IMA Journal of Numerical Analysis (1998) **18**, 273–285

The method of fundamental solutions for Signorini problems

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[Received 2 September 1996 and in revised form 7 August 1997]

We investigate the use of the method of fundamental solutions (MFS) for the numerical solution of Signorini boundary value problems. The MFS is an ideal candidate for solving such problems because inequality conditions alternating at unknown points of the boundary can be incorporated naturally into the least-squares minimization scheme associated with the MFS. To demonstrate its efficiency, we apply the method to two Signorini problems. The first is a groundwater flow problem related to percolation in gently sloping beaches, and the second is an electropainting application. For both problems, the results are in close agreement with previously reported numerical solutions.

1. Introduction

When boundary methods are used to solve problems involving partial differential equations only the boundary of the domain is discretized. As a consequence, compared to domain discretization methods, such as finite difference and finite element methods, the size of the linear system of equations resulting from boundary methods is much smaller and the required data manipulation is considerably reduced. Boundary methods are particularly suitable for the solution of free surface problems since, in these, it is the free boundary which is of prime interest (Wrobel & Brebbia (1991)).

The method of fundamental solutions (MFS) has become popular in recent years and has proven to be an excellent alternative to the standard boundary element method for the numerical solution of certain elliptic boundary value problems. The method has already been applied to various harmonic and biharmonic problems with boundary singularities and free surfaces (Karageorghis (1992a, b), Poullikkas *et al* (1998)). The MFS can be viewed as an indirect boundary element method based on the approximation of the solution in terms of fundamental solutions of the governing equation involving sources located outside the domain of the problem. The unknown coefficients of the fundamental solutions and the final location of the sources are calculated so that the boundary conditions are satisfied in a least-squares sense. The MFS offers the advantages of boundary methods over domain discretization methods; in addition, it is adaptive in the sense that it takes into account sharp changes in the solution or in the geometry of the domain, and is relatively easy to implement. As will be demonstrated, another advantage of the MFS is that it very easily

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accommodates difficult boundary conditions involving inequalities, as occur in Signorini problems.

Signorini-type problems form a special class of free boundary problems (Elliott & Ockendon (1982), Spann (1993)). On part of the boundary, two types of conditions alternate in conjunction with certain inequality constraints. The main difficulty in these problems arises from the fact that the points where a change from one type of condition to the other occurs are unknown (Spann (1993)). Signorini-type problems, including the special cases studied in this paper, are the subject of a recent review article by Howison *et al* (1997). In other methods, enforcing these boundary conditions is usually achieved by special iterative schemes (Aitchison *et al* (1983, 1984)). In the MFS, however, the boundary inequalities can be incorporated into the minimization scheme in a natural way, and the need to design an appropriate iterative algorithm is thus avoided. Our objective is to demonstrate the use of the MFS for solving Signorini problems. We apply the method to two such problems, namely the shallow dam problem (Aitchison *et al* (1983)) and an electropainting problem (Aitchison *et al* (1984)). For both cases the results agree well with previously reported numerical solutions.

In Section 2 we present the formulation of the MFS for Signorini problems. The steady-state shallow dam and electropainting problems are defined and solved in Sections 3 and 4, respectively, where we also make comparisons with numerical results reported in the literature. Our conclusions are summarized in Section 5.

2. Application of the MFS to Signorini problems

Consider the problem

$$\nabla^2 u = 0 \qquad \text{in} \quad \Omega, \tag{2.1}$$

where ∇^2 denotes the Laplace operator, u is the dependent variable, and Ω is a bounded domain in the plane. We assume that the boundary $\partial\Omega$ consists of two parts, $\partial\Omega_I$ and $\partial\Omega_{II}$. On $\partial\Omega_I$, we assume that u is prescribed by a given function g, i.e., in Cartesian coordinates:

$$u = g(x, y) \qquad \text{on} \quad \partial\Omega_I. \tag{2.2}$$

On $\partial\Omega_{II}$, we assume that either u or its normal derivative $\partial u/\partial n$ are prescribed, according to the following conditions:

$$u = h(x, y) \quad \text{when} \quad \frac{\partial u}{\partial n} < f(x, y) \qquad \text{on} \quad \partial\Omega_{II}, \tag{2.3}$$

or

$$\frac{\partial u}{\partial n} = f(x, y) \quad \text{when} \quad u < h(x, y) \qquad \text{on} \quad \partial\Omega_{II}, \tag{2.4}$$

where h and f are known functions.

