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# The role of Cheeger sets in the steady flows of viscoplastic fluids in pipes: A survey

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**Note:** This paper is part of the special topic, Tanner: 90 Years of Rheology.
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## ABSTRACT

Given a convex or a Jordan domain $\Omega$, let $\Omega'$ be a subset of this domain, with $P(\Omega')$ denoting its perimeter and $A(\Omega')$ its area. If a subset $\Omega_c$ exists such that $h = P(\Omega_c)/A(\Omega_c)$ is a minimum, the subset $\Omega_c$ is called the Cheeger set of $\Omega$ and $h$, the Cheeger constant of the given domain. If one considers the reciprocal of this minimum or the maximum ratio of the area of the subset to its perimeter, $t^* = 1/h$. It follows from the work of Mosolov and Miasnikov that the minimum pressure gradient $G$ to sustain the steady flow of a viscoplastic fluid in a pipe, with a cross section defined by $\Omega$, is given by $G > \tau_y/t^*$, where $\tau_y$ is the constant yield stress of the fluid. In this survey, we summarize several results to determine the constant $h$ when the given domain is self-Cheeger or a Cheeger-regular set that touches each boundary of a convex polygon and when the Cheeger-irregular set does not do so. The determination of the constant $h$ for an arbitrary ellipse, a strip, and a region with no necks is also mentioned.





## I. INTRODUCTION

The Cheeger set $\Omega_c$ of a domain $\Omega$ in a plane is the one that minimizes the ratio of its perimeter to its area among all subsets $\Omega'$ of $\Omega$. It arose in the work of Cheeger,[1] who proved a lower bound for the smallest eigenvalue for the Laplacian under Dirichlet conditions in 1970. If $\Omega$ is convex, the Cheeger set exists and is unique.[2] It touches the boundary, although not everywhere. In places where $\Omega_c$ does not touch the boundary $\partial\Omega$, its free boundary $\partial\Omega_c \cap \Omega$ has constant curvature, or it is the arc of a circle. Moreover, this free boundary touches the boundary of $\Omega$ tangentially. Hence, to determine the Cheeger set of a given domain, it suffices to consider those subsets whose free boundary consists of circular arcs. For example, see the various figures in Ref. 3 and Fig. 9.1 in Ref. 4. The ratio of the perimeter to the area of $\Omega_c$, denoted by $h$, is called the *Cheeger constant* of $\Omega$.

Much earlier in 1965, Mosolov and Miasnikov[5] had proved that the minimum pressure gradient, $G$, for a steady flow of a Bingham fluid to exist in a pipe of a given cross section was determined by the maximum ratio of the area of a subset of the domain to its perimeter. This value, $t^*$, is clearly given by $t^* = 1/h$. In addition, they also proved[5] that $G > \tau_y/t^*$, where $\tau_y$ is the yield stress of the Bingham fluid. In fact, this result of Mosolov and Miasnikov[5] remains valid for all viscoplastic fluids provided the yield stress is constant, as shown in Sec. II.

Hence, in order to solve the minimum pressure gradient problem, it suffices to determine the Cheeger constant of the domain defining the cross section of a pipe. Numerous publications have appeared to determine the Cheeger constant since 1970, and we describe some of them here when the given domain is convex or the inclusive, Jordan domain. In doing so, we rely on the results of Kawohl and Lachand-Robert,[2] Leonardi and Pratelli,[6] Leonardi *et al.*,[7] Leonardi and Saracco,[8] and Saracco.[9] Applications include the determination of the Cheeger constant when the domain is itself, i.e., self-Cheeger;[2,9] when the Cheeger set touches each side of a convex polygon and is called Cheeger-regular;[2] when it does not do so and is called Cheeger-irregular.[2] Additionally, we mention the determination of the constant $h$ for an arbitrary ellipse,[10] a strip,[6] and a domain with no necks.[7]

Next, some new results concerning the existence of stagnant zones in the flow of viscoplastic fluids in a pipe, the upper and lower bounds on the size, and the velocity of the rigid core are derived in Sec. III. In Sec. IV, a summary of the methods to determine the Cheeger constant for a convex domain is provided with applications to an ellipse[10] in Sec. V and a Cheeger-regular convex polygon[2] in Sec. VI.

In Sec. VII, we look at the specific example of a trapezium inscribed in an equilateral triangle and describe how it can be Cheeger-regular, i.e., its Cheeger set touches each one of its sides, and how it can be turned into a Cheeger-irregular set by increasing its size. Conversely, this example illustrates how one can begin with a Cheeger-irregular convex polygon and turn it into a regular one using the algorithm due to Kawohl and Lachand-Robert.[2] Next, in Sec. VIII, we summarize the determination of the Cheeger constants for Jordan domains, such as strips,[6,11] domains with no necks,[7,8] and domains that are self-Cheeger.[9] Finally, we draw attention to domains where the Cheeger set is not unique, such as the Pinocchio set,[6,9] a face with two ears, and a dumbbell with a thin, long arm.[6]

In sum, apart from recalling some old results, the goal of this survey is to show that the minimum pressure gradient problem in viscoplastic fluids can be solved for a wider class of ducts.

## II. THE MINIMUM PRESSURE DROP PER UNIT LENGTH TO INITIATE A FLOW IN A PIPE

Consider a pipe of arbitrary cross section defined through $\Omega$ in $(x, y)$ coordinates, with its boundary defined by $\partial\Omega$. Suppose that a steady axial flow of an incompressible viscoplastic fluid exists in this pipe with the velocity field defined through

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = w(x, y) \geq 0, \quad w(x, y)|_{\partial\Omega} = 0. \tag{2.1}$$

This flow is assumed to occur under a constant pressure drop, i.e., $\partial p/\partial z = -G$, $G > 0$. The components of the relevant Rivlin–Ericksen tensor[12] for the velocity field in Eq. (2.1) are given by

$$A_{13} = A_{31} = w_{,x} \quad A_{23} = A_{32} = w_{,y}, \tag{2.2}$$

where the commas denote the respective partial derivatives. It is also known that $A_{11} = A_{22} = A_{33} = A_{12} = A_{21} = 0$.

