davis and D. Gaier) as the standard procedure for constructing Bergman polynomials. It was subsequently used by researchers in this area (e.g. Burbea, Kokkinos, Papamichael, Sideridis and Warby). It has been even employed in the numerical conformal FORTRAN package BKMPACK of Warby.

Instability Indicator

The GS method is notorious for its instability. For measuring it, when orthonormalizing a system $S_n := \{u_0, u_1, \ldots, u_n\}$ of functions, the following instability indicator has been proposed by J.M. Taylor, (Proc. R.S. Edin., 1978):

$$\boxed{I_n := \frac{\|u_n\|^2_{L^2(G)}}{\min_{u\in\text{span}(S_{n-1})} \|u_n - u\|^2_{L^2(G)}}, \quad n \in \mathbb{N}.}$$

Note that, when S_n is an orthonormal system, then $\boxed{I_n = 1}$. When S_n is linearly dependent then $\boxed{I_n = \infty}$. Also, if $G_n := [\langle u_m, u_k\rangle]^n_{m,k=1}$, denotes the Gram matrix associated with S_n then,

$$\boxed{\kappa_2(G_n) \geq I_n},$$

where $\kappa_2(G_n)$ is the spectral condition number of G_n.

Instability of the Conventional GS

In the single-component case $N = 1$, consider the monomials $u_j := z^j$, $j = 0, 1, \ldots, n$. Then, for the conventional GS we have the following result:

Theorem (N. Papamichael and M. Warby, Numer. Math., 1986.)

Assume that the curve Γ is piecewise-analytic without cusps and let

$$\boxed{L := \|z\|_{L^\infty(\Gamma)}/\text{cap}(\Gamma)} \quad (\geq 1),$$

where $\text{cap}(\Gamma)$ *denotes the capacity of* Γ. *Then,*

$$\boxed{c_1(\Gamma)\, L^{2n} \leq I_n \leq c_2(\Gamma)\, L^{2n}}.$$

Note that $L = 1$, iff $G \equiv \mathbb{D}$ and that I_n is sensitive to the relative position of G w.r.t. the origin. When G is the 8×2 rectangle centered at the origin, then $L = 3/\sqrt{2} \approx 2.12$. In this case, $I_{25} \asymp 10^{16}$ and the method breaks down in MATLAB, for $\boxed{n = 25}$.

The Arnoldi algorithm in Numerical Liner Algebra

Let $A \in \mathbb{C}^{m,m}$, $b \in \mathbb{C}^m$ and consider the Krylov subspace

$$K_k := \text{span}\{b, Ab, A^2b \ldots, A^{k-1}b\}.$$

The Arnoldi algorithm produces an orthonormal basis $\{v_1, v_2, \ldots, v_k\}$ of K_k as follows:

W. Arnoldi (Quart. Appl. Math., 1951)

At the n-th step, apply GS to orthonormalize the vector Av_{n-1} (instead of $A^{n-1}b$) against the (already computed) orthonormal vectors $\{v_1, v_2, \ldots, v_{n-1}\}$.

The Arnoldi algorithm for OP's

Let μ be a (non-trivial) finite Borel measure with compact support $\boxed{\Sigma := \text{supp}(\mu)}$ on $\mathbb{C}$ and consider the series of orthonormal polynomials

$$p_n(z,\mu) := \lambda_n(\mu)z^n + \cdots, \quad \lambda_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

generated by the inner product

$$\langle f, g\rangle_\mu = \int f(z)\overline{g(z)}d\mu(z).$$

Arnoldi GS for Orthonormal Polynomials

At the n-th step, apply GS to orthonormalize the polynomial zp_{n-1} (instead of z^n) against the (already computed) orthonormal polynomials $\{p_0, p_1, \ldots, p_{n-1}\}$.

Used in B. Gragg & L. Reichel, Linear Algebra Appl. (1987), for the construction of Szegö polynomials.