Equation (2.1) and boundary conditions (2.2)–(2.4) constitute a harmonic Signorini problem (Elliott & Ockendon (1982), Spann (1993)). In such problems, we do not know in advance the parts of the boundary $\partial\Omega_{II}$ on which conditions (2.3) apply and its remaining parts where conditions (2.4) apply. The number of and the positions of the points separating these different parts of $\partial\Omega_{II}$ are unknown. The separation points must be found as part of the solution. In order to obtain conditions valid everywhere on $\partial\Omega_{II}$, one can combine conditions (2.3) and (2.4) as follows:

$$u \leqslant h \quad \text{on} \quad \partial\Omega_{II}, \tag{2.5}$$

$$\frac{\partial u}{\partial n} \leqslant f \quad \text{on} \quad \partial\Omega_{II}, \tag{2.6}$$

and

$$(u-h)\left(\frac{\partial u}{\partial n} - f\right) = 0 \quad \text{on} \quad \partial\Omega_{II}. \tag{2.7}$$

In the MFS, the solution u is approximated in terms of fundamental solutions of the governing equation which involve sources located outside the domain Ω (Mathon & Johnston (1977)). We use N sources, the positions of which are to be determined, and choose M fixed points along the boundary $\partial\Omega$. Let $\mathbf{t}_j = (t_{j_x}, t_{j_y})$ denote the coordinates of source j and let $\mathbf{p}_i = (p_{i_x}, p_{i_y})$ be the coordinates of boundary point i. We seek the following approximation of the solution at the point $\mathbf{p}_i$:

$$\bar{u}_i = \bar{u}(\mathbf{c}, \mathbf{t}, \mathbf{p}_i) = \sum_{j=1}^{N} c_j\, k(\mathbf{t}_j, \mathbf{p}_i), \tag{2.8}$$

where $k(\mathbf{t}_j, \mathbf{p}_i) = \log r_{ij}$ is the fundamental solution of Laplace's equation, and r_{ij} is the distance between boundary point i and source j, i.e.,

$$r_{ij} = \sqrt{(p_{i_x} - t_{j_x})^2 + (p_{i_y} - t_{j_y})^2}.$$

The vector $\mathbf{c} = [c_1, c_2, ..., c_N]^T$ contains the unknown coefficients of the fundamental solutions, and the vector $\mathbf{t} = [t_{1_x}, t_{1_y}, t_{2_x}, t_{2_y}, ..., t_{N_x}, t_{N_y}]^T$ contains the unknown coordinates of the sources. Hence, the total number of unknowns is $3N$.

Because $\bar{u}$ is a solution of the governing equation (2.1), the unknowns are determined so that the boundary conditions (2.2) and (2.7) are satisfied in a least-squares sense. To achieve this, we minimize the functional

$$F(\mathbf{c}, \mathbf{t}) = \sum_{i=1}^{M_1} (\bar{u}_i - g_i)^2 + \sum_{i=M_1+1}^{M} \left[(\bar{u}_i - h_i)\left(\frac{\partial \bar{u}_i}{\partial n} - f_i\right)\right]^2, \tag{2.9}$$

where M_1 is the number of fixed boundary points along $\partial\Omega_I$; we require $\bar{u}$ to satisfy the remaining inequality constraints (2.5) and (2.6), i.e.,

$$\bar{u} - h \leqslant 0 \quad \text{and} \quad \frac{\partial \bar{u}}{\partial n} - f \leqslant 0 \;\; \text{on} \;\; \partial\Omega_{II}. \tag{2.10}$$

Note that enforcing the inequality constraints (2.10) may be directly accommodated in the minimization scheme with no difficulty, and that the total number of unknowns is not altered.

