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The method of fundamental solutions for inhomogeneous elliptic problems

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Abstract We investigate the use of the Method of Fundamental Solutions (MFS) for solving inhomogeneous harmonic and biharmonic problems. These are transformed to homogeneous problems by subtracting a particular solution of the governing equation. This particular solution is taken to be a Newton potential and the resulting homogeneous problem is solved using the MFS. The numerical calculations indicate that accurate results can be obtained with relatively few degrees of freedom. Two methods for the special case where the inhomogeneous term is harmonic are also examined.

1 Introduction

Numerical methods for solving homogeneous elliptic partial differential equations can be divided into domain discretization methods and boundary methods. The main advantage of boundary methods over domain discretization methods is the reduction of the problem dimension by one. Also, they are relatively easy to program and offer program compactness. However, when boundary methods are applied to inhomogeneous boundary value problems it is necessary to evaluate a domain integral. The reduction of the dimension of the problem is thus lost and there is a considerable increase in the amount of work involved in the solution of the problem.

In the MFS, which may be viewed as an indirect boundary element method (Mathon and Johnston 1977, Fairweather and Johnston 1982), the solution is approximated by a set of fundamental solutions of the governing equation which are expressed in terms of sources located outside the domain of the problem. The unknown coefficients in the linear combination of the fundamental solutions and the final locations of the sources are determined so that the boundary conditions are satisfied in a least squares sense. The method is relatively easy to implement, is adaptive, in the sense that it takes into account sharp changes in the solution and/or in the geometry of the domain, and can easily incorporate difficult boundary conditions. This has been demonstrated with its successful application to various homogeneous harmonic and biharmonic problems (Karageorghis and Fairweather 1987, Karageorghis 1992).

In a boundary method formulation of an inhomogeneous problem, the evaluation of the domain integral is clearly of fundamental importance (Benitez and Wideberg 1996). In the case a direct numerical domain integration is performed (Burgess and Mahajerin 1987), the computational cost is generally high. There are, however, two different schemes which resolve the domain integral problem. In the first, the domain integral is transformed to a series of line integrals as in the dual reciprocity method (Partridge et al. 1992). The approach of Atkinson (1985) is an alternative to this scheme. In it, the inhomogeneous term is eliminated by finding an appropriate particular solution, constructed from the Newton potential which can be evaluated numerically without having to discretize the domain. The method assumes that the inhomogeneous term can be extended smoothly to a suitable region containing the original domain of the problem. In theory, this is always possible for regions with a smooth boundary (Atkinson 1985). In particular, the inhomogeneous equation $\mathscr{L}u = f$, where $\mathscr{L}$ is a linear elliptic partial differential operator, is transformed to the homogeneous equation $\mathscr{L}v = 0$, by constructing a particular solution u_0 and introducing the new variable $v = u - u_0$. Golberg (1995) compared this approach with the dual reciprocity method by solving Poisson problems with Dirichlet boundary conditions and showed that it gives more accurate results. In both Burgess and Mahajerin (1987) and Golberg (1995), the resulting Laplace equation is solved by the MFS with fixed sources.

The objective of this paper is to extend the MFS to the solution of inhomogeneous elliptic problems, based on the particular solution approach of Atkinson (1985). In Section 2, we solve Poisson problems with mixed boundary conditions. In Section 3, the method is extended to biharmonic problems. Finally, in Section 4, we consider a special class of Poisson problems in which the inhomogeneous term is harmonic. Two different methods are examined; the first is based on the particular solution approach and the second is a direct application of the MFS. In all cases, numerical results are presented for a number of test problems. Accurate results are obtained

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with relatively few degrees of freedom. Our conclusions are summarized in Section 5.