Hence, only two shear stresses exist in the yielded region. They are

$$\begin{aligned} S_{xz} = S_{zx} &= \eta(|\nabla w|)w_{,x} + \frac{\tau_y}{|\nabla w|}w_{,x}, \\ S_{yz} = S_{zy} &= \eta(|\nabla w|)w_{,y} + \frac{\tau_y}{|\nabla w|}w_{,y}, \end{aligned} \tag{2.3}$$

since

$$II(\mathbf{A}) = |\nabla w| \tag{2.4}$$

and

$$\nabla w = w_{,x}\mathbf{i} + w_{,y}\mathbf{j}. \tag{2.5}$$

In Eq. (2.3), $\eta(|\nabla w|)$ is the shear rate dependent viscosity and $\tau_y$ is the aforementioned yield stress. Obviously, shear stresses exist in the unyielded regions as well, and in them, they obey the following inequality:

$$0 \leq II(\mathbf{S}) = \left[S_{xz}^2 + S_{yz}^2\right]^{1/2} \leq \tau_y. \tag{2.6}$$

The main problem for the flow in a pipe of arbitrary cross section may now be posed: Is there a minimum pressure drop per unit length to initiate the flow? The answer to this question can be found in Lemmas 2.2 and 2.3, proved by Mosolov and Miasnikov.[5] In order to apply these Lemmas, one has to begin with the energy equation for the flow of a viscoplastic fluid in a pipe of arbitrary cross section.

To derive the energy equation for a viscoplastic fluid, we begin with the velocity potential, $\phi(\xi)$. This function satisfies the following conditions [see Eqs. (8.1.6) and (8.1.7) in Ref. 4]:

$$\phi(\xi) = 0, \quad \xi = 0, \tag{2.7}$$

$$\frac{d\phi}{d\xi} = \eta(\xi)\xi + \tau_y, \quad \xi > 0. \tag{2.8}$$

Thus,

$$\phi(\xi) = \int_0^{\xi} \left[\eta(\zeta)\zeta + \tau_y\right] d\zeta. \tag{2.9}$$

For convenience, let

$$\int_0^{\xi} \eta(\zeta)\zeta \, d\zeta = V(\xi), \tag{2.10}$$

where $V(\xi)$ is the viscous power. Using this, the energy equation now becomes

$$\int_\Omega V(|\nabla w|)\, da + \tau_y \int_\Omega |\nabla w|\, da = G\int_\Omega w\, da. \tag{2.11}$$

The basic idea is to turn the above into an inequality by replacing the right side with an upper bound, as discovered by Mosolov and Miasnikov.[5]

1. Since the velocity field vanishes on the boundary of a pipe, we consider the following class of smooth functions only:

$$f(x, y)|_{\partial\Omega} = 0. \tag{2.12}$$

Then, there is a subset $\Omega' \subseteq \Omega$ such that

$$t^* \int_\Omega |\nabla f|\, da \geq \int_\Omega f\, da, \tag{2.13}$$

where

$$t^* = \sup_{\Omega' \subseteq \Omega} \frac{A(\Omega')}{P(\Omega')}. \tag{2.14}$$

Here, $A(\Omega')$ is the area of this sub-domain, and $P(\Omega')$ is its perimeter. Employing Eqs. (2.13) and (2.14) on the right side of Eq. (2.11) results in the following inequality:

$$\int_\Omega V(|\nabla w|)\, da + \tau_y \int_\Omega |\nabla w|\, da \leq Gt^* \int_\Omega |\nabla w|\, da. \tag{2.15}$$

2. Obviously, there will be an infinite number of sub-domains of $\Omega$. Let the subset where $t^*$ is attained be denoted by $\Omega_c$, which is called the Cheeger set.[1] Mosolov and Miasnikov[5] proved that, in a Bingham fluid, if $\partial\Omega_c$ meets $\partial\Omega$, it must do so tangentially with a circular arc of radius $t^*$. It will be seen below in the section on Cheeger sets that this result is valid for all viscoplastic fluids.

The constant $t^*$ is important, for it determines the minimum pressure gradient to initiate the flow of a viscoplastic fluid in a pipe. To see this, one rewrites Eq. (2.15) as

$$\int_\Omega V(|\nabla w|)\,da \le (Gt^* - \tau_y)\int_\Omega |\nabla w|\,da. \tag{2.16}$$

If $Gt^* - \tau_y \le 0$, it follows that $V(|\nabla w|) \le 0$. Since the viscosity $\eta(|\nabla w|) > 0$, one deduces from Eq. (2.10) that $|\nabla w| = 0$. That is, the flow must have a constant velocity across the cross section of the pipe. However, the velocity field $w = 0$ on the boundary of the pipe, which means that $w = 0$ in the pipe. Hence, we conclude that a steady flow of the viscoplastic fluid will exist in a pipe provided the pressure drop per unit length $G$ satisfies the following inequality:

$$G > \frac{\tau_y}{t^*}. \tag{2.17}$$

Thus, to find this value of $G$, one must determine $t^*$ for a given cross section. Equivalently, one must find its Cheeger set.

## III. STAGNANT ZONES AND THE MOVING RIGID CORE

As shown above, Eq. (2.17) determines a lower bound on the pressure drop per unit length, $G$, in order for a steady flow to exist. Physically speaking, when the flow is steady, the boundary $\partial\Omega_c$ is one part of the yield surface with the stagnant zones lying outside it. The other part of the yield surface surrounds the core, moving as a rigid body in the interior.

Here, we shall derive a restriction on $G$ for the initiation of a flow. Obviously, when the flow commences, the magnitude of the shear stress $\tau$ on the boundary $\partial\Omega$ will be less than or equal to the yield stress $\tau_y$. Considering a pipe of unit length, the flow will not commence if the pressure drop per unit length $G$ cannot overcome this shearing force. That is, if

$$\tau_y P(\Omega) > GA(\Omega), \tag{3.1}$$

the fluid cannot move. Hence, a stagnant zone will exist near the boundary of the pipe.