To minimize the nonlinear functional F defined by equation (2.9), subject to the inequality constraints (2.10), we use the least-squares subroutine E04UPF of the NAG Library (1991). This routine is designed to minimize an arbitrary sum of squares subject to constraints, which may include simple bounds on the variables, linear constraints, and nonlinear constraints. It employs a sequential algorithm in which the search direction is determined by the solution of a quadratic programming problem. The user must supply an initial estimate of the solution and subroutines that define the functions and the nonlinear constraints; the subroutine E04UPF performs best if the user supplies as many first partial derivatives as possible or, preferably, the exact Jacobian. Derivatives not provided by the user are approximated internally by finite differences. The subroutine E04UPF terminates when either the user-specified tolerance, τ, is achieved or the user-specified maximum number of iterations, $NITER$, is reached. The tolerance can be supplied through the optional input parameter 'Optimality Tolerance' which specifies the accuracy to which the user wishes the final iterate to approximate the solution of the problem. The maximum number of iterations can be supplied to E04UPF by the optional input parameter 'Major Iteration Limit'.

In a recent paper (Poullikkas *et al* (1998)), we solved harmonic and biharmonic problems and compared the computational efficiency of E04UPF with that of two MINPACK subroutines, LMDIF and LMDER. Both MINPACK subroutines use a Levenberg–Marquardt algorithm to minimize a sum of squares of nonlinear functions (Garbow *et al* (1980)). Our comparisons showed that the MINPACK routines generally perform slightly more efficiently than E04UPF. Unlike E04UPF, however, they cannot accommodate constraints, which option is necessary in solving Signorini problems. The success of the method depends on the quality of the software used to solve the constrained problem. An alternative to using the nonlinear least-squares E04UPF would have been to use a penalty function approach.

The initial placement of the N moving sources is important in the least-squares procedure. Usually the sources are initially distributed uniformly around the boundary at a fixed distance d from the boundary (Karageorghis & Fairweather (1987)). The fixed boundary points are distributed uniformly on the boundary and following the recommendations of Oliveira (1968), their number is chosen to be approximately three times the number of unknowns. The tendency of the sources to move into the interior of the domain is overcome by introducing an additional nonlinear constraint into the minimization scheme that excludes all points from Ω.

3. The steady-state shallow dam problem

We consider the potential flow problem shown in Fig. 1, which is related to percolation in gently sloping beaches and is known as the steady-state shallow dam problem. As shown in Fig. 1, the sand is divided into saturated and dry regions. The base is impermeable and, therefore, water seeps slowly from the left to the right through the sand. Since the velocity is very small, laminar flow is assumed, governed by Darcy's law. The domain of interest is the saturation region ABCDEF in Fig. 1. More details of the problem are given by Aitchison *et al* (1982).

Following Aitchison *et al* (1983) the model for the dam problem is made dimensionless in which case $|AB| = 1$, the surface profile is of the form $y(x) = 1 + \delta G(x)$ and the free boundary is of the form $y(x) = 1 + \delta W(x)$ where $W(x) \leqslant G(x)$. Then, we consider the asymptotic expansions of the pressure u and W in powers of δ. The leading term is trivial. Equating terms of order δ yields the problem we are examining. In particular, the top part

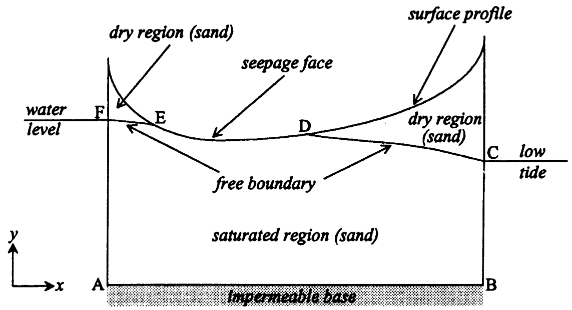

FIG. 1. Percolation in gently sloping beaches.

of the boundary is the projection of the boundary FEDC (see Fig. 1) on $y = 1$ because of the asymptotic expansion of both the field equations and the boundary conditions. This is a Signorini boundary value problem on a square domain with the pressure u as the unknown in the saturated region. The problem is then

$$\nabla^2 u = 0 \quad \text{in} \quad \Omega = (0, 1) \times (0, 1). \tag{3.1}$$