2 Inhomogeneous harmonic problems

2.1 Formulation

Consider the Poisson equation

$$\nabla^2 u = f(x,y) \quad \text{in } \Omega \ , \tag{1}$$

subject to the boundary conditions

$$u = g(x,y) \quad \text{on } \partial\Omega_1 \ , \tag{2}$$

and

$$\frac{\partial u}{\partial n} = h(x,y) \quad \text{on } \partial\Omega_2 \ , \tag{3}$$

where ∇^2 denotes the Laplace operator, u is the dependent variable, f, g and h are given functions, $\partial u/\partial n$ denotes the outward normal derivative of u on the boundary, Ω is a bounded domain and $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ denotes its boundary. If u_0 is a particular solution of (1), i.e., $\nabla^2 u_0 = f(x,y)$, then the function $v = u - u_0$ satisfies the Laplace equation

$$\nabla^2 v = 0 \quad \text{in } \Omega \ , \tag{4}$$

subject to the modified boundary conditions

$$v = g(x,y) - u_0 \quad \text{on } \partial\Omega_1 \tag{5}$$

and

$$\frac{\partial v}{\partial n} = h(x,y) - \frac{\partial u_0}{\partial n} \quad \text{on } \partial\Omega_2 \ . \tag{6}$$

The problem (4)–(6) can now be solved using the standard MFS (Mathon and Johnston 1977, Fairweather and Johnston 1982).

One technique for obtaining a particular solution of Poisson's equation is based on the Newton potential (Atkinson 1985, Kellog 1954, Gilbarg and Trudinger 1983, Golberg and Chen 1994), which is given by the integral

$$u_0(\mathbf{p}) = \frac{1}{2\pi}\int_{\Omega} \log|\mathbf{p}-\mathbf{q}| f(\mathbf{q})\, \mathrm{d}V(\mathbf{q}), \qquad \mathbf{p} = (p_x, p_y) \in \Omega \ . \tag{7}$$

The difficulties involved in the evaluation of this domain integral can be avoided by using a method proposed by Atkinson (1985). If Ω_0 is a larger region containing $\bar{\Omega} = \Omega \cup \partial\Omega$ and $f(\mathbf{q})$ can be extended smoothly to Ω_0, then

$$u_0(\mathbf{p}) = \frac{1}{2\pi}\int_{\Omega_0} \log|\mathbf{p}-\mathbf{q}| f(\mathbf{q})\, \mathrm{d}V(\mathbf{q}), \quad \mathbf{p} \in \bar{\Omega} \ , \tag{8}$$

is another particular solution of (1). The region Ω_0 is chosen so that the calculation of the integral (8) is facilitated. Moreover, in order to avoid the singularity at $\mathbf{q} = \mathbf{p}$, we use the following change of variables

$$\mathbf{q} = \mathbf{p} + R[\mathbf{b}(\theta) - \mathbf{p}], \quad 0 \le R \le 1, \quad 0 \le \theta \le 2\pi, \qquad \mathbf{q} = (q_x, q_y) \in \Omega_0 \ , \tag{9}$$

where $\mathbf{b}(\theta) = [b_x(\theta), b_y(\theta)] \in \partial\Omega_0$. The Jacobian of the transformation is given by

$$|J| = R\, H(\mathbf{p}, \theta) \ , \tag{10}$$

where

$$H(\mathbf{p},\theta) = b_y'(\theta)[b_x(\theta) - p_x] - b_x'(\theta)[b_y(\theta) - p_y] \ , \tag{11}$$

and Eq. (8) becomes

$$u_0(\mathbf{p}) = \frac{1}{2\pi}\int_{\Omega_0} \log\left(R\sqrt{[b_x(\theta)-p_x]^2 + [b_y(\theta)-p_y]^2}\right) \times R\,H(\mathbf{p},\theta) f_0(\mathbf{p},R,\theta)\, \mathrm{d}R\, \mathrm{d}\theta \ , \tag{12}$$

where $f_0(\mathbf{p},R,\theta) = f(\mathbf{q})$. For simplicity, we let Ω_0 be an ellipse with axes R_x and R_y, such that

$$\mathbf{b}(\theta) = (R_x \cos\theta,\ R_y \sin\theta) \ , \tag{13}$$

and

$$H(\mathbf{p},\theta) = R_x R_y \left(1 - \frac{p_x}{R_x}\cos\theta - \frac{p_y}{R_y}\sin\theta\right) \ . \tag{14}$$