Stability of the Arnoldi GS

In the case of the Arnoldi GS, the instability indicator is given by:

$$I_n := \frac{\|zp_{n-1}\|^2_{L^2(G)}}{\min_{p\in\mathbb{P}_{n-1}} \|zp_{n-1} - p\|^2_{L^2(G)}}, \quad n \in \mathbb{N}.$$

Theorem

It holds,

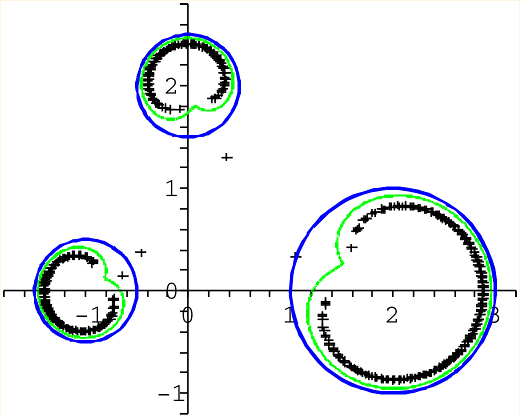
$$1 \leq I_n \leq \|z\|_{L^\infty(\Sigma)} \frac{\lambda^2_{n-1}(\mu)}{\lambda^2_n(\mu)}, \quad n \in \mathbb{N}.$$

Typically: When $d\mu \equiv |dz|$ (Szegö polynomials), or $d\mu \equiv dA$ (Bergman polynomials), then

$$\boxed{c_1(\Gamma) \leq \frac{\lambda_{n-1}(\mu)}{\lambda_n(\mu)} \leq c_2(\Gamma)}, \quad n \in \mathbb{N}.$$

When $d\mu \equiv w(x)dx$ on $[a, b] \subset \mathbb{R}$, this ratio tends to a constant.

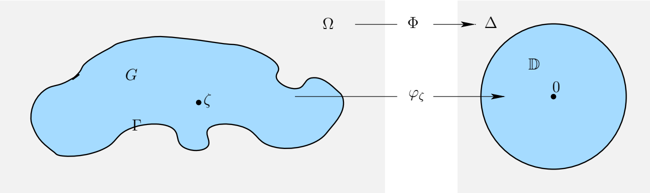
Zeros of Bergman polys: Three Disks



Zeros of the Bergman polynomials p_n, $n = 140$, 150 and 160.

Theory in: Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

Single-component case $N = 1$



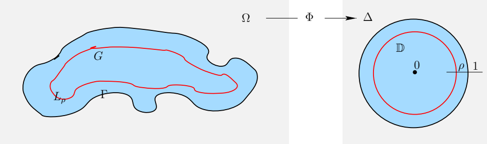
$$\boxed{\Omega := \overline{\mathbb{C}} \setminus \overline{G}}$$

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \boxed{\mathrm{cap}(\Gamma) = 1/\gamma}$$

The Bergman polynomials of G:

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

Strong asymptotics when Γ is analytic



T. Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\text{ext}(L_\rho)$, then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\rho^{2n},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\},} \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\sqrt{n}\,\rho^n, \quad z \in \overline{\Omega}.$$

Strong asymptotics when Γ is smooth

We say that $\Gamma \in C(p, \alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if Γ is given by $z = g(s)$, where s is the arclength, with $g^{(p)} \in \text{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \text{Lip}\alpha$.