Through the base $S_1 = \{(x, y) : 0 \leqslant x \leqslant 1,\ y = 0\}$, no flow occurs, i.e., the pressure gradient is zero:

$$\frac{\partial u}{\partial y} = 0 \quad \text{on} \quad S_1. \tag{3.2}$$

On the vertical sides $S_2 = \{(x, y) : x = 1,\ 0 \leqslant y \leqslant 1\}$ and $S_4 = \{(x, y) : x = 0,\ 0 \leqslant y \leqslant 1\}$, hydrostatic conditions are assumed, i.e., the pressure is constant. Along S_4 the pressure is set to zero:

$$u = 0 \quad \text{on} \quad S_4. \tag{3.3}$$

We assume that the surface profile is given by a known function $G(x)$ (Aitchison *et al* (1983)). Then, along S_2 we have

$$u = G(1) \quad \text{on} \quad S_2. \tag{3.4}$$

Conditions (3.3) and (3.4) are chosen in order to model the physical situation satisfactorily by considering a single surface profile adjacent to regions generating a small horizontal flow (Tayler (1986)).

On $S_3 = \{(x, y) : 0 \leqslant x \leqslant 1,\ y = 1\}$, the number and the positions of the separation points between the free boundary (the projection of FE and DC onto $y = 1$) and the seepage face, the projection of ED onto $y = 1$ (Fig. 1), obviously depend on the surface profile $G(x)$. On the parts of S_3 corresponding to the free boundary, the kinematic condition requires the pressure gradient to vanish. On the remaining parts of S_3, corresponding to the seepage face, momentum balance requires the pressure u to be equal to $G(x)$ in the absence of surface tension. Therefore, we have

$$\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad u < G(x) \quad \text{on} \quad S_3 \tag{3.5}$$

or

$$u = G(x) \quad \text{and} \quad \frac{\partial u}{\partial y} < 0 \quad \text{on} \quad S_3. \tag{3.6}$$

As already mentioned, the number and positions of the separation points depend on the surface profile $G(x)$. In order to make comparisons with the numerical solutions of Aitchison *et al* (1983) and Karageorghis (1987), we considered the following two profiles:

$$G_1(x) = \left(\frac{1}{2} - x\right)(1 - x) - x, \tag{3.7}$$

$$G_2(x) = \left(\frac{1}{2} - \frac{5}{2}x\right)\left(1 - \frac{5}{3}x\right)(1 - x) - x. \tag{3.8}$$

From the numerical results of Aitchison *et al* (1983) and Karageorghis (1987), we know that $G_1(x)$ has one separation point with one free boundary segment, whereas $G_2(x)$ has two separation points with two free boundary segments. Aitchison *et al* (1983) solved the steady-state shallow dam problem using a finite element method, whereas Karageorghis (1987) developed a technique for determining the number and positions of the separation points and solved the problem using a boundary element method.

In the MFS, the approximate solution is given by equation (2.8). Let M_1, M_2, M_3 and M_4 be the numbers of boundary points along the four parts of the boundary, the total number of boundary points being $M = M_1 + M_2 + M_3 + M_4$. As in equation (2.9), the boundary conditions are satisfied in a least-squares sense by minimizing the functional

$$F(\mathbf{c}, \mathbf{t}) = \sum_{i=1}^{M_1} \left(\frac{\partial \bar{u}_i}{\partial y}\right)^2 + \sum_{i=M_1+1}^{M_1+M_2} [\bar{u}_i - G(1)]^2 + \sum_{i=M_1+M_2+1}^{M_1+M_2+M_3} \left\{[\bar{u}_i - G(x_i)] \frac{\partial \bar{u}_i}{\partial y}\right\}^2 + \sum_{i=M_1+M_2+M_3+1}^{M} \bar{u}_i^2, \tag{3.9}$$

subject to the three inequality constraints:

$$\bar{u} - G(x) \leqslant 0 \quad \text{on} \quad S_3, \tag{3.10}$$

$$\frac{\partial \bar{u}}{\partial y} \leqslant 0 \quad \text{on} \quad S_3, \tag{3.11}$$