Next, consider two different pressure drops per unit length $G_1 > G_2$, with the respective velocity fields $w_1(x,y)$ and $w_2(x,y)$ vanishing on the boundary of the pipe. Physically speaking, one would expect that $w_1(x,y) \ge w_2(x,y)$, with the stagnant zones of the former lying outside that of the latter. This result was proved by Mosolov and Miasnikov[13] for Bingham fluids, and we shall extend it to other viscoplastic fluids. To begin, consider the following functional:

$$J_i(w) = \int_\Omega \{V(|\nabla w|) + \tau_y|\nabla w| - G_i w\}\,da, \quad i = 1, 2. \tag{3.2}$$

It is well known from the velocity minimization principle[4] that $w_1 = w_1(x,y)$ minimizes $J_1(w)$, while $w_2 = w_2(x,y)$ minimizes $J_2(w)$. Hence,

$$J_1(w_1) \le J_1(w_2), \quad J_2(w_2) \le J_2(w_1). \tag{3.3}$$

These inequalities lead to

$$\int_\Omega \{V(|\nabla w_1|) + \tau_y|\nabla w_1|\}\,da \le \int_\Omega \{V(|\nabla w_2|) + \tau_y|\nabla w_2|\}\,da + \int_\Omega G_1(w_1 - w_2)\,da \tag{3.4}$$

and

$$\int_\Omega \{V(|\nabla w_2|) + \tau_y|\nabla w_2|\}\,da \le \int_\Omega \{V(|\nabla w_1|) + \tau_y|\nabla w_1|\}\,da + \int_\Omega G_2(w_2 - w_1)\,da. \tag{3.5}$$

Adding the two, we obtain

$$\int_\Omega (G_1 - G_2)(w_1 - w_2)\,da \ge 0. \tag{3.6}$$

Hence, if $G_1 > G_2$, it is clear that $w_1 \ge w_2$. Now, consider the stagnant zone of $w_2 = w_2(x,y)$, where $w_2 = 0$. In this part of the flow domain, the velocity field $w_1 = w_1(x,y) \ge 0$, which means that the stagnant zones of $w_1$ must lie outside that of $w_2$, as mentioned earlier.

Turning to the size of the moving rigid core when the domain of flow is simply connected, it is known that the magnitude of the shear stress on the boundary of the core is equal to the yield stress, $\tau_y$. Let the area of the core be $F$ and $L$ its perimeter, where we use the notation employed in isoperimetric inequalities.[14] From the usual force balance of the steady flow of the core under a pressure drop per unit length, $G$, one finds that

$$GF = \tau_y L. \tag{3.7}$$

One can derive an upper bound to the size of the core as follows. Let $R$ be the radius of the largest circle that can be inscribed within the core; such a circle is called an *incircle*. On the circumference of this circle, the magnitude of the shear stress $\tau \le \tau_y$. Thus,

$$\pi R^2 G \le 2\pi R \tau_y \tag{3.8}$$

from which we find that an upper bound, $R_2$, to the radius of the largest circle, $R$, is given by

$$R \le R_2 = \frac{2\tau_y}{G}. \tag{3.9}$$

To obtain a lower bound to $R$, one appeals to the following isoperimetric inequality:

$$R \ge \frac{2F}{L + \sqrt{L^2 - 4\pi F}}. \tag{3.10}$$

A simple proof of Eq. (3.10) can be found in Sec. 1.3.4 of the monograph by Burago and Zalgaller.[14] The bound in Eq. (3.10) is difficult to determine, for the area $F$ and the perimeter $L$ of the core are both unknown. However, it is possible to circumvent this problem as follows. From Eq. (3.7), it follows that

$$\frac{2F}{L + \sqrt{L^2 - 4\pi F}} = \frac{2\tau_y}{G}\frac{1}{1 + \sqrt{1 - [4\pi F/L^2]}}. \tag{3.11}$$

Next,

$$1 - [4\pi F/L]^2 = 1 - \left[4\pi\tau_y^2/G^2 F\right] \le 1 - \left[4\pi\tau_y^2/G^2 A(\Omega)\right], \tag{3.12}$$

since the area of the core $F \le A(\Omega)$, where $A(\Omega)$ is the area of the cross section of the pipe. Thus, we obtain the result due to Mosolov and Miasnikov:[13]

$$R \ge R_1 = \frac{2\tau_y}{G}\frac{1}{1 + \sqrt{1 - \left[4\pi\tau_y^2/G^2 A(\Omega)\right]}} > \frac{\tau_y}{G}. \tag{3.13}$$

Hence, the radius of the core, $R$, satisfies the inequality $R_1 \le R \le R_2$.

Turning next to the maximum velocity $w_c$ of the core, it is easy to see that the flow rate through the core of area $F$ satisfies the following inequality:

$$Fw_c \le Q = \int_\Omega w\, da, \tag{3.14}$$

where $Q$ is the flow rate through the pipe. Using the Cauchy–Schwarz inequality, one finds that

$$\int_\Omega |\nabla w|\, da \le \left(\int_\Omega |\nabla w|^2\, da\right)^{1/2} A(\Omega)^{1/2}, \tag{3.15}$$

where $A(\Omega)$ is the area of the cross section of the pipe.

While an upper bound on $w_c$ cannot be obtained through Eq. (3.14) for all viscoplastic fluids, it is possible to obtain one for a Bingham fluid with a constant viscosity, $\eta_0$. For such a fluid, Eq. (2.16) takes the simple form:

$$\frac{1}{2}\eta_0 \int_\Omega |\nabla w|^2\, da \le (Gt^* - \tau_y)\int_\Omega |\nabla w|\, da, \tag{3.16}$$

where $(Gt^* - \tau_y) > 0$ from Eq. (2.17). Next, employing Eq. (3.15), we find that

$$\frac{1}{2}\frac{\eta_0}{A(\Omega)}\left(\int_\Omega |\nabla w|\, da\right)^2 \le (Gt^* - \tau_y)\int_\Omega |\nabla w|\, da, \tag{3.17}$$

whence,

$$\int_\Omega |\nabla w|\, da \le 2A(\Omega)\frac{Gt^* - \tau_y}{\eta_0}. \tag{3.18}$$

Now, from Eq. (2.13), we see that

$$\int_\Omega w\, da \le t^* \int_\Omega |\nabla w|\, da. \tag{3.19}$$

Combining Eqs. (3.14), (3.18), and (3.19), we obtain a result slightly better than that in Ref. 5

$$w_c \le \frac{2A(\Omega)}{F}\frac{(Gt^* - \tau_y)t^*}{\eta_0} \le \frac{2A(\Omega)}{\pi R_1^2}\frac{(Gt^* - \tau_y)t^*}{\eta_0}. \tag{3.20}$$

## IV. CHEEGER SETS FOR CONVEX CROSS SECTIONS

As mentioned earlier, the set $\Omega_c$ is also known as a Cheeger set. It arose in the work of Cheeger,[1] who proved a lower bound for the smallest eigenvalue, $\lambda_1$, for the Laplacian under Dirichlet boundary conditions. Let $h(\Omega)$ be defined through

$$h(\Omega) = \inf_{\Omega' \subseteq \Omega} \frac{P(\Omega')}{A(\Omega')}, \tag{4.1}$$

provided it exists. Here, $P(\Omega')$ is the perimeter, and $A(\Omega')$ is the area of the subset $\Omega' \subseteq \Omega$, as defined earlier in Eq. (2.14). In the sequel, whenever the Cheeger set exists, we shall denote it by $\Omega_c$.