Then, Eq. (12) yields the following expression for $u_0(\mathbf{p})\ (\mathbf{p} \in \bar{\Omega})$

$$u_0(\mathbf{p}) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \log\left(R\sqrt{[b_x(\theta)-p_x]^2 + [b_y(\theta)-p_y]^2}\right) \times R\,H(\mathbf{p},\theta) f_0(\mathbf{p},R,\theta)\, \mathrm{d}R\, \mathrm{d}\theta \ . \tag{15}$$

The normal derivative of $u_0(\mathbf{p})\ (\mathbf{p} \in \bar{\Omega})$ is given by

$$\frac{\partial u_0}{\partial n}(\mathbf{p}) = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \left[[b_x(\theta)-p_x]\left(\frac{\partial p_x}{\partial n}\right) + [b_y(\theta)-p_y]\left(\frac{\partial p_y}{\partial n}\right)\right] \times \frac{1}{[b_x(\theta)-p_x]^2 + [b_y(\theta)-p_y]^2} \times H(\mathbf{p},\theta) f_0(\mathbf{p},R,\theta)\, \mathrm{d}R\, \mathrm{d}\theta \ , \tag{16}$$

where $(\partial p_x/\partial n,\ \partial p_y/\partial n)$ are the direction cosines at the boundary point $\mathbf{p}$. The integral in Eq. (16) is free of singularities and can be evaluated at any point $\mathbf{p}$ by standard subroutines for double integrals. For this purpose, we use the subroutine D01DAF, from the NAG library (NAG (UK) Ltd 1991). This subroutine evaluates the double integral by calculating two single integrals by the method of Patterson (1968), which optimizes the number of points in Gauss quadrature formulae. The subroutine requires the user to specify a tolerance which is defined as the absolute accuracy to which the results are evaluated.

Equation (15) for $u_0(\mathbf{p})$ may also be written as follows:

$$u_0(\mathbf{p}) = \frac{1}{2\pi}\int_0^{2\pi} H(\mathbf{p},\theta)\int_0^1 R \log R\, f_0(\mathbf{p},R,\theta)\, \mathrm{d}R\, \mathrm{d}\theta + \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \log\left([b_x(\theta)-p_x]^2 + [b_y(\theta)-p_y]^2\right) \times R\,H(\mathbf{p},\theta) f_0(\mathbf{p},R,\theta)\, \mathrm{d}R\, \mathrm{d}\theta \equiv I_1 + I_2 \ , \tag{17}$$

where the inner integral of I_1 contains a weak singularity due to the term $R \log R$. The particular solution $u_0(\mathbf{p})$ can be calculated using a combination of NAG library subroutines, so that the weak singularity in the first integral is taken into account. For the inner integral of I_1, we use D01APF, a subroutine for single integrals with logarithmic-type end-point singularities. This subroutine starts by bisecting the original interval and applying modified Clenshaw-Curtis integration of orders 12 and 24 on both halves. Clenshaw-Curtis integration is then used on all subintervals in which the end-points are included. On the other subintervals, Gauss-Kronrod integration is carried out. For the outer integral of I_1 we use D01AHF, a standard subroutine for single integrals based, as D01DAF, on the method of Patterson (1968). Finally, for the double integral I_2 which is not singular we again use D01DAF. For comparison purposes, we set $f(x,y) = 2e^{x-y}$ and $(R_x, R_y) = (2.5, 1.5)$ and calculated $u_0(\mathbf{p})$ at various specified points using D01DAF for Eq. (15) and the combination of the three NAG routines for the evaluation of the integrals in Eq. (17). In Table 1, representative CPU times required in both cases are tabulated. We observe that D01DAF failed for tolerances less than 10^{-10} as both the outer and the inner integral failed to converge. The combination of the three routines which proved less costly was used hereafter.