and

$$\max(|x - 0{\cdot}5|, |y - 0{\cdot}5|) > 0{\cdot}5. \tag{3.12}$$

The last constraint ensures that the sources remain outside the domain. The fact that this constraint is not differentiable does not appear to be causing internal difficulties for the solver, as convergence was achieved in all the cases considered. In the documentation for E04UPF it is stated explicitly that the method will usually work if there are isolated singularities away from the solution. In order to confirm this, the constraint was replaced by the smooth constraint $(x - 0{\cdot}5)^{2m} + (y - 0{\cdot}5)^{2m} > 0{\cdot}5^{2m}$, where m is a positive integer (clearly as m increases the above curve tends to the square) and tests were run for various

TABLE 1

Convergence with N of the solution near the separation point for $G_1(x) = (\frac{1}{2} - x)(1 - x) - x$ *and* $\tau = 10^{-6}$

		x	0·6000	0·6500	0·7000	0·7500	0·8000	0·8500	0·9000
		$G_1(x)$	−0·6400	−0·7025	−0·7600	−0·8125	−0·8600	−0·9025	−0·9400
M	N	$NITER$	Approximate solution						
28	3	72	−0·8045	−0·8676	−0·9146	−0·9418	−0·9481	−0·9348	−0·9055
48	5	119	−0·6935	−0·7392	−0·7828	−0·8244	−0·8641	−0·9023	−0·9391
64	7	893	−0·6618	−0·7151	−0·7663	−0·8148	−0·8603	−0·9027	−0·9424
84	9	946	−0·6599	−0·7148	−0·7664	−0·8125	−0·8601	−0·9027	−0·9417
			free boundary			seepage face			

TABLE 2

Convergence with N of the solution near the separation point for $G_1(x) = (\frac{1}{2} - x)(1 - x) - x$ *and* $NITER = 700$

	x	0·6000	0·6500	0·7000	0·7500	0·8000	0·8500	0·9000
	$G_1(x)$	−0·6400	−0·7025	−0·7600	−0·8125	−0·8600	−0·9025	−0·9400
M	N	Approximate solution						
28	3	−0·8045	−0·8676	−0·9146	−0·9418	−0·9481	−0·9348	−0·9055
48	5	−0·7055	−0·7495	−0·7912	−0·8309	−0·8688	−0·9051	−0·9400
64	7	−0·6619	−0·7153	−0·7665	−0·8149	−0·8603	−0·9027	−0·9423
84	9	−0·6602	−0·7153	−0·7667	−0·8126	−0·8599	−0·9027	−0·9420
		free boundary			seepage face			

values of m. We also ran E04UPF with no nonlinear constraint to keep the sources outside the domain (for cases when no sources entered the domain). The results obtained both ways were almost identical to the results already obtained with the original constraint. Also, the number of iterations required to reach a specific tolerance was essentially the same with all three methods.

The problem was solved for various numbers of sources N, values of the tolerance τ and numbers of iterations $NITER$. The sources were initially placed outside the domain at a distance $d = 0{\cdot}1$ from the boundary. In Tables 1 and 2, we illustrate the convergence of the MFS with N (and M, recall that $M \approx 9N$). In both tables, we tabulate the values of the surface profile $G_1(x)$ and the calculated values of $\bar{u}$ on S_3 near the separation point. Recall that the separation point is the point where we have a change from conditions (3.5) to conditions (3.6) or vice versa. As mentioned in Section 2, the point at which the subroutine E04UPF terminates is controlled by either a user specified tolerance, τ, or a user specified number of iterations, $NITER$. In order to demonstrate the convergence of the method while increasing the number of sources N (and boundary points M), in Table 1 we present

