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A SINGULAR FUNCTION BOUNDARY INTEGRAL METHOD FOR LAPLACIAN PROBLEMS WITH BOUNDARY SINGULARITIES*

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Abstract. A singular function boundary integral method for Laplacian problems with boundary singularities is analyzed. In this method, the solution is approximated by the truncated asymptotic expansion for the solution near the singular point and the Dirichlet boundary conditions are weakly enforced by means of Lagrange multiplier functions. The resulting discrete problem is posed and solved on the boundary of the domain, away from the point of singularity. The main result of this paper is the proof of convergence of the method; in particular, we show that the method approximates the generalized stress intensity factors, i.e., the coefficients in the asymptotic expansion, at an exponential rate. A numerical example illustrating the convergence of the method is also presented.

Key words. boundary singularities, boundary approximation methods, Lagrange multipliers, stress intensity factors

AMS subject classifications. 65N15, 65N38, 65N12, 65N30

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1. Introduction.

In many engineering problems (e.g., fracture mechanics applications) governed by elliptic partial differential equations (PDEs), boundary singularities arise when there is a sudden change in the boundary conditions and/or on the boundary itself. Singularities are known to affect adversely the accuracy and the convergence of standard numerical methods, such as finite element, boundary element, finite difference, and spectral methods. Grid refinement and high-order discretizations are common strategies aimed at improving the convergence rate and accuracy of the above-mentioned standard methods. If, however, the form of the singularity is taken into account and is properly incorporated into the numerical scheme, then a more effective method may be constructed (see, e.g., [10, 15]). (For a recent survey of treatment of singularities in elliptic boundary value problems see [17] and the references therein.)

In the case of the two-dimensional Laplace equation, for example, the form of the singularity is visible through the asymptotic expansion for the solution u near the singular point. When the boundaries sharing the singular point are not curved, u is given by [12]

$$u(r,\theta) = \sum_{j=1}^{\infty} \alpha_j r^{\beta_j} \phi_j(\theta), \tag{1.1}$$

where (r,θ) are polar coordinates centered at the singular point, $\alpha_j \in \mathbb{R}$ are the unknown singular coefficients, and $\beta_j \in \mathbb{R}, \phi_j(\theta)$ are the eigenvalues and eigenfunctions of the problem, respectively, which are uniquely determined by the geometry and the boundary conditions along the boundaries sharing the singular point. The constants

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α_j are often called *generalized stress intensity factors* (GSIFs) or flux intensity factors. In the case of elasticity problems with cracks, these constants are called stress intensity factors (SIFs) and serve as a measure of the stress at which failure occurs. In most commonly used methods, such as the finite element method (FEM), the SIFs are calculated as a postsolution operation (see, e.g., [4, 26, 27]). If, however, the goal of the computation is the calculation of the SIFs, then methods which calculate these quantities *directly* may be preferable. The singular function boundary integral method (SFBIM), which we will analyze in this article, falls in the latter category. (See also [19, 20, 22, 23] and the references therein for additional information on the determination and importance of the aforementioned coefficients.)

The SFBIM was originally developed by Georgiou and coworkers [10, 11] and was subsequently refined and expanded by Elliotis, Georgiou, and Xenophontos [6, 7, 8, 9]. The method uses the leading terms in the local asymptotic expansion for the solution near the singular point as an approximation, while any Dirichlet boundary conditions are weakly enforced by means of Lagrange multiplier functions. The resulting problem is posed on the boundary of the domain; hence the dimension of the problem is reduced by one, leading to considerable computational cost reduction. We should also mention here the works of Li et al. [14, 15, 16] and Arad et al. [1], who also developed similar methods. (See also [21] for a review of SIF evaluation and modeling of singularities in boundary integral methods.)

The SFBIM has been successfully applied to a number of problems in solid and fluid mechanics and excellent numerical results have been obtained thus far [6, 7, 8, 9]. In particular, it was observed that the method (i) approximates the SIFs at an exponential rate of convergence, (ii) is very efficient, and (iii) compares extremely well with other accurate methods found in the literature. Our main goal in this article is to prove the observed convergence rates of the method.

The rest of the paper is organized as follows: In section 2 we present the formulation of the method for a two-dimensional Laplacian problem with a boundary singularity. In section 3 we present the convergence analysis, and in section 4 we comment on how the method can be efficiently implemented. Finally, section 5 includes the results of some numerical computations illustrating the convergence of the method. Throughout the paper the usual notation $H^k(\Omega)$ will be used for spaces containing functions on the domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$, having k generalized derivatives in $L^2(\Omega)$; the norm and seminorm on $H^k(\Omega)$ will be denoted by $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. Also, the letters C and c will be used to denote generic positive constants independent of any discretization parameters and possibly having different values in each occurrence.

2. The model problem and its formulation.

For simplicity we consider the Laplacian problem stated below and depicted graphically in Figure 2.1. Find u such that

$$\Delta u = 0 \text{ in } \Omega, \tag{2.1}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } S_1, \tag{2.2}$$

$$u = 0 \text{ on } S_2, \tag{2.3}$$

$$u = f(r,\theta) \text{ on } S_3, \tag{2.4}$$

$$\frac{\partial u}{\partial n} = g(r,\theta) \text{ on } S_4, \tag{2.5}$$

where $\partial\Omega = \bigcup_{i=1}^4 S_i$.