Since the particular solution $u_0(\mathbf{p})$ and its normal derivative on the boundary can be evaluated at any point, the standard MFS can be applied for solving the homogeneous problem (4)–(6). We assume that we have N sources outside the domain Ω. Their coordinates $\mathbf{t}_j = (t_{j_x}, t_{j_y})$, $j = 1, 2, \ldots, N$, are unknown and must be calculated as part of the solution. Along the boundary $\partial\Omega$, M fixed points with coordinates $\mathbf{p}_i = (p_{i_x}, p_{i_y})$, $i = 1, 2, \ldots, M$, are also chosen. The solution v at a boundary point $\mathbf{p}_i$ is approximated by a linear combination of fundamental solutions of the Laplace equation, say $\bar{v}_i$:

$$\bar{v}_i = \bar{v}(\mathbf{c}, \mathbf{t}, \mathbf{p}_i) = \sum_{j=1}^{N} c_j \, k(\mathbf{t}_j, \mathbf{p}_i) \ , \tag{18}$$

where $\mathbf{c} = [c_1, c_2, \ldots, c_N]^{\mathrm{T}}$ is the vector of the unknown coefficients, $\mathbf{t} = [t_{1_x}, t_{1_y}, \ldots, t_{N_x}, t_{N_y}]^{\mathrm{T}}$ is the vector containing the unknown coordinates of all the sources, and $k(\mathbf{t}_j, \mathbf{p}_i) = \log r_{ij}$ is the fundamental solution of Laplace's equation with $r_{ij} = \sqrt{(p_{i_x} - t_{j_x})^2 + (p_{i_y} - t_{j_y})^2}$.

Table 1. CPU times (in s) required by the integration routines

Tolerance	D01DAF	Combination of the three NAG subroutines
10^{-2}	0.06	0.11
10^{-4}	0.16	0.17
10^{-6}	0.38	0.20
10^{-8}	1.32	0.24
10^{-10}	2.71	0.37
10^{-12}	No convergence	0.46
10^{-14}	No convergence	0.53
10^{-16}	No convergence	0.64

Since $\bar{v}_i$ is a solution of the differential equation (4), the coefficients c_j and the positions of the sources $\mathbf{t}_j$ are chosen so that the boundary conditions are satisfied in a least-squares sense, namely by minimizing the nonlinear functional

$$F(\mathbf{c}, \mathbf{t}) = \sum_{i=1}^{M_1} (\bar{v}_i - g_i + u_{o_i})^2 + \sum_{i=M_1+1}^{M} \left(\frac{\partial \bar{v}_i}{\partial n} - h_i + \frac{\partial u_{o_i}}{\partial n} \right)^2 \ , \tag{19}$$

where M_1 is the number of boundary points on $\partial\Omega_1$.

In order to minimize the functional F, we used the least squares routine LMDIF from MINPACK (Garbow et al. 1980, see also Karegeorghis and Fairweather 1987). This routine employs a modified version of the Levenberg-Marquardt algorithm, and minimizes the sum of squares of M nonlinear functions in N variables. LMDIF terminates when either a user-specified tolerance is achieved or the user specified maximum number of function evaluations is reached.

2.2 Numerical examples

2.2.1 Example 1

We first considered the Poisson problem on a square domain,

$$\nabla^2 u = f \quad \text{in } \Omega = (-1, 1) \times (-1, 1) \ , \tag{20}$$

subject to the Dirichlet boundary condition

$$u = 4xy(1-x)(1-y) \quad \text{on } \partial\Omega \ . \tag{21}$$

The exact solution of the problem for

$$f = 8x(x-1) + 8y(y-1)$$

is

$$u = 4xy(1-x)(1-y) \ .$$

The above problem was solved using the method described in Section 2.1. The particular solution (17) was evaluated on a circle, by setting $R_x = R_y = R_0$. The domain Ω_0 was chosen to be a circle, in all the examples examined, due to the symmetry of the domain Ω. When Ω_0 was taken to be an ellipse, we observed a slight deterioration in the accuracy of the solution as the eccentricity of the ellipse increased. Results have been obtained for various values of R_0 and various numbers of sources N and function evaluations $NFEV$. The initial distance of the sources from the boundary was taken to be $d = 0.1$.