TABLE 3

Convergence with τ of the solution near the separation point for $G_1(x) = (\frac{1}{2} - x)(1 - x) - x$, $N = 7$ and $M = 64$

x		0·6000	0·6500	0·7000	0·7500	0·8000	0·8500	0·9000
$G_1(x)$		−0·6400	−0·7025	−0·7600	−0·8125	−0·8600	−0·9025	−0·9400
τ	*NITER*	Approximate solution						
10^{-2}	77	−0·6640	−0·7183	−0·7686	−0·8149	−0·8596	−0·9045	−0·9502
10^{-3}	84	−0·6625	−0·7171	−0·7678	−0·8144	−0·8594	−0·9046	−0·9506
10^{-4}	270	−0·6587	−0·7129	−0·7647	−0·8138	−0·8599	−0·9033	−0·9446
10^{-5}	345	−0·6622	−0·7162	−0·7675	−0·8157	−0·8606	−0·9025	−0·9422
10^{-6}	893	−0·6618	−0·7151	−0·7663	−0·8148	−0·8603	−0·9027	−0·9424
		free boundary			seepage face			

TABLE 4

Convergence with NITER of the solution near the separation point for $G_1(x) = (\frac{1}{2} - x)(1 - x) - x$, $N = 7$ and $M = 64$

x	0·6000	0·6500	0·7000	0·7500	0·8000	0·8500	0·9000
$G_1(x)$	−0·6400	−0·7025	−0·7600	−0·8125	−0·8600	−0·9025	−0·9400
NITER	Approximate solution						
100	−0·6711	−0·7266	−0·7764	−0·8202	−0·8614	−0·9026	−0·9049
300	−0·6627	−0·7170	−0·7685	−0·8166	−0·8610	−0·9025	−0·9418
500	−0·6623	−0·7158	−0·7670	−0·8153	−0·8605	−0·9025	−0·9419
700	−0·6619	−0·7153	−0·7665	−0·8150	−0·8603	−0·9027	−0·9423
900	−0·6618	−0·7151	−0·7663	−0·8148	−0·8603	−0·9027	−0·9424
	free boundary			seepage face			

the solutions obtained with the surface profile $G_1(x)$, when keeping τ fixed. We observe that the values of $\bar{u}$ converge with N. On the section corresponding to the surface profile, the solution is in excellent agreement with the values of $G_1(x)$. In order to demonstrate the convergence of the method with N (and M) in the case where the number of iterations is fixed instead of the tolerance, in Table 2 we tabulate results obtained with the surface profile $G_1(x)$ when $NITER$ is kept fixed. In order to observe the change in the results as the stopping criteria of E04UPF are made stricter (i.e., when either the tolerance is made smaller or the number of iterations is increased) in Tables 3 and 4 we present the results obtained for fixed N (and M) when varying τ and $NITER$, respectively. Again, the $\bar{u}$ values calculated on the surface profile agree well with the values of $G_1(x)$.

In Fig. 2 we show the numerical results obtained with the surface profile $G_1(x)$, $N = 7$, $M = 64$ and $NITER = 700$. We observe that there is only one separation point between the free boundary and the surface profile with one free boundary curved compo-

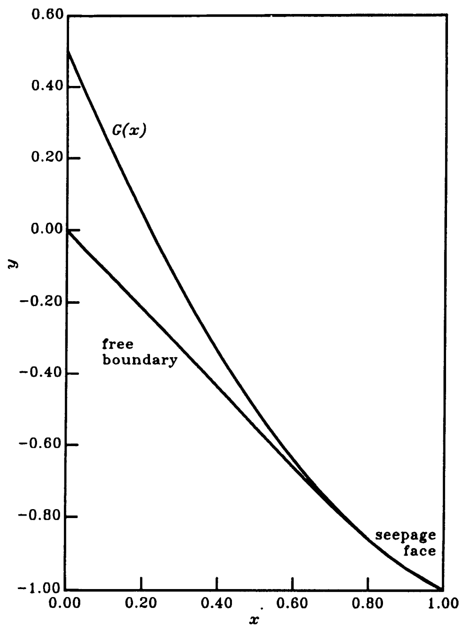

FIG. 2. Approximate solution of the shallow dam problem for $N = 7$, $M = 64$, $NITER = 700$ and $G_1(x) = (\frac{1}{2} - x)(1 - x) - x$.

nent. Similar results were obtained with the surface profile $G_2(x)$, $N = 7$, $M = 64$ and $NITER = 2000$. For this profile, there correspond two separation points with two free boundary segments, as shown in Fig. 3. The free boundary segments are slightly curved. The results are again in excellent agreement with those of Aitchison *et al* (1983) and Karageorghis (1987).