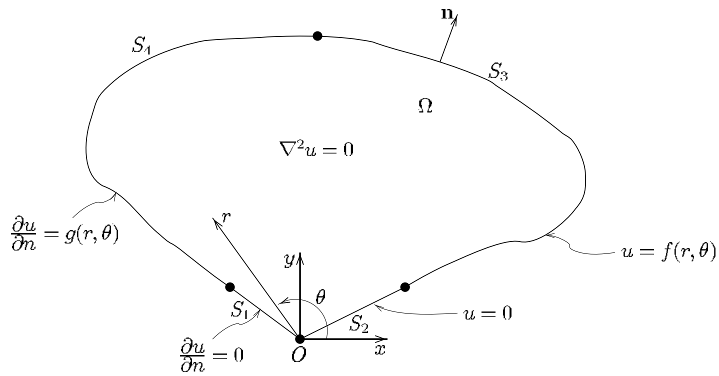

FIG. 2.1. *A two-dimensional Laplace equation problem with one boundary singularity.*

In (2.1)–(2.5), Δ denotes the Laplacian operator, the variables (r, θ) denote polar coordinates centered at O, and the functions f and g are known. It is assumed that f, g, and the boundary $\partial\Omega$ are such that there is only one boundary singularity at O. The local solution near O is given by an asymptotic expansion of the form (1.1) [12].

Note. Even though the presentation and analysis of the method will be restricted to the type of boundary singularity shown in Figure 2.1, any type of boundary singularity for which an asymptotic expansion (1.1) exists can be treated.

Multiplying (2.1) by a test function $v \in V_1$ (to be specified shortly) and integrating over Ω, we get

$$\iint_\Omega v\Delta u = 0,$$

and then, by means of Green's theorem, we obtain

$$-\iint_\Omega \nabla v \cdot \nabla u + \int_{\partial\Omega} v\frac{\partial u}{\partial n} = 0.$$

Since $\frac{\partial u}{\partial n} = 0$ on S_1 and $\frac{\partial u}{\partial n} = g(r,\theta)$ on S_4, we further have

$$\iint_\Omega \nabla v \cdot \nabla u - \int_{S_2} v\frac{\partial u}{\partial n} - \int_{S_3} v\frac{\partial u}{\partial n} = \int_{S_4} vg. \tag{2.6}$$

Suppose v is chosen to satisfy

$$\Delta v = 0 \text{ in } \Omega, \; v = 0 \text{ on } S_2, \; \frac{\partial v}{\partial n} = 0 \text{ on } S_1, \tag{2.7}$$

one such choice being $v \equiv r^{\beta_j}\phi_j(\theta)$ (see (1.1)). Then (2.6) becomes

$$\iint_\Omega \nabla v \cdot \nabla u - \int_{S_3} v\frac{\partial u}{\partial n} = \int_{S_4} vg. \tag{2.8}$$

Now, since $u = f$ on S_3 we have

$$-\int_{S_3} \frac{\partial v}{\partial n}(u - f) = 0,$$

so adding this to (2.8), we get

$$\iint_{\Omega} \nabla v \cdot \nabla u - \int_{S_3} v \frac{\partial u}{\partial n} - \int_{S_3} \frac{\partial v}{\partial n}(u - f) = \int_{S_4} vg$$

or, equivalently,

$$\iint_{\Omega} \nabla v \cdot \nabla u - \int_{S_3} v \frac{\partial u}{\partial n} - \int_{S_3} \frac{\partial v}{\partial n} u = \int_{S_4} vg - \int_{S_3} \frac{\partial v}{\partial n} f.$$

Letting

$$\lambda = \left.\frac{\partial u}{\partial n}\right|_{S_3}, \; \mu = \left.\frac{\partial v}{\partial n}\right|_{S_3}, \tag{2.9}$$

the above equation becomes

$$\iint_{\Omega} \nabla v \cdot \nabla u - \int_{S_3} v\lambda - \int_{S_3} \mu u = \int_{S_4} vg - \int_{S_3} \mu f.$$

Hence, the *variational problem* to be solved reads as follows: Find $(u, \lambda) \in V_1 \times V_2$ such that

$$B(u, v) + b(u, v; \lambda, \mu) = F(v, \mu) \quad \forall \; (v, \mu) \in V_1 \times V_2, \tag{2.10}$$

where

$$B(u, v) = \iint_{\Omega} \nabla v \cdot \nabla u, \tag{2.11}$$

$$b(u, v; \lambda, \mu) = -\int_{S_3} v\lambda - \int_{S_3} \mu u, \tag{2.12}$$

$$F(v, \mu) = \int_{S_4} vg - \int_{S_3} \mu f. \tag{2.13}$$

The spaces V_1 and V_2 are chosen as follows. Let the *trace space* of functions in $H^1(\Omega)$ be denoted by

$$H^{1/2}(\partial\Omega) = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} \in L^2(\partial\Omega)\right\}. \tag{2.14}$$

With $T : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ denoting the trace operator, the norm of $H^{1/2}(\partial\Omega)$ is defined as

$$\|\psi\|_{1/2,\partial\Omega} = \inf_{u \in H^1(\Omega)} \left\{ \|u\|_{1,\Omega} : Tu = \psi \right\}. \tag{2.15}$$

Then, we define $H^{-1/2}(\partial\Omega)$ as the *closure* of $H^0(\partial\Omega) \equiv L^2(\partial\Omega)$ with respect to the norm

$$\|\varphi\|_{-1/2,\partial\Omega} = \sup_{\psi \in H^{1/2}(\partial\Omega)} \frac{\int_{\partial\Omega} \varphi\psi}{\|\psi\|_{1/2,\partial\Omega}}. \tag{2.16}$$

(See, e.g., [24] for more details.) We then define

$$H^1_*(\Omega) = \{u \in H^1(\Omega) : u|_{S_2} = 0\}, \tag{2.17}$$

and we take

$$V_1 = H^1_*(\Omega), \; V_2 = H^{-1/2}(S_3). \tag{2.18}$$

The discrete problem corresponding to (2.10) then reads as follows: Find $(u_N, \lambda_h) \in [V_1^N \times V_2^h] \subset [H^1_*(\Omega) \times H^{-1/2}(S_3)]$ such that

$$B(u_N, v) + b(u_N, v; \lambda_h, \mu) = F(v, \mu) \;\; \forall \;\; (v, \mu) \in V_1^N \times V_2^h, \tag{2.19}$$

with $B(u, v), b(u, v; \lambda, \mu)$, and $F(v, \mu)$ given by (2.11)–(2.13), and with V_1^N, V_2^h finite dimensional spaces to be chosen shortly.