In Table 2, we tabulate the maximum absolute errors in the solution for various values of N, on a 0.25×0.25 grid on Ω with $R_0 = 2.0$ and $NFEV = 20\,000$. We observe that the accuracy of the approximation improves as N is increased. We also examined the effect of R_0 on the maximum absolute error in the solution. As R_0 becomes larger, the calculated values of the particular solution become larger and the absolute accuracy of the minimization process is reduced. This is due to the fact that the magnitude of the particular solutions in Eq. (17) – and their normal derivatives in Eq. (16) – increases with R_0. As a

Table 2. Maximum absolute errors in the solution for various values of N; Poisson test problems, 0.25×0.25 grid, $R_o = 2.0$, $NFEV = 20\,000$

N	Problem 1	Problem 2
28	0.87D-4	0.89D-3
32	0.23D-4	0.65D-3
36	0.84D-5	0.29D-4

result, the quantities to be minimized in (19) also increase in magnitude and this leads to a reduction in the absolute accuracy of the minimization process (for a fixed number of function evaluations). On the other hand, we observed that for values of R_0 smaller than 2.0, i.e. when Ω_0 is very close to Ω, the integration routines failed to converge for relatively high values of the tolerance. For instance, for $R_0 < 2.0$, the integration routines failed to converge for a tolerance of 10^{-10}, but converged for larger values of the tolerance. This is because as Ω_0 tends to Ω, further singularities tend to appear in both integrals (16) and (17). In Table 3, we give the maximum absolute errors in the solution for $N = 36$ and various values of R_0. In Table 4, we tabulate the standard error of the MFS (based on the boundary approximation where standard error is defined as $\sqrt{F/M}$, where F is given in (19)) for $R_0 = 2.0, N = 36$ and $NFEV = 20\,000$. We observe, that the standard error is essentially the same for tolerances, less than 10^{-4}. In all subsequent results, the value of the tolerance was taken equal to 10^{-10}.

2.2.2 Example 2

We subsequently solved the boundary value problem (20)–(21) when some of the Dirichlet boundary conditions are replaced by Neumann boundary conditions. In particular, we considered the case

$$u = 4xy(1-x)(1-y) \quad \text{on } -1 \le x \le 1,\, y = -1, \quad x = -1,\, -1 \le y \le 1$$

Table 3. Maximum absolute errors in the solution for various values of R_o; Poisson test problems, 0.25×0.25 grid, $N = 36$, $NFEV = 20\,000$

R_o	Problem 1	Problem 2
1.8	0.18D-4	0.69D-4
2.0	0.84D-5	0.29D-4
3.0	0.71D-4	0.34D-3
4.0	0.34D-3	0.72D-3

Table 4. Effect of integration tolerance on the MFS standard error; $R_o = 2.0$, $N = 36$, $NFEV = 20\,000$

Integration tolerance	MFS standard error
10^{-2}	0.31D-03
10^{-4}	0.98D-04
10^{-6}	0.98D-04
10^{-8}	0.98D-04
10^{-10}	0.98D-04

and

$$\frac{\partial u}{\partial n} = [4y(1-y)(1-2x)]\left(\frac{\partial p_x}{\partial n}\right) + [4x(1-x)(1-2y)]\left(\frac{\partial p_y}{\partial n}\right)$$
$$\text{on } -1 \le x \le 1,\, y = 1, \quad x = 1,\, -1 \le y \le 1 \ ,$$

which has the exact solution of example 1.

The problem was solved for $d = 0.1$ and various values of N, R_0 and $NFEV$. As was the case in example 1, the maximum absolute error in the solution decreases with N (Table 2) and increases with R_0 (Table 3), when R_0 is greater than 2.0.

3 Inhomogeneous biharmonic problems

3.1 Formulation

The formulation for Poisson problems presented in Section 2.1 can be extended for the solution of inhomogeneous biharmonic problems. For example, consider the problem

$$\nabla^4 u = f(x,y) \quad \text{in } \Omega \ , \tag{22}$$

subject to either

$$u = g_1(x,y), \quad \frac{\partial u}{\partial n} = h_1(x,y) \quad \text{on } \partial\Omega \ , \tag{23}$$

or

$$u = g_2(x,y), \quad \nabla^2 u = h_2(x,y) \quad \text{on } \partial\Omega \ , \tag{24}$$

where g_1, g_2, h_1 and h_2 are prescribed functions.