4. The steady-state electropainting problem

We apply the MFS to another potential problem known as the steady-state electropainting problem (Aitchison *et al* (1984)). In the electropaint process, the object to be coated is immersed in a bath containing an electrolyte paint solution. A potential difference is applied between the object and the tank walls and a current flows through the solution. The paint particles thus become charged and are deposited on the surface of the object. Special attention is required near corners to achieve sufficient paint thickness.

Aitchison *et al* (1984) formulated the problem as a Signorini problem on a square domain for the electric potential u in the paint solution. The electric potential in the paint solution is a harmonic function, i.e., the governing equation is

$$\nabla^2 u = 0 \quad \text{in} \quad \Omega = (-0{\cdot}5, 0{\cdot}5) \times (0, 1). \tag{4.1}$$

The base $S_1 = \{(x, y) : -0{\cdot}5 \leqslant x \leqslant 0{\cdot}5,\ y = 0\}$ is the anode where the voltage V is applied. Hence,

$$u = V \quad \text{on} \quad S_1. \tag{4.2}$$

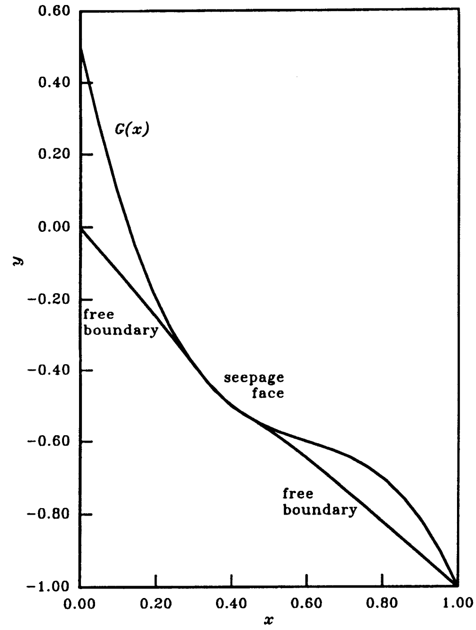

FIG. 3. Approximate solution of the shallow dam problem for $N = 7$, $M = 64$, $NITER = 2000$ and $G_2(x) = (\frac{1}{2} - \frac{5}{2}x)(1 - \frac{5}{3}x)(1 - x) - x$.

The other sides $S_2 = \{(x, y) : x = 0{\cdot}5,\ 0 \leqslant y \leqslant 1\}$, $S_3 = \{(x, y) : -0{\cdot}5 \leqslant x \leqslant 0{\cdot}5,\ y = 1\}$ and $S_4 = \{(x, y) : x = -0{\cdot}5,\ 0 \leqslant y \leqslant 1\}$ make up the object to be painted which is the cathode. On these boundaries, we do not know which sections are painted and which are unpainted. On the parts corresponding to unpainted surfaces, the boundary conditions are

$$u = 0, \qquad \frac{\partial u}{\partial n} + \varepsilon > 0 \quad \text{on} \quad S_2 \cup S_3 \cup S_4, \tag{4.3}$$

and on the parts corresponding to painted surfaces, the boundary conditions are

$$\frac{\partial u}{\partial n} + \varepsilon = 0, \qquad u > 0 \quad \text{on} \quad S_2 \cup S_3 \cup S_4, \tag{4.4}$$

where ε is the critical cut-off current which is a property of the paint. Compared to the other length scales of the problem, the paint thickness, $h = u/\varepsilon$, is very small. Aitchison *et al* (1984) used a finite element method to solve the resulting problem. They also showed the existence of a unique solution.