The number, $h(\Omega)$ in Eq. (4.1), is called the *Cheeger constant of* $\Omega$. Cheeger's result[1] that $\lambda_1 \ge h(\Omega)^2/4$ is not of relevance to the contents of this paper, whereas the constant $h(\Omega)$ is crucial, since $t^* = 1/h(\Omega)$ [see Eqs. (2.14) and (4.1)]. To be specific, when $\Omega$ is *convex*, it is known that the Cheeger set $\Omega_c$ exists. Hence, the determination of $h$ for a given convex set is of importance here.

In order to determine $h(\Omega)$ for convex sets in the plane, we shall describe the two theorems proved by Kawohl and Lachand-Robert[2] to determine their Cheeger sets. Some modifications in the wording of the theorems occur due to the subsequent work of Leonardi and Partelli,[6] who extended these theorems to nonconvex sets (see Sec. VIII). Whereas Theorem 1 of this section applies to convex polygons as well, a different and more specific procedure is to be found in Sec. V.

For any given convex set $\Omega$ in the plane, we denote by dist $(x, \partial\Omega)$ the distance from $x \in \Omega$ to a point on the boundary $\partial\Omega$. For any $t \ge 0$, we denote the points of distance at least $t$ by

$$\Omega^t := \{x \in \Omega |\ \mathrm{dist}(x, \partial\Omega) > t\}. \tag{4.2}$$

So, the boundary $\partial\Omega^t$ is the inner parallel set to $\partial\Omega$ at a distance $t$ at least. We have the following:

**Theorem 1**: *There exists a unique value* $t = t^* > 0$ *such that* $|\Omega^t| = \pi t^2$. *Then, the unique Cheeger set* $\Omega_c$ *of* $\Omega$ *is the union of all balls of radius* $t^* = 1/h(\Omega)$. *Equivalently,*

$$\Omega_c = \bigcup_{x\in\Omega^{t^*}} B_{t^*}(x), \quad |\Omega^{t^*}| = \pi t^{*2}. \tag{4.3}$$

The set $\Omega^{t^*}$ is parallel to the set $\Omega$ and lies inside of it, at a distance of $t^*$ at least from the boundary $\partial\Omega$. This set $\Omega^{t^*}$ is called the *inner Cheeger set* of $\Omega$.[2] The ball, of radius $t^*$, may have its center on the boundary $\partial\Omega^{t^*}$. When it touches the boundary $\partial\Omega$, this disk is tangential to it.

As an example, we apply Theorem 1 to the case when $\Omega$ is a square with a side of length $a$. It is easy to see that the set $\Omega^{t^*}$ is also a square, with its sides parallel to the boundary $\partial\Omega$ and placed symmetrically inside. This inner square has a side of length $\sqrt{\pi}\, t^*$. The disk has a radius of length $t^*$. Hence, from Fig. 1, we see that the Cheeger set $\Omega_c$ is the union of $\Omega^{t^*}$ and the set generated by the union of a rolling disk of radius $t^*$ touching the boundary $\partial\Omega$, rounding off at the corners. It is essential to realize that the Cheeger set touches each side of the square.

Returning to Fig. 1, we see quite easily that

$$\left(2 + \sqrt{\pi}\right)t^* = a. \tag{4.4}$$

It is known from the work of Mosolov–Miasnikov that the value of $t^* = a/\left(2 + \sqrt{\pi}\right)$ (see Refs. 3–5). The calculation of $t^*$ from Theorem 1 is much more direct, however.

If the domain $\Omega$ is convex and the curvature of its boundary is not finite, such as in the case of an ovoid, Theorem 1 can be applied to find its Cheeger set (see Fig. 1 in Ref. 2).

Suppose that the set $\Omega^{t^*}$ is difficult to find for a given convex domain $\Omega$. Here, Theorem 2 from Ref. 2 is helpful in some cases. This is stated as follows:

**Theorem 2**: *Let* $\Omega$ *be any convex set* and $\bar{\kappa}$ *the maximum value of its curvature. Then* $\Omega_c = \Omega$ *if and only if*

$$\bar{\kappa}|\Omega| \le |\partial\Omega|. \tag{4.5}$$

That is, the Cheeger set of $\Omega$ is itself, or self-Cheeger, provided the product of $\bar{\kappa}$ and its area, $|\Omega|$, is less than or equal to its perimeter $|\partial\Omega|$. It has been shown by Saracco[9] that under certain conditions, a Jordan domain can be self-Cheeger. These results are discussed in Sec. VIII.

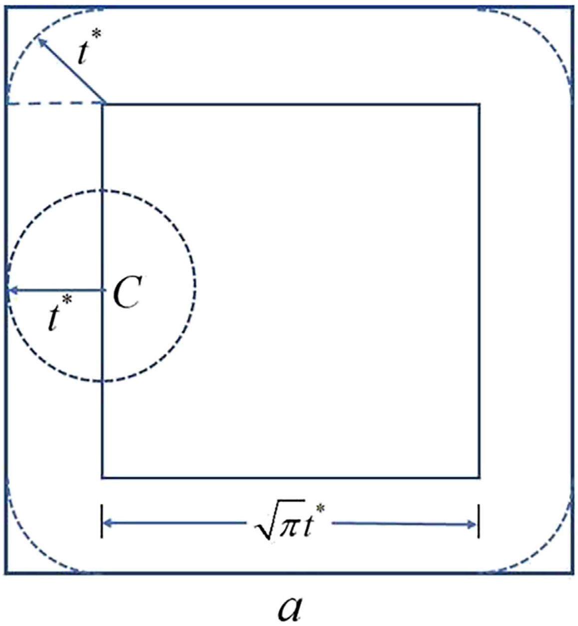


**FIG. 1.** Cheeger set for a square.

This theorem can be applied to find the Cheeger constant $h$ for a restricted class of ellipses as follows. First of all, one observes that the boundary of an ellipse does not have a constant curvature, for it varies along its boundary. Indeed, let the ellipse have a semi-major axis of length $a$ and a semi-minor axis of length $b < a$. The maximum curvature occurs at the endpoints of the major axis and is given by $\bar{\kappa} = a/b^2$.