Remark 1. The above formulation will be used in the analysis of the method; for the implementation, we will use an equivalent boundary integral formulation in which all integrations will be one-dimensional and carried out away from the point causing the singularity. Details will be given in section 4.

3. Error analysis. We begin by defining the finite dimensional spaces V_1^N and V_2^h which will be used in the approximate problem (2.19). First, with

$$v_i \equiv r^{\beta_i} \phi_i(\theta) \tag{3.1}$$

denoting the singular functions, we define the finite dimensional space

$$V_1^N = span \{v_i\}_{i=1}^N . \tag{3.2}$$

Next, let S_3 be divided into quasiuniform sections $\Gamma_i, i = 1, \ldots, n$, such that $S_3 = \bigcup_{i=1}^n \Gamma_i$. Let $h_i = |\Gamma_i|$ and set $h = \max_{1 \le i \le n} h_i$. We assume that for each segment Γ_i, there exists an invertible mapping $\mathcal{F}_i : I \to \Gamma_i$ which maps the interval $I = [-1, 1]$ to Γ_i, and we define the space

$$V_2^h = \{\lambda_h : \lambda_h|_{\Gamma_i} \circ \mathcal{F}_i^{-1} \in \mathcal{P}_p(I), i = 1, \ldots, n\}, \tag{3.3}$$

where $\mathcal{P}_p(I)$ is the set of polynomials of degree $\le p$ on $I = [-1, 1]$. In practice, the representation of the boundary S_3 determines the mappings $\mathcal{F}_i$; i.e., if S_3 is represented by a polynomial, then an isoparametric mapping may be used, and if S_3 has some (general) parametric representation, then the *blending map* technique may be used (see Ch. 6 in [25]).

We have the following theorem.

THEOREM 3.1. *Let* (u, λ) *and* (u_N, λ_h) *be the solutions to* (2.10) *and* (2.19), *respectively. Suppose there exist positive constants* c_0, c, β, *and* γ, *independent of* N *and* h, *such that the following hold:*

$$B(v, v) \ge c_0 \|v\|^2_{1,\Omega} \;\; and \;\; |B(u, v)| \le c \|u\|_{1,\Omega} \|v\|_{1,\Omega} \;\; \forall v \in V_1^N, \tag{3.4}$$

$$\exists\, 0 \ne v_N \in V_1^N \;\; s.t. \left| \int_{S_3} \mu_h v_N \right| \ge \beta \|\mu_h\|_{-1/2,S_3} \|v_N\|_{1,\Omega} \;\; \forall \mu_h \in V_2^h, \tag{3.5}$$

$$\left| \int_{S_3} \lambda v \right| \le \gamma \|\lambda\|_{-1/2,S_3} \|v\|_{1,\Omega} \;\; \forall v \in V_1^N. \tag{3.6}$$

Then

(3.7)
$$\|u-u_N\|_{1,\Omega}+\|\lambda-\lambda_h\|_{-1/2,S_3}\le C\left\{\inf_{v\in V_1^N}\|u-v\|_{1,\Omega}+\inf_{\eta\in V_2^h}\|\lambda-\eta\|_{-1/2,S_3}\right\},$$

with $C\in\mathbb{R}^+$ *independent of* N *and* h*.*

This is similar to the classical result for saddle point problems (cf. [5]), and its proof for our problem appears in the appendix; see also Theorem 6.1 in [14].

Before verifying that (3.4)–(3.6) hold for the problem under consideration, we will make certain assumptions that will aid in the analysis. First, we note that for any function w which can be expressed in the form given by (1.1), we can always write

$$w=w_N+r_N,$$

where

$$w_N=\sum_{j=1}^{N}\alpha_j v_j\in V_1^N,\ r_N=\sum_{j=N+1}^{\infty}\alpha_j v_j. \tag{3.8}$$

We will assume that there exists $a\in(0,1)$ such that

$$|r_N|\le Ca^N \tag{A1}$$

and

$$\left|\frac{\partial r_N}{\partial r}\right|\le CNa^N. \tag{A2}$$

If $r<1$ in (1.1), assumptions (A1), (A2) may be replaced with the assumption that $|\alpha_j|<\infty\ \forall\, j$, since then, by (3.1) and the fact that ϕ_j is harmonic,

$$|r_N|\le\sum_{j=N+1}^{\infty}|\alpha_j|\,r^{\beta_j}\le C\frac{r^{\beta_{N+1}}}{1-r}\le Ca^N,$$

with $r<a<1$, and $C\in\mathbb{R}^+$ independent of a and N. Similarly,

$$\begin{aligned}\left|\frac{\partial r_N}{\partial r}\right|&\le\sum_{j=N+1}^{\infty}|\alpha_j|\,r^{\beta_j-1}=\sum_{j=N+1}^{\infty}|\alpha_j|\left\{\frac{d}{dr}\int_0^r\rho^{\beta_j-1}d\rho\right\}=\frac{d}{dr}\sum_{j=N+1}^{\infty}|\alpha_j|\left\{\int_0^r\rho^{\beta_j-1}d\rho\right\}\\&\le\frac{d}{dr}\sum_{j=N+1}^{\infty}|\alpha_j|\,r^{\beta_j}\le C\frac{d}{dr}\left(\frac{r^{\beta_{N+1}}}{1-r}\right)\le CNa^N.\end{aligned}$$

If $r\ge1$, one may partition the domain Ω into subdomains in which separate approximations may be obtained, including one near O which is valid for $r<1$. The solution over the entire domain can then be composed by combining the solutions from each subdomain and properly dealing with their interactions across the interfaces separating them (see, e.g., [18]).