As before, in the MFS we approximate the solution using equation (2.8). The boundary conditions are satisfied in a least-squares sense by minimizing the functional

$$F(\mathbf{c}, \mathbf{t}) = \sum_{i=1}^{M_1} (\bar{u}_i - 1)^2 + \sum_{i=M_1+1}^{M} \left[\bar{u}_i \left(\frac{\partial \bar{u}_i}{\partial n} + \varepsilon\right)\right]^2, \tag{4.5}$$

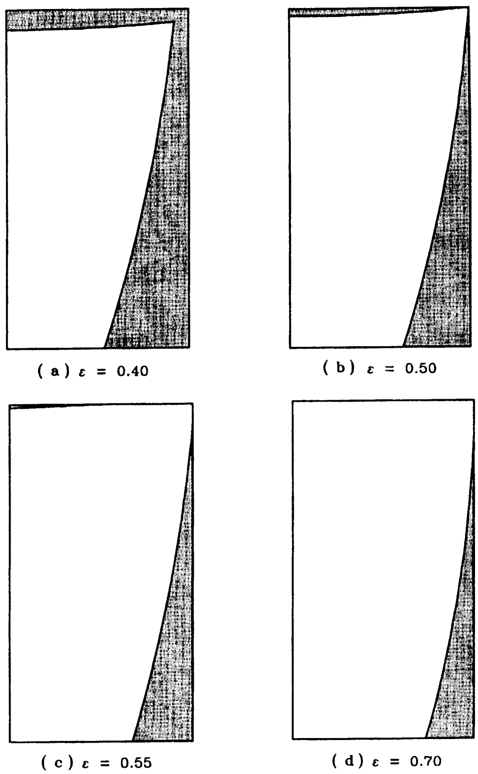

FIG. 4. Paint distributions for (a) $\varepsilon = 0{\cdot}40$, (b) $\varepsilon = 0{\cdot}50$, (c) $\varepsilon = 0{\cdot}55$, and (d) $\varepsilon = 0{\cdot}70$ with $N = 12$, $M = 108$ and $NITER = 600$.

subject to the following three inequality constraints:

$$\bar{u} \geqslant 0, \qquad \text{on} \qquad S_2 \cup S_3 \cup S_4, \tag{4.6}$$

$$\frac{\partial \bar{u}}{\partial n} + \varepsilon \geqslant 0 \qquad \text{on} \qquad S_2 \cup S_3 \cup S_4, \tag{4.7}$$

and

$$\max\left(\left|\frac{|x| - 0{\cdot}25}{0{\cdot}25}\right|, \left|\frac{y - 0{\cdot}5}{0{\cdot}5}\right|\right) > 1{\cdot}0. \tag{4.8}$$

As before, the latter constraint ensures that the sources remain outside the domain.

The problem was solved for various numbers of sources N, tolerances τ and numbers of iterations $NITER$, with the sources initially placed outside the domain at a distance $d = 0{\cdot}1$ from the boundary. In Fig. 4, we show the paint distribution over half of the domain (because of the symmetry), calculated for various values of ε and $N = 12$, $M = 108$ and $NITER = 600$. For the smaller values of ε, the object is completely painted with the corner receiving the least amount of paint, as shown in Fig. 4(a) where we plot the results for $\varepsilon = 0{\cdot}40$. If we increase the value of ε, the paint film near the corner becomes thinner (Fig. 4(b)), and the corner eventually becomes unpainted (Fig. 4(c)). Finally, at an even higher value of ε, both the corner and the top surface of the object are unpainted (Fig. 4(d)). The results are identical to those of Aitchison *et al* (1984).

5. Conclusions

We have demonstrated that the MFS is ideally suited for the numerical solution of Signorini problems. The inequality boundary conditions can be incorporated into the solution process in a natural way, when using a state-of-the-art constrained least-squares minimization routine. The separation points, where the type of boundary condition changes, are automatically generated by the method. These factors make the MFS an ideal candidate for the numerical solution of this type of problem. It takes into account all the particular features of the problem in a natural way which is very easy to implement. This renders the MFS extremely efficient when compared to previously used numerical methods such as finite element methods or boundary element methods. The advantages of the MFS (when applicable) over these methods are well documented (Karageorghis & Fairweather (1987)). The method is applied to the steady-state shallow dam problem and to a steady-state electropainting problem. The results for both problems are in close agreement with previously reported numerical solutions.

Acknowledgements

The authors wish to thank Professor Graeme Fairweather of the Colorado School of Mines for his useful comments and suggestions, and the two anonymous referees for their constructive remarks.

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