The area of an ellipse has a simple result $|\Omega| = \pi ab$. Its eccentricity $k$ is given by $k = \sqrt{a^2 - b^2}/a$. The perimeter $|\partial\Omega|$ is defined through the following integral:

$$|\partial\Omega| = 4a\int_0^{\pi/2}\sqrt{1 - k^2\sin^2\xi}\,d\xi, \quad 0 \le k < 1, \tag{4.6}$$

which is an elliptical integral of the second kind

$$E(k, \pi/2) = \int_0^{\pi/2}\sqrt{1 - k^2\sin^2\xi}\,d\xi. \tag{4.7}$$

Since $b = a\sqrt{1 - k^2}$, one finds from Eq. (4.4) that whenever

$$E(k, \pi/2) \ge \frac{\pi}{4\sqrt{1 - k^2}}, \tag{4.8}$$

the Cheeger set of the ellipse is itself, leading to $h = |\partial\Omega|/|\Omega|$. Numerical computations of $E(k, \pi/2)$ for various values of $k$ can be found in Ref. 4, and one finds that Eq. (4.7) holds provided $0 \le k < 0.791\,17$, or $(b/a) > 0.6116$, confirming the result in Ref. 2.

**Remark**: Regardless of the exact value of $b/a$, every elliptical cross section is convex, and Theorem 1 determines its Cheeger set. Thus, as $b/a$ decreases from its maximum value of 1, the area of the inner Cheeger set of the ellipse is given by $\pi t^{*2}$, and the distance from its boundary to that of the surrounding ellipse is at least $t^*$. This *squeeze* causes the ellipse to lose its self-Cheeger property as $b/a$ decreases. Numerical modeling of the ellipses with $b/a$ decreasing from 0.61115 to 1/30 in Ref. 10 confirms this observation.

In Sec. V, we shall derive the shape of the Cheeger set for an arbitrary ellipse.

## V. CHEEGER SET FOR AN ELLIPSE

Let the equation of an ellipse be given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0. \tag{5.1}$$

Referring to Fig. 2, which is based on Fig. 7 in Ref. 3, we consider a point $Q$ on the ellipse in the first quadrant. Here, it is simpler to work in terms of the eccentric angle $\theta$.[15] The coordinates of the point $Q$ are $(a\cos\theta, b\sin\theta)$. The area under the ellipse in $0 \le x \le a\cos\theta$ is given by

$$E(\theta) = \frac{ab}{4}\int_\theta^{\pi/2}\sin^2\xi\,d\xi = \frac{ab}{4}[\pi - 2\theta + \sin 2\theta]. \tag{5.2}$$

The length $L(\theta)$ of the arc is given by

$$L(\theta) = a\int_0^{\pi/2-\theta}\sqrt{1 - k^2\sin^2\xi}\,d\xi, \tag{5.3}$$

where $k$ is the eccentricity of the ellipse. The integral in Eq. (5.3) is the elliptical integral of the second kind. MATLAB, for example, can be used to determine it.

The normal to the ellipse through the point $Q$, with the coordinates $(a\cos\theta, b\sin\theta)$, intersects the $x$-axis at the point $G$ with its coordinates given by $(ak^2\cos\theta, 0)$,[15] provided $0 < \theta \le \pi/2$. Let us denote the point with the coordinates $(a\cos\theta, 0)$ by $N$. Hence, the points ($G$, $N$, $Q$) form a right-angled triangle (see Fig. 2 again). The area of the triangle is given by

$$T(\theta) = \frac{1}{2}(a\cos\theta - ak^2\cos\theta)b\sin\theta = \frac{ab\sin 2\theta(1 - k^2)}{4}. \tag{5.4}$$

Let the angle formed by the line $GQ$ with the $x$-axis be given by $\alpha$. It is easy to see that

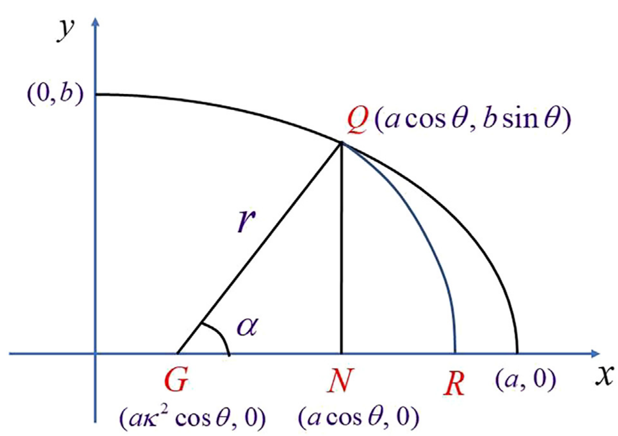


**FIG. 2.** Relevant points of the ellipse in the first quadrant.

$$\tan\alpha = \frac{b\sin\theta}{(a\cos\theta - ak^2\cos\theta)} = \frac{b}{a(1-k^2)}\tan\theta. \qquad (5.5)$$

Next, the area of the sector passing through $Q$ with $G$ as its center and meeting the $x$-axis at $R = (ak^2\cos\theta + r, 0)$ is given by $r^2\alpha/2$, where the radius $r$ is the length of the line $GQ$. Note that we are using the property of the Cheeger set that the free boundary must be the arc of a circle tangential to the boundary of the ellipse. Thus

$$r^2 = a^2(1-k^2)[1 - k^2\cos^2\theta]. \qquad (5.6)$$

Hence, the area of a candidate for the Cheeger set in the first quadrant is given by

$$A(\theta) = \frac{ab}{4}[\pi - 2\theta + k^2\sin 2\theta] + \frac{b^2}{2}(1 - k^2\cos^2\theta)\tan^{-1}\left(\frac{a}{b}\tan\theta\right). \qquad (5.7)$$

Its perimeter is the sum of the arc from $(0, b)$ to $Q$ along the ellipse plus that of the circular arc. This is given by

$$P(\theta) = L(\theta) + r\alpha, \qquad (5.8)$$

which yields the following:

$$P(\theta) = a\int_0^{\pi/2-\theta}\sqrt{1 - k^2\sin^2\xi}\, d\xi + b\sqrt{1 - k^2\cos^2\theta}\tan^{-1}\left(\frac{a}{b}\tan\theta\right). \qquad (5.9)$$

So, given $a$ and $b$, one has to find the minimum of $F(\theta) = P(\theta)/A(\theta)$ in order to determine the location of the optimal point $Q$ on the ellipse. This is straightforward to compute numerically by finding the root of $F'(\theta) = 0$, or, equivalently, the root of $A'(\theta)P(\theta) - A(\theta)P'(\theta) = 0$ in $(0, \pi/2)$. Several results, when $a = 1$ and $b = 1/30, 1/10, 1/4, 0.45$, can be found in Ref. 10.