Let us now verify that (3.4)–(3.6) hold for the problem given by (2.19). First, note that $B(v,v)=|v|^2_{1,\Omega}$ so that, by Poincaré's inequality,

$$B(v,v)\ge c_0\|v\|^2_{1,\Omega}\ \ \forall v\in H^1_*(\Omega). \tag{3.9}$$

By the Cauchy–Schwarz inequality,

$$B(u,v) \leq c \left\| u \right\|_{1,\Omega} \left\| v \right\|_{1,\Omega} \;\forall u,v \in H^1_*(\Omega), \tag{3.10}$$

so that (3.9) and (3.10) give (3.4). To verify (3.5), consider the following auxiliary problem (for which a unique solution exists): Find w such that

$$\Delta w = 0 \text{ in } \Omega, \tag{3.11}$$

$$\frac{\partial w}{\partial n} = \mu_h \text{ on } S_3, \tag{3.12}$$

$$w = 0 \text{ on } S_2, \tag{3.13}$$

$$\frac{\partial w}{\partial n} = 0 \text{ on } S_1 \cup S_4, \tag{3.14}$$

where $\mu_h \in V_2^h$ in (3.12). From (3.11) and (3.12) we obtain (using Green's formula and Poincaré's inequality)

$$\int_{S_3} \mu_h w = \int_{S_3} w \frac{\partial w}{\partial n} = \iint_\Omega w \Delta w + \iint_\Omega \left| \nabla w \right|^2 = \left| w \right|^2_{1,\Omega} \geq c_0 \left\| w \right\|^2_{1,\Omega}, \tag{3.15}$$

with $c_0 \in \mathbb{R}^+$. Also (cf. [2])

$$\left\| \mu_h \right\|_{-1/2,S_3} = \left\| \frac{\partial w}{\partial n} \right\|_{-1/2,S_3} \leq C \left\| w \right\|_{1,\Omega}, \tag{3.16}$$

so that by (3.15) and (3.16)

$$\int_{S_3} \mu_h w \geq c_0 \left\| w \right\|^2_{1,\Omega} \geq \beta \left\| w \right\|_{1,\Omega} \left\| \mu_h \right\|_{-1/2,S_3}, \tag{3.17}$$

with $\beta \in \mathbb{R}^+$ independent of w and h. Now, let $w_N \in V_1^N$ be such that $w = w_N + r_N$, as given by (3.8). We have

$$\int_{S_3} \mu_h w_N = \int_{S_3} \mu_h w - \int_{S_3} \mu_h r_N, \tag{3.18}$$

and also

$$\int_{S_3} \mu_h r_N \leq \left\| \mu_h \right\|_{-1/2,S_3} \left\| r_N \right\|_{1/2,S_3} \leq C_1 \left\| \mu_h \right\|_{-1/2,S_3} \left\| r_N \right\|_{1,\Omega}, \tag{3.19}$$

so that, combining (3.17)–(3.19), we get

$$\int_{S_3} \mu_h w_N \geq \beta \left\| w \right\|_{1,\Omega} \left\| \mu_h \right\|_{-1/2,S_3} - C_1 \left\| \mu_h \right\|_{-1/2,S_3} \left\| r_N \right\|_{1,\Omega}. \tag{3.20}$$

Now, using the reverse triangle inequality, we have

$$\left\| w \right\|_{1,\Omega} = \left\| w_N + r_N \right\|_{1,\Omega} \geq \left\| w_N \right\|_{1,\Omega} - \left\| r_N \right\|_{1,\Omega},$$

which along with (3.20) gives

$$\int_{S_3} \mu_h w_N \geq \beta \left(\|w_N\|_{1,\Omega} - \|r_N\|_{1,\Omega} \right) \|\mu_h\|_{-1/2,S_3} - C_1 \|\mu_h\|_{-1/2,S_3} \|r_N\|_{1,\Omega}$$

$$\geq \beta \|w_N\|_{1,\Omega} \|\mu_h\|_{-1/2,S_3} - (C_1 + \beta) \|\mu_h\|_{-1/2,S_3} \|r_N\|_{1,\Omega} . \tag{3.21}$$

Since by assumption (A1), r_N converges to 0 exponentially (or, equivalently, w_N converges to w exponentially), we have

$$\lim_{N\to\infty} \frac{\|r_N\|_{1,\Omega}}{\|w_N\|_{1,\Omega}} = 0,$$

which means that for any $\varepsilon > 0$ there exists N^* such that

$$\frac{\|r_N\|_{1,\Omega}}{\|w_N\|_{1,\Omega}} < \varepsilon$$

whenever $N > N^*$. Hence, for N sufficiently large we may write

$$\frac{\|r_N\|_{1,\Omega}}{\|w_N\|_{1,\Omega}} < \frac{\beta}{2\,(C_1 + \beta)}, \tag{3.22}$$

where C_1 and β are the constants from above. Combining (3.21) and (3.22) leads to

$$\int_{S_3} \mu_h w_N \geq \frac{\beta}{2} \|\mu_h\|_{-1/2,S_3} \|w_N\|_{1,\Omega} ,$$

which in turn gives (3.5) once we replace w_N by v_N and $\beta/2$ by β. Condition (3.6) follows directly from the definition of the $H^{-1/2}$-norm (see also (3.19)). The preceding discussion leads to the following theorem.