P.K. Suetin, Proc. Steklov Inst. Math. AMS (1974)

Assume that $\Gamma \in C(p + 1, \alpha)$, with $p + \alpha > 1/2$. Then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \leq \alpha_n \leq c_1(\Gamma)\,\frac{1}{n^{2(p+\alpha)}},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\},} \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \leq c_2(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$

Strong asymptotics for Γ non-smooth

Theorem (St, C. R. Acad. Sci. Paris, 2010)

Assume that Γ *is piecewise analytic without cusps. Then,*

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1-\alpha_n}, \quad \textit{where} \quad 0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

and for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c_1(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

Ratio asymptotics for λ_n

Corollary (St, C. R. Acad. Sci. Paris, 2010)

$$\boxed{\sqrt{\frac{n+1}{n}}\,\frac{\lambda_{n-1}}{\lambda_n} = \mathrm{cap}(\Gamma) + \xi_n}, \quad \textit{where} \quad |\xi_n| \le c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

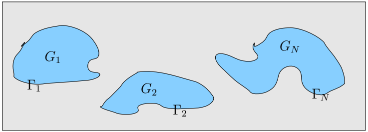
The above relation provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the Bergman polynomials. In addition it yiels:

Corollary

$$c_1(\Gamma) \le I_n \le c_2(\Gamma), \quad n \in \mathbb{N}.$$

Hence, under the assumptions of the previous theorem, the Arnoldi GS for Bergman polynomials, in the single component case, is stable.

Leading coefficients in an archipelago



Theorem (Gustafsson, Putinar, Saff & St, Adv. Math., 2009)

Assume that every Γ_j is analytic, $j = 1, 2, \ldots, N$. Then,

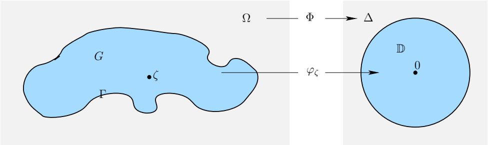
$$c_1(\Gamma)\sqrt{\frac{n+1}{\pi}}\frac{1}{\mathrm{cap}(\Gamma)^{n+1}} \leq \lambda_n \leq c_2(\Gamma)\sqrt{\frac{n+1}{\pi}}\frac{1}{\mathrm{cap}(\Gamma)^{n+1}}, \quad n \in \mathbb{N}.$$

Corollary

$$c_3(\Gamma) \leq I_n \leq c_4(\Gamma), \quad n \in \mathbb{N}.$$

Hence, the Arnoldi GS, for Bergman polynomials on an archipelago, is stable.

The inverse conformal map Ψ



Recall that

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots ,$$

and let $\Psi := \Phi^{-1} : \{w : |w| > 1\} \to \Omega$, denote the inverse conformal map. Then,

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots , \quad |w| > 1,$$

where

$$b = \text{cap}(\Gamma) = 1/\gamma.$$

Faber polynomials of G

The Faber polynomial $F_n(z)$ ($n \in \mathbb{N}$) of G, is the polynomial part of the Laurent series expansion of $\Phi^n(z)$ at ∞:

$$F_n(z) = \Phi^n(z) + O\left(\frac{1}{z}\right), \quad z \to \infty.$$

The Faber polynomial of the 2nd kind $G_n(z)$, is the polynomial part of the expansion of the Laurent series expansion of $\Phi^n(z)\Phi'(z)$ at ∞:

$$G_n(z) = \Phi^n(z)\Phi'(z) + O\left(\frac{1}{z}\right), \quad z \to \infty.$$

Note:

$$\boxed{G_n(z) = \frac{F'_{n+1}(z)}{n+1}}.$$

The Faber matrix $\mathcal{G}$

The Faber polynomials of the 2nd kind satisfy the recurrence relation,

$$\boxed{zG_n(z) = bG_{n+1}(z) + \sum_{k=0}^{n} b_k G_{n-k}(z)}, \quad n = 0, 1, \ldots,$$

and induce the upper Hessenberg Toeplitz matrix:

$$\mathcal{G} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\ b & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \cdots \\ 0 & b & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ 0 & 0 & b & b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & 0 & 0 & b & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & 0 & 0 & b & b_0 & b_1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & b & b_0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