## VI. GENERAL CONVEX POLYGONS

Suppose a polygon is convex; generally it need not be symmetric. For such a domain, Theorem 3 in Ref. 2, quoted below, provides the required tools to find the Cheeger constant. Hence, one can determine the minimum pressure drop to maintain a steady flow of a viscoplastic fluid in pipes of convex, polygonal cross sections using the results presented in this section and next in Sec. VII.

Let the convex polygon have $n$ sides, with vertices at $x_0, x_1, \ldots, x_{n-1}, x_n = x_0$, ordered counterclockwise. Let the interior angle at $x_i$ be given by $\pi - 2\alpha_i$, with $\alpha_0 = \alpha_n$. We note that $\alpha_i \in (0, \pi/2)$ because the polygon is convex. Let $l_i = |x_i - x_{i-1}|$ be the length of the $i$th side, so that $|\partial\Omega| = \sum_{i=1}^n l_i$. Define the sum of the tangents of the angles $\alpha_i$, $i = 1, \ldots, n$, through

$$T(\Omega) = \sum_{i=1}^n \tan\alpha_i. \qquad (6.1)$$

Since $\sum_{i=1}^n \alpha_i = \pi$, it follows that $T(\Omega) > \pi$ because for any $x \in (0, \pi/2)$, we have $\tan x > x$. With these preliminaries, we can ask whether the Cheeger set of the polygon touches every side of it, which is called a *Cheeger-regular* set. The following Theorem has been proved by Kawohl and Lachand-Robert.[2]

**Theorem 3**: *A convex polygon $\Omega$ is Cheeger-regular if and only if*

$$|\Omega| - r_0|\partial\Omega| + r_0^2(T(\Omega) - \pi) \le 0, \qquad (6.2)$$

$$r_0 = \min_{1\le i\le n}\frac{l_i}{\tan\alpha_i + \tan\alpha_{i-1}}. \qquad (6.3)$$

In this case, the perimeter and area of the Cheeger-regular set are given, respectively, by

$$|\partial\Omega_c| = |\partial\Omega| - 2(T(\Omega) - \pi)r, \qquad (6.4)$$

$$|\Omega_c| = |\Omega| - (T(\Omega) - \pi)r^2 = r|\partial C_\Omega|. \qquad (6.5)$$

Here, $r$ is the smaller root of

$$(T(\Omega) - \pi)r^2 - r|\partial\Omega| + |\Omega| = 0, \qquad (6.6)$$

whence,

$$r = |\partial\Omega| - \sqrt{|\partial\Omega|^2 - 4(T(\Omega) - \pi)|\Omega|}. \qquad (6.7)$$

Thus, the Cheeger constant has the following simple formula:

$$h(\Omega) = \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 - 4(T(\Omega) - \pi)|\Omega|}}{2|\Omega|}. \qquad (6.8)$$

As an example, consider the case when the polygon is a square, with a side of length $a$ each. Here, $T(\Omega) = 4$, $r_0 = a/2$. The left side of Eq. (6.2) is $(-\pi a^2/4) < 0$. Thus, Eq. (6.8) leads to the following result:

$$h = \frac{2 + \sqrt{\pi}}{a}, \qquad (6.9)$$

which is the same as in Eq. (4.4).

In order to find $h$ through Eq. (6.8), the only difficult task is to determine $T(\Omega)$ for a given polygon since $|\partial\Omega|$ and $|\Omega|$ are quite easy to find. For instance, when $\Omega$ is a triangle, the three bisectors of the included angles meet at a point. Using this as the center, an inscribed circle can be drawn, touching all three sides. See Fig. 3, which appears as Fig. 4 in Ref. 2. It shows that the radius of this circle is $r_0$, and, from Eq. (6.3), it follows that $r_0$ is the same for each $i = 1, 2, 3$. It is also quite easy to see from the six small triangles that $|\Omega| = r_0|\partial\Omega|/2$. Next,

$$|\partial\Omega| = \sum_{i=0}^2 r_0(\tan\alpha_i + \tan\alpha_{i+1}) = 2r_0T(\Omega). \qquad (6.10)$$

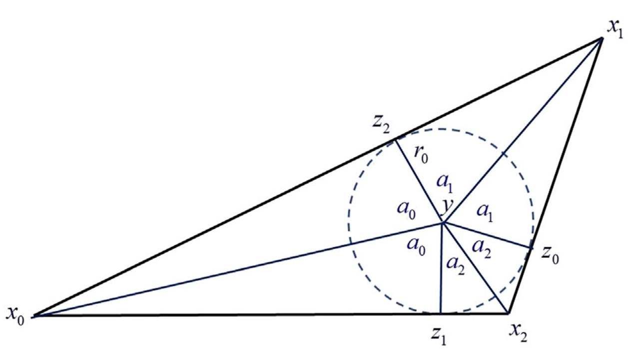


**FIG. 3.** Cheeger set for a triangle.

Hence, for any triangle,

$$h(\Omega) = \frac{|\partial\Omega| + \sqrt{4\pi|\Omega|}}{2|\Omega|}. \tag{6.11}$$

For an equilateral triangle of side $a$ each, it is obvious that $|\partial\Omega| = 3a$, $|\Omega| = \sqrt{3}\,a^2/4$. Hence, Eq. (6.11) provides the following result:

$$h(\Omega) = \frac{2\left(3 + \sqrt{\pi\sqrt{3}}\right)}{\sqrt{3}\,a}. \tag{6.12}$$

This value has been determined earlier by Huilgol[3] through the Mosolov–Miasnikov Lemmas.