THEOREM 3.2. *Let* (u, λ) *and* (u_N, λ_h) *be the solutions to* (2.10) *and* (2.19), *respectively. If* $\lambda \in H^k(S_3)$ *for some* $k \geq 1$, *then there exists a positive constant* C, *independent of* N, h, *and* $a \in (0,1)$, *such that*

$$\|u - u_N\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,S_3} \leq C \left\{ \sqrt{N} a^N + h^m p^{-k} \right\}, \tag{3.23}$$

where $m = \min\{k, p+1\}$.

Proof. From Theorem 3.1 we have

$$\|u - u_N\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,S_3} \leq C \left\{ \inf_{v \in V_1^N} \|u - v\|_{1,\Omega} + \inf_{\eta \in V_2^h} \|\lambda - \eta\|_{-1/2,S_3} \right\}.$$

Now,

$$\inf_{v \in V_1^N} \|u - v\|_{1,\Omega} \leq \|u - w_N\|_{1,\Omega} = \|r_N\|_{1,\Omega} ,$$

with w_N, r_N given by (3.8). Using (A1) and (A2) we get

$$\inf_{v \in V_1^N} \|u - v\|_{1,\Omega} \leq \|r_N\|_{0,\Omega} + |r_N|_{1,\Omega} \leq C \left\{ a^N + \sqrt{N} a^N \right\} \leq C \sqrt{N} a^N, \tag{3.24}$$

with $C \in \mathbb{R}^+$ independent of N. Next, let λ_I be the pth-order interpolant of λ on the partition $\{\Gamma_i\}_{i=1}^n$ of S_3. We have (cf. [3])

$$\|\lambda - \lambda_I\|_{0,S_3} \le Ch^m p^{-k} \|\lambda\|_{k,S_3}, \tag{3.25}$$

where $m = \min\{k, p+1\}$, and $C > 0$ a constant independent of h and p. Now,

$$\inf_{\eta \in V_2^h} \|\lambda - \eta\|_{-1/2,S_3} \le \|\lambda - \lambda_I\|_{-1/2,S_3} \le \|\lambda - \lambda_I\|_{0,S_3}, \tag{3.26}$$

so that, since $\lambda \in H^k(S_3)$, (3.24)–(3.26) give the desired result. □

Remark 2. The above theorem shows that if the number of singular functions N is increased then u_N converges to u at an exponential rate. The theorem also shows that the convergence of λ_h to λ can occur in one of three ways: (i) by keeping p fixed and reducing h, (ii) by keeping h fixed and increasing p, or (iii) by doing both. These three "options" loosely correspond to the three versions of the FEM, namely, the h, p, and hp versions (cf. [3]).

Remark 3. Based on the above theorem, one may obtain the "optimal" matching between N and h, i.e., the relationship between the number of singular functions and the number of Lagrange multipliers used in the method, by choosing them in such a way so that the error in (3.23) is balanced. For example, in the case when p is kept fixed and $h \to 0$, we take $h^{p+1} \approx \sqrt{N} a^N$. This leads to the following approximate expression for N:

$$N \approx (p+1) \left| \frac{\ln h}{\ln a} \right|. \tag{3.27}$$

The approximation of the GSIFs is given by the following corollary.

COROLLARY 3.3. *Let*

$$u = \sum_{j=1}^{\infty} \alpha_j r^{\beta_j} \phi_j(\theta)$$

and

$$u_N = \sum_{j=1}^{N} \alpha_j^N r^{\beta_j} \phi_j(\theta)$$

satisfy (2.10) *and* (2.19), *respectively, with* α_j, α_j^N *denoting the true and approximate GSIFs. Then, there exists a positive constant* C, *independent of* N *and* $a \in (0,1)$, *such that*

$$\left|\alpha_j - \alpha_j^N\right| \le C a^N. \tag{3.28}$$

Proof. This is a direct consequence of (A1) and the fact that

$$\left|\alpha_j - \alpha_j^N\right| \le C \|u - u_N\|_{0,\Omega}.$$

(See also (3.24).) □

4. Implementation. We now give a description of the implementation in order to emphasize the properties of the method. As mentioned in Remark 1, the discretized equation will be solved on the boundary of the domain; here we describe how this is done. Recall the discrete version of the problem given by (2.19), which may be rewritten in *mixed* form as follows: Find $(u_N, \lambda_h) \in [V_1^N \times V_2^h]$ such that

$$\iint_\Omega \nabla v \cdot \nabla u_N - \int_{S_3} v\lambda_h = \int_{S_4} vg \;\forall\; v \in V_1^N, \tag{4.1}$$

$$\int_{S_3} \mu\,(u_N - f) = 0 \;\forall\; \mu \in V_2^h. \tag{4.2}$$

We wish to reduce the double integral in (4.1) to a boundary one. To this end, using Green's theorem, we have

$$\iint_\Omega \nabla v \cdot \nabla u_N = \int_{\partial\Omega} \frac{\partial v}{\partial n} u_N - \iint_\Omega u_N \Delta v = \int_{\partial\Omega} \frac{\partial v}{\partial n} u_N$$

by (2.7). Moreover, since $\frac{\partial v}{\partial n}\big|_{S_1} = 0$ by (2.7), and $u_N|_{S_2} = 0$ by (2.17), we get