The upper Hessenberg matrix $\mathcal{M}$

The Bergman polynomials satisfy the recurrence relation,

$$\boxed{zp_n(z) = \sum_{k=0}^{n+1} a_{k,n} p_k(z)}, \quad n = 0, 1, \ldots,$$

where $a_{k,n}$ are Fourier coefficients: $\boxed{a_{k,n} = \langle zp_n, p_k \rangle}$, and induce the (infinite) upper Hessenberg matrix:

$$\mathcal{M} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & a_{06} & \cdots \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\ 0 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\ 0 & 0 & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & \cdots \\ 0 & 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} & \cdots \\ 0 & 0 & 0 & 0 & a_{54} & a_{55} & a_{56} & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_{65} & a_{66} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

Eigenvalues = Zeros

Note

The eigenvalues of the finite $n \times n$ Faber matrix are the zeros of the Faber polynomial $G_n(z)$.

Note

The eigenvalues of the finite $n \times n$ Hessenberg matrix are the zeros of the Bergman polynomial $p_n(z)$.

Main subdiagonal

Consider the main subdiagonal of the Hessenberg matrix:

$$a_{n+1,n} = \langle zp_n, p_{n+1} \rangle = \langle \lambda_n z^{n+1} + \cdots , p_{n+1} \rangle = \langle \lambda_n z^{n+1}, p_{n+1} \rangle = \frac{\lambda_n}{\lambda_{n+1}}.$$

Since $\text{cap}(\Gamma) = b$, it follows from the ratio asymptotics for λ_n, that:

Lemma

$$\sqrt{\frac{n+2}{n+1}}\, a_{n+1,n} = b + O\left(\frac{1}{n}\right), \quad n \in \mathbb{N}.$$

That is, *the main subdiagonal of the Hessenberg matrix tends to the main subdiagonal of the Faber matrix*.

$\mathcal{M} \to \mathcal{G}$

More generally, using the theory on strong asymptotics for non-smooth curves we have:

Theorem (Saff & St)

Assume that Γ *is piecewise analytic w/o cusps, then for any fixed* $k \in \mathbb{N} \cup \{0\}$,

$$\sqrt{\frac{n+1}{n+k+1}}\, a_{n,n+k} = b_k + O\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty.$$

That is, *the k-th diagonal of the Hessenberg matrix tends to the k-th diagonal of the Faber matrix*.

Ratio asymptotics for $p_n(z)$

Theorem (St, C. R. Acad. Sci. Paris, 2010)

Assume that Γ is piecewise analytic without cusps. Then, for any $z \in \Omega$, and sufficiently large $n \in \mathbb{N}$,

$$\boxed{\sqrt{\frac{n+1}{n+2}}\,\frac{p_{n+1}(z)}{p_n(z)} = \Phi(z)\{1 + B_n(z)\}},$$

where

$$|B_n(z)| \leq \frac{c_1(\Gamma)}{\sqrt{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

Only ellipses carry finite-term recurrences for $\{p_n\}$

Definition

We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a $(M+1)$-term recurrence relation, if for any $n \geq M-1$,

$$zp_n(z) = a_{n+1,n}p_{n+1}(z) + a_{n,n}p_n(z) + \ldots + a_{n-M+1,n}p_{n-M+1}(z).$$

Theorem (St, C. R. Acad. Sci. Paris, 2010)

Assume that:

- $\Gamma = \partial G$ *is piecewise analytic without cusps;*
- *the Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy a* $(M+1)$*-term recurrence relation, with some* $M \geq 2$*.*

Then $M = 2$ *and* Γ *is an ellipse.*

The above theorem refines results of Putinar & St (CAOT, 2007) and Khavinson & St (Springer, 2010).

Banded Hessenberg matrices are tridiagonal

Corollary

If the Hessenberg matrix is banded with constant bandwidth ≥ 3, then is tridiagonal.

This result should put an end to the long search in Numerical Linear Algebra, for practical polynomial iteration methods, based on short-term recurrence relations of orthogonal polynomials,.