## VII. CHEEGER-IRREGULAR CONVEX POLYGONS

If a convex polygonal domain $\Omega$ is Cheeger-irregular, i.e., the Cheeger set does not touch each side, an algorithm has been proven by Kawohl and Lachand-Robert[2] to find $h(\Omega)$. We shall employ this next, with a specific example exploring the connection between an equilateral triangle with an inscribed trapezium (see Fig. 4).

Let the length of each side of the equilateral triangle be $a$. Its inner Cheeger set is also an equilateral triangle with a rolling disk of radius $t^*$ moving along its boundary. From Eq. (6.12), it follows that

$$t^* = \frac{1}{h(\Omega)} = \frac{a}{2\left(\sqrt{3} + \sqrt{\pi/\sqrt{3}}\right)}. \tag{7.1}$$

The distance from the top vertex of the inner Cheeger set to that of the equilateral triangle can be found easily. It is given by $2t^*$ (see Fig. 4).

Thus, one can construct a trapezium with the top side parallel to the base of the equilateral triangle, with its two inclined sides lying along the sides of the equilateral triangle. When the distance, $d$, of the top edge from the base is given by

$$d = \frac{\sqrt{3}}{2}a - t^*, \tag{7.2}$$

the upper side of the trapezium is tangential to the circular arc bounding the Cheeger set of the equilateral triangle. Obviously, any trapezium with its top edge at a distance less than $d$ will be Cheeger-regular; conversely, one with a top edge at a distance greater than $d$ will be Cheeger-irregular. Indeed, numerical modeling of a series of trapezia inscribed in the equilateral triangle is depicted in Fig. 5, where $b$ is the length of the top edge, showing the evolution of the relevant Cheeger sets.

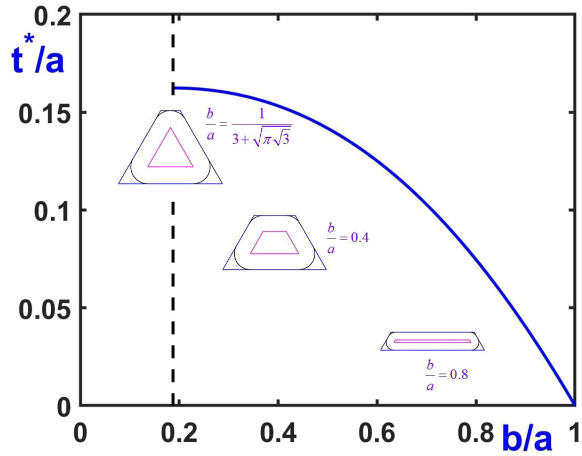


FIG. 5. Variation of $t^*$ for trapezia with the top side of length $b$ obtained by truncating an equilateral triangle of side $a$. The Cheeger sets for various values of $b$ with the corresponding Cheeger sets $\Omega^{t^*}$ are shown.

We can now discuss how to convert a Cheeger-irregular trapezium, with its top side at a distance $d_t > d$, into a regular one. Here, one simply removes the top side of the trapezium and extends its two inclined sides till they meet at a point, yielding an equilateral triangle (see Fig. 6). In producing this figure, we have followed the procedure recommended by Kawohl and Lachand-Robert.[2] The Cheeger set of the equilateral triangle will also be that of this Cheeger-irregular trapezium.

Turning to the more general case of a convex, Cheeger-irregular polygon with $n$-sides, one can determine its Cheeger-regular convex polygon as follows.[2] Locate the side of the $n$-gon satisfying Eq. (6.3);

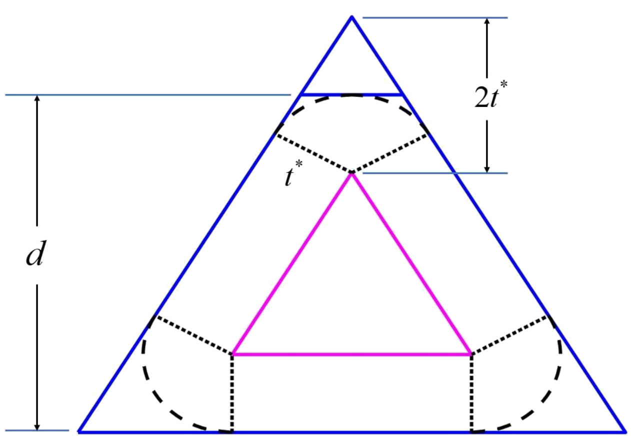


FIG. 4. Trapezium inscribed within an equilateral triangle.

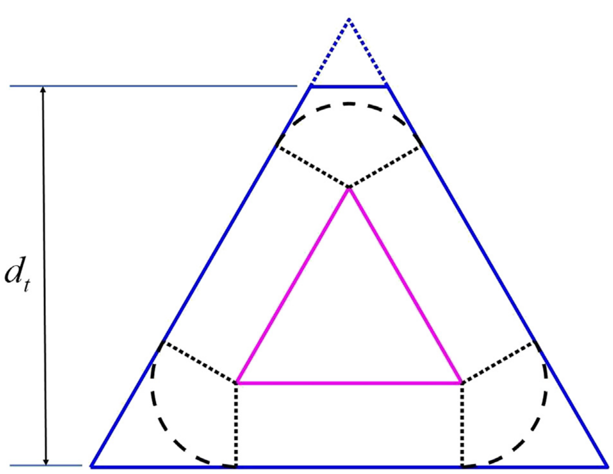


FIG. 6. Turning a Cheeger-irregular trapezium into a Cheeger-regular triangle.

it is possible that there may be one or more such sides. In each instance, extend the two adjacent sides to form a triangle, as in Fig. 6. This process of excision and extension produces a new polygon which is convex, once again.

If this polygon, with its modified side(s), satisfies Eq. (6.2), it is Cheeger regular, and its Cheeger constant can be found through Eq. (6.8). If this convex polygon is, once again, Cheeger-irregular, we repeat the procedure just described. The maximum number of iterations is $(n-3)$ when the polygon turns into a triangle, which is convex and Cheeger-regular. The construction described here produces the Cheeger set of the original $n$-gon.

It has to be understood that the Cheeger set of a convex polygon, whether it is Cheeger-regular or not, is unique. The real problem lies in determining the Cheeger set. The material summarized here in Secs. VI and VII provide the answer.