$$\int_{\partial\Omega} \frac{\partial v}{\partial n} u_N = \int_{S_3} \frac{\partial v}{\partial n} u_N + \int_{S_4} \frac{\partial v}{\partial n} u_N,$$

and (4.1)–(4.2) may be written as follows: Find $(u_N, \lambda_h) \in [V_1^N \times V_2^h]$ such that

$$\int_{S_3} \frac{\partial v}{\partial n} u_N + \int_{S_4} \frac{\partial v}{\partial n} u_N - \int_{S_3} v\lambda_h = \int_{S_4} vg \;\forall\; v \in V_1^N, \tag{4.3}$$

$$\int_{S_3} \mu\,(u_N - f) = 0 \;\forall\; \mu \in V_2^h. \tag{4.4}$$

Obviously if $(u_N, \lambda_h) \in [V_1^N \times V_2^h]$ solves (2.19) (or (4.1)–(4.2)), then it also solves (4.3)–(4.4); it is straightforward to reverse the above steps and see that the two-dimensional formulation analyzed in the previous section is in fact equivalent to the one-dimensional formulation given by (4.3)–(4.4). The latter are the equations which we discretize, since they are posed on the boundary of the domain Ω. This reduces the dimension of the problem by one and leads to significant computational savings. Now, to obtain the linear system of equations corresponding to (4.3)–(4.4), we write

$$u_N = \sum_{j=1}^{N} \alpha_j^N v_j \in V_1^N, \tag{4.5}$$

and

$$\lambda_h = \sum_{i=1}^{M} \gamma_i \psi_i \in V_2^h, \tag{4.6}$$

with α_j^N and γ_k the unknowns in the system, and $V_1^N = span\,\{v_j\}_{j=1}^N$, $V_2^h = span\,\{\psi_i\}_{i=1}^N$. Upon inserting these into (4.3)–(4.4), we obtain the $(N+M)\times(N+M)$ linear system

$$\begin{bmatrix} K_1 & K_2 \\ K_2^T & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{\alpha} \\ \overrightarrow{\gamma} \end{bmatrix} = \begin{bmatrix} \overrightarrow{G} \\ \overrightarrow{F} \end{bmatrix}, \tag{4.7}$$

where $\overrightarrow{\alpha} = \left[\alpha_1^N, \ldots, \alpha_N^N\right]^T$, $\overrightarrow{\gamma} = [\gamma_1, \ldots, \gamma_M]^T$, and

$$[K_1]_{j,k} = \int_{S_3} v_k \frac{\partial v_j}{\partial n} + \int_{S_4} v_k \frac{\partial v_j}{\partial n}\ ,\ j = 1, \ldots, N,\ k = 1, \ldots, N,$$

$$[K_2]_{j,i} = -\int_{S_3} v_j \psi_i\ ,\ j = 1, \ldots, N,\ i = 1, \ldots, M,$$

$$\left[\overrightarrow{G}\right]_j = \int_{S_4} v_j g,\ j = 1, \ldots, N,$$

$$\left[\overrightarrow{F}\right]_i = \int_{S_3} f \psi_i,\ i = 1, \ldots, M.$$

It is easily shown that the coefficient matrix in (4.7) is symmetric. This matrix, however, is singular if $N < M$. Hence, we should choose N larger than M, but not too large since for excessively large values of N the linear system (4.7) becomes ill conditioned and the results obtained are unreliable. The relationship between N and h (hence M) described in Remark 2 should be our guide in deciding how large to choose these values; see also section 5. As a final remark in this section, we should point out that all integrals involved in the determination of the coefficient matrix (and right-hand side) in (4.7) are along the sides of the domain that *do not contain the singularity*. Moreover, they are one-dimensional and can be approximated by standard techniques, such as Gaussian quadrature.

5. Numerical results. Even though the analysis of the method was carried out for the rather general case of a domain having curved boundaries (except the ones intersecting at the singular point), and the nonhomogeneous boundary conditions can be given by any smooth function, we will consider for simplicity the model problem depicted in Figure 5.1, originally studied in [13]. The numerical results presented here correspond to the following choices of all relevant parameters: The Lagrange multiplier function λ_h used to impose the Dirichlet condition along S_3 is expanded in terms of *quadratic* basis functions ψ_i (see (4.6)), and boundary S_3 is divided into $2n$ quadratic elements of equal size. For the integration, boundary S_4 is also subdivided into n intervals of equal size. All integrals involved are calculated numerically by subdividing each interval above into 10 subintervals and using a 15-point Gauss–Legendre quadrature over each one. In computing the coefficient matrix in (4.7), its symmetry is taken into account.

To determine the relationship between the number of singular functions N and the number of Lagrange multipliers M, we proceed as follows: Since for λ_h we are using $p = 2$ and $h = 2/n$, we have $M = 2n + 1$. For the moment, we fix $n = 8$ (say), which amounts to $M = 17$, and solve the linear system (4.7) for various values of $N > M$ (e.g., $N = 19, 21, 23, \ldots$). We concentrate only on the calculation of the first GSIF α_1^N and record our results in Table 5.1. From the table we see that, for this choice for M, the value of α_1^N is converged up to 14 significant digits once $N = 35$. Moreover, from (3.27) we have

$$N \approx (p+1)\frac{\ln h}{\ln a} = (p+1)\frac{\ln(2/n)}{\ln a} = (p+1)\frac{\ln\left(\frac{2p}{M-1}\right)}{\ln a}$$

$$\Rightarrow \ln a \approx \frac{p+1}{N} \ln\left(\frac{2p}{M-1}\right) \Rightarrow a \approx \left(\frac{2p}{M-1}\right)^{\frac{p+1}{N}};$$