As far as the steady flow of a viscoplastic fluid in a pipe is concerned, it is clear that if the domain of flow is a Cheeger-regular convex polygon, the circular arcs bounding the stagnant zones exist at the corners only. If the domain is Cheeger-irregular, the stagnant zones can be at a distance from the walls of the pipe as well. Here, we have chosen a trapezium to exemplify this situation.

## VIII. CHEEGER CONSTANTS FOR JORDAN DOMAINS

The Jordan–Schoenflies theorem states that every simple closed curve in the plane separates the plane into two regions, the inside one bounded and the outside one unbounded. The inside one is usually described as a Jordan domain. Such domains need not be convex; they include those with fractal boundaries, such as the Koch snowflake. However, we shall assume that the boundary of each Jordan domain is sufficiently smooth to proceed and consider the extensions of Theorems 1 and 2 of Kawohl and Lachand-Robert,[2] described above in Sec. IV, to regions with and without necks. The results summarized here are based on the research of Krejĉirík and Pratelli,[11] Leonardi and Pratelli,[6] Leonardi *et al.*,[7] Leonardi and Saracco,[8] and Saracco.[9]

### A. Strips

A planar strip $S$ is of finite length and constant width. One can visualize it as a rectangle, an L-shaped region, or a curved one. While the rectangular domain is convex, the L-shaped domain is nonconvex. For a given strip $S$ of length $L$ and width 2, the following estimates of the Cheeger constant are available. The first is due to Krejĉirík and Pratelli[11]

$$1+\frac{1}{400L} \leq h(S) \leq 1+\frac{2}{L}. \tag{8.1}$$

This was improved by Leonardi and Pratelli,[6] and the following optimal asymptotic estimate was obtained:

$$h(S)=1+\frac{\pi}{2L}+O(L^{-2}) \quad \text{as}\, L \to +\infty. \tag{8.2}$$

Second, in Ref. 6, we find that the Cheeger set $h(S)$ satisfies Theorem 1 in Sec. IV. That is, the Cheeger set of the strip $S$ is the union of all balls of radius $t^*$. For instance, consider an L-shaped region with arms of arbitrary length. In order to calculate $t^*$, one follows the optimization procedure in Ref. 3. For example, see Eqs. (4.14)–(4.19) in Ref. 3 to find $t^*$ when both arms are of equal length.

### B. Regions with no necks

In simple terms, a domain does not have a neck of radius $r$ if one can roll a ball of radius $r$ from one end to the other of the given region. Using this idea, Leonardi *et al.*[7] proved that if a set $\Omega$ is a Jordan domain with no necks of radius $r=1/h(\Omega)$, it satisfies Theorem 1 in Sec. IV.

There is another way of approaching the minimization problem of the Jordan domain with no necks of radius $r=1/\kappa$. Given a domain $\Omega$ and a constant $\kappa>0$, consider the minimization of the curvature functional:

$$\mathscr{F}_\kappa(\Omega')=P(\Omega')-\kappa A(\Omega'), \tag{8.3}$$

for all subsets $\Omega' \subseteq \Omega$. If $E_\kappa$ is a nontrivial minimizer of Eq. (8.3), its internal boundary $\partial E_\kappa \cap \Omega$ is smooth and made up of circular arcs of curvature equal to $\kappa$. The problem is to find the maximal subset $\Omega_c$, which is the Cheeger set. On this set, it has been proved by Leonardi and Saracco[8] that the maximal minimizer $E_\kappa^M$ of Eq. (8.3) satisfies Theorem 1 in Sec. IV.

### C. Self-Cheeger sets

For a convex domain, Theorem 2 in Sec. IV provides a sufficient condition for it to be self-Cheeger. The extension of this sufficient condition to Jordan domains has been proved by Saracco;[9] a summary follows next.

Given a Jordan domain $\Omega$ in the plane, let a disk, $B_R$, of radius $R$ roll along the boundary $\partial\Omega$ inside the given domain. If the disk touches the boundary at $z \in \partial\Omega$ and its antipodal point on the disk does not touch the boundary, then the set $\Omega$ is said to have the *strict interior rolling property*. If this property for a given disk $B_R$ holds when $R=|\Omega|/P(\Omega)$, it follows that $\Omega$ is a minimal self-Cheeger, i.e., it is unique.

The set $\Omega$ is self-Cheeger even if it has the (weaker) nonstrict interior rolling property for $R=|\Omega|/P(\Omega)$. The strictness says on top of that $\Omega$ is the only Cheeger set of $\Omega$. An example of a set that has the nonstrict property and uniqueness fails is that of a Pinocchio set. See Fig. 1.2 in Ref. 9 and for the relevant computations (see Example 4.6 in Ref. 6). From this example, one can deduce that the Cheeger set is not unique for a face with two ears; it is also non-unique for a dumbbell with a long, thin arm.[6] These results are useful only when one encounters pipes of uniform cross section, which are odd-shaped and not convex.

## IX. CONCLUDING REMARKS

Recognizing the importance of the Cheeger constant in determining the minimum pressure gradient to sustain the steady flow of a Bingham fluid in pipes of convex cross section, we have been able to extend the method to more general viscoplastic fluids with constant yield stress. In addition, we have determined the location of stagnant zones and obtained bounds on the size of the rigid core. In a Bingham fluid, an upper bound on the velocity of the rigid core has also been derived. The determination of the Cheeger constant for convex and nonconvex regions, as well as when a given domain is self-Cheeger, Cheeger-regular, or Cheeger-irregular has been explained in full. Extension of these methods to a Jordan domain is also mentioned.

While some of the results mentioned here have appeared earlier in Refs. 3,4, and 10, they are not totally accurate nor complete. This survey should be accepted as the final version concerned with the flows of viscoplastic fluids in pipes of several cross sections of interest in engineering applications.

Finally, for a given flow with two or three velocity components, one can produce an energy equation similar to Eq. (2.11) using the relevant velocity potential. However, an inequality similar to Eq. (2.15) cannot be obtained, and the Cheeger set does not play a role in these flows, for the pressure gradient is not constant in one direction. Second, the results of this paper are not applicable to thixotropic yield stress fluids since thixotropy manifests in the initiation of a shearing flow.

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

### Author Contributions

**Raja Ramesh Huilgol:** Conceptualization (equal); Formal analysis (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Georgios C. Georgiou:** Conceptualization (equal); Formal analysis (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

## DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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