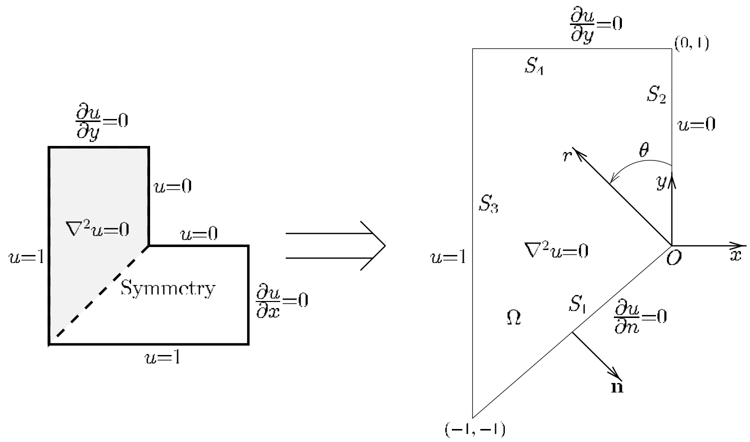

FIG. 5.1. *Geometry and boundary conditions for the model problem.*

TABLE 5.1
Approximate GSIF α_1^N with $M = 17$.

N	α_1^N
19	1.12797929883135
21	1.12798071689013
23	1.12798444682112
25	1.12798040013968
27	1.12798040107216
29	1.12798040105877
31	1.12798040105983
33	1.12798040105939
35	1.12798040105939

TABLE 5.2
Approximate GSIFs α_j^N, $j = 1, \ldots, 5$.

j	α_j^N (SFBIM)	α_j^N (hp-FEM)	α_j^N (Ref. [13])
1	1.12798040105939	1.12798010	1.1280
2	0.16993386650225	0.16993387	0.1699
3	−0.02304097399348	−0.0230419	−0.0230
4	0.0034711966582	0.0034755	0.0035
5	0.0009151570991	0.0009126	0.0009

hence, using $M = 17$ and $N = 35$, we find that $a \approx 0.89$. With a known, we may now compute the rest of the GSIFs and/or any other quantities of interest by choosing N and M via (3.27). For example, for $M = 41$ (i.e., $h = 1/10$), we find that $N \approx 60$.

Table 5.2 above shows the converged approximate values for the first five GSIFs obtained using the SFBIM (with $M = 41$ and $N = 60$), as well as the hp version of the FEM, which is considered to be the state-of-the-art method for problems with singularities [25]; the values from reference [13] are also included for comparison.

These results suggest that the SFBIM can be an attractive (and often preferable) method for problems in which the GSIFs are the main goal of the computation. Figure 5.2 shows the convergence of the leading GSIFs with N; in particular, the figure shows a semilog plot of the estimated percentage relative error in α_j^N, $j = 1, \ldots, 5$; since no

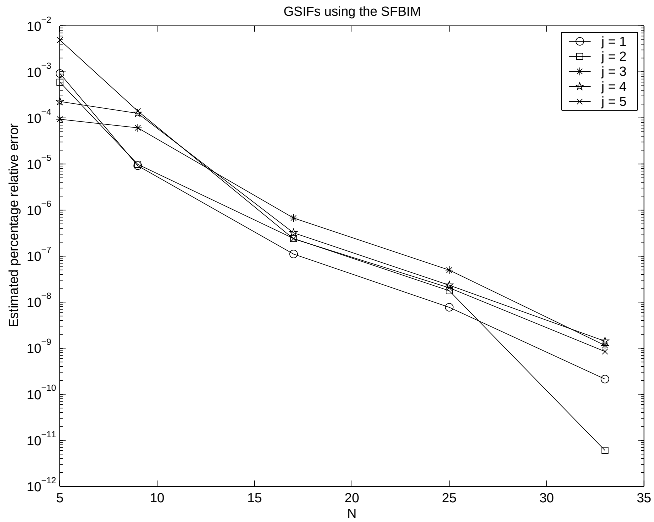

FIG. 5.2. *Convergence of* α_j^N, $j = 1, \ldots, 5$, *using the SFBIM.*

exact values exist for the true GSIFs, we use α_j^{60} as our reference/"exact" value. Since each error curve is essentially straight, this illustrates the exponential convergence of the method, as predicted by Corollary 3.3.

Appendix. Proof of Theorem 3.1. Here, for completeness, we give a proof of Theorem 3.1. First, we note that since (u, λ) also satisfy (2.19), we have $\forall\ (v, \mu) \in V_1^N \times V_2^h$

$$B(u - u_N, v) = -b(u - u_N, v; \lambda - \lambda_h, \mu) = \int_{S_3} v(\lambda - \lambda_h) + \int_{S_3} \mu(u - u_N). \tag{A.1}$$

Since $u = f$ on S_3 and $\int_{S_3} \mu(u_N - f) = 0\ \forall\ \mu \in V_2^h$, we have

$$\int_{S_3} \mu u_N = \int_{S_3} \mu u\ \forall \mu \in V_2^h, \tag{A.2}$$

and thus the last integral in (A.1) is zero. Hence,

$$B(u - u_N, v) = \int_{S_3} v(\lambda - \lambda_h). \tag{A.3}$$

Letting $w = u_N - v, v \in V_1^N$, we get

$$\begin{aligned} B(u_N - v, w) = B(u_N - u, w) + B(u - v, w) &= B(u - v, w) - \int_{S_3} w(\lambda - \lambda_h) \\ &= B(u - v, w) - \int_{S_3} w(\lambda - \eta) - \int_{S_3} w(\eta - \lambda_h). \end{aligned}$$

By means of (A.2), the above equation becomes

$$B(u_N - v, w) = B(u - v, w) - \int_{S_3} w(\lambda - \eta) - \int_{S_3} (\lambda_h - \eta)(u - v). \tag{A.4}$$

Now, from (3.4), (3.6), and (A.4) we get

$$\begin{aligned} c_0 \|u_N - u\|_{1,\Omega}^2 &= c_0 \|w\|_{1,\Omega}^2 \\ &\leq \|u-v\|_{1,\Omega} \|w\|_{1,\Omega} + \gamma \|w\|_{1,\Omega} \|\lambda - \eta\|_{-1/2,S_3} + \gamma \|u-v\|_{1,\Omega} \|\lambda_h - \eta\|_{-1/2,S_3} \\ &\leq C\big\{ (\|u-v\|_{1,\Omega} + \|\lambda-\eta\|_{-1/2,S_3}) \|w\|_{1,\Omega} + \|u-v\|_{1,\Omega} \|\lambda_h - \eta\|_{-1/2,S_3} \big\}. \end{aligned}$$

This is an inequality of order 2:

$$c_0 x^2 \leq bx + d\ ,\ b, d > 0,$$

where

$$x = \|w\|_{1,\Omega}\ ,\ b = C(\|u-v\|_{1,\Omega} + \|\lambda - \eta\|_{-1/2,S_3}),$$

$$d = C\|u-v\|_{1,\Omega} \|\lambda_h - \eta\|_{-1/2,S_3} \leq \frac{C}{2}\left(\varepsilon \|\lambda_h - \eta\|_{-1/2,S_3} + \frac{1}{\varepsilon}\|u-v\|_{1,\Omega} \right)^2,\ \varepsilon > 0.$$

Therefore, we obtain the bound

$$x \leq \frac{b + \sqrt{b^2 + 4c_0 d}}{2c_0}$$

or, equivalently,

(A.5)
$$\|w\|_{1,\Omega}\ \leq C\left\{ \|u-v\|_{1,\Omega} + \|\lambda-\eta\|_{-1/2,S_3} + \frac{1}{\varepsilon}\|u-v\|_{1,\Omega} \right\} + C\varepsilon \|\lambda_h - \eta\|_{-1/2,S_3}.$$

Next, using (3.5) with $\mu_h = \lambda_h - \eta$, we find that there exists $0 \neq v_N \in V_1^N$ such that

$$\|\lambda_h - \eta\|_{-1/2,S_3} \leq \beta \frac{\left|\int_{S_3} (\lambda_h - \eta) v_N\right|}{\|v_N\|_{1,\Omega}}. \tag{A.6}$$

Also, it follows from (A.3) that

$$\begin{aligned} \int_{S_3} (\lambda_h - \eta) v_N &= \int_{S_3} (\lambda_h - \lambda) v_N + \int_{S_3} (\lambda - \eta) v_N = B(u - u_N, v_N) + \int_{S_3} (\lambda - \eta) v_N \\ &\leq c\|u - u_N\|_{1,\Omega} \|v_N\|_{1,\Omega} + \gamma \|v_N\|_{1,\Omega} \|\lambda - \eta\|_{-1/2,S_3}, \end{aligned} \tag{A.7}$$

so that by (A.6) and (A.7),

$$\begin{aligned} \|\lambda_h - \eta\|_{-1/2,S_3} &\leq C\left\{ \|u - u_N\|_{1,\Omega} + \|\lambda - \eta\|_{-1/2,S_3} \right\} \\ &\leq C\left\{ \|u-v\|_{1,\Omega} + \|v - u_N\|_{1,\Omega} + \|\lambda-\eta\|_{-1/2,S_3} \right\}. \end{aligned}$$

Since $\|v - u_N\|_{1,\Omega} = \|w\|_{1,\Omega}$, we have from (A.5)

$$\begin{aligned} \|\lambda_h - \eta\|_{-1/2,S_3} \leq C_1 &\left\{ \|u-v\|_{1,\Omega} + \|\lambda - \eta\|_{-1/2,S_3} + \varepsilon^{-1}\|u-v\|_{1,\Omega} \right\} \\ &+ \varepsilon \|\lambda_h - \eta\|_{-1/2,S_3} \end{aligned}$$

with $C_1 \in \mathbb{R}^+$ independent of v, η, and ε. Letting $C_1\varepsilon \leq 1/2$ we get

$$\|\lambda_h - \eta\|_{-1/2,S_3} \leq C\left\{\|u - v\|_{1,\Omega} + \|\lambda - \eta\|_{-1/2,S_3}\right\}. \tag{A.8}$$

Using the triangle inequality, we immediately get

$$\begin{aligned}\|\lambda - \lambda_h\|_{-1/2,S_3} &\leq \|\lambda - \eta\|_{-1/2,S_3} + \|\lambda_h - \eta\|_{-1/2,S_3}\\ &\leq C\left\{\|u - v\|_{1,\Omega} + \|\lambda - \eta\|_{-1/2,S_3}\right\}.\end{aligned} \tag{A.9}$$

Similarly, using (A.5) and (A.9), we get

$$\begin{aligned}\|u - u_N\|_{1,\Omega} &\leq \|u - v\|_{1,\Omega} + \|v - u_N\|_{1,\Omega} = \|u - v\|_{1,\Omega} + \|w\|_{1,\Omega}\\ &\leq C\left\{\|u - v\|_{1,\Omega} + \|\lambda - \eta\|_{-1/2,S_3}\right\},\end{aligned}$$

which gives the desired result.

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