If u_0 is a particular solution of (22), then $v = u - u_0$ satisfies the homogeneous biharmonic equation

$$\nabla^4 v = 0 \quad \text{in } \Omega \ , \tag{25}$$

subject to either

$$v = g_1(x,y) - u_0, \quad \frac{\partial v}{\partial n} = h_1(x,y) - \frac{\partial u_0}{\partial n} \text{ on } \partial\Omega \ , \tag{26}$$

or

$$v = g_2(x,y) - u_0, \quad \nabla^2 v = h_2(x,y) - \nabla^2 u_0 \text{ on } \partial\Omega \ . \tag{27}$$

A particular solution of Eq. (22) is given by (see Golberg and Chen 1994)

$$u_0(\mathbf{p}) = \frac{1}{8\pi}\int_\Omega |\mathbf{p}-\mathbf{q}|^2 \log|\mathbf{p}-\mathbf{q}|\, f(\mathbf{q})\, dV(\mathbf{q}), \quad \mathbf{p} \in \Omega \ . \tag{28}$$

If $f(\mathbf{q})$ can be extended smoothly to Ω_0 which contains $\bar{\Omega}$, then

$$u_0(\mathbf{p}) = \frac{1}{8\pi}\int_{\Omega_0} |\mathbf{p}-\mathbf{q}|^2 \log|\mathbf{p}-\mathbf{q}| f(\mathbf{q})\, dV(\mathbf{q}), \quad \mathbf{p} \in \bar{\Omega} \ . \tag{29}$$

After using the change of variables (9), (29) becomes

$$u_0(\mathbf{p}) = \frac{1}{8\pi}\int_{\Omega_0}\left([b_x(\theta)-p_x]^2+[b_y(\theta)-p_y]^2\right) \times \log\left(R\sqrt{[b_x(\theta)-p_x]^2+[b_y(\theta)-p_y]^2}\right) \times R^3 H(\mathbf{p},\theta) f_0(\mathbf{p},R,\theta)\,\mathrm{d}R\,\mathrm{d}\theta \ . \tag{30}$$

In the case when Ω_0 is an ellipse, from (13) we obtain

$$\begin{aligned} u_0(\mathbf{p}) &= \frac{1}{8\pi}\int_0^{2\pi}\left([b_x(\theta)-p_x]^2+[b_y(\theta)-p_y]^2\right)H(\mathbf{p},\theta) \\ &\quad\times\int_0^1 R^3\log R f_0(\mathbf{p},R,\theta)\,\mathrm{d}R\,\mathrm{d}\theta \\ &\quad+\frac{1}{16\pi}\int_0^{2\pi}\int_0^1\left([b_x(\theta)-p_x]^2+[b_y(\theta)-p_y]^2\right) \\ &\quad\times\log\left([b_x(\theta)-p_x]^2+[b_y(\theta)-p_y]^2\right) \\ &\quad\times R^3H(\mathbf{p},\theta)f_0(\mathbf{p},R,\theta)\,\mathrm{d}R\,\mathrm{d}\theta \ , \\ &\equiv I_1+I_2 \ . \end{aligned} \tag{31}$$

Note that the inner integral of I_1 has a R^3 log R (mild) singularity. The normal derivative and the Laplacian of $u_0(\mathbf{p})(\mathbf{p}\in\bar{\Omega})$ are given by

$$\begin{aligned} \frac{\partial u_0}{\partial n}(\mathbf{p}) &= -\frac{1}{4\pi}\int_0^{2\pi}\left[[b_x(\theta)-p_x]\left(\frac{\partial p_x}{\partial n}\right)\right. \\ &\quad\left.+[b_y(\theta)-p_y]\left(\frac{\partial p_y}{\partial n}\right)\right]H(\mathbf{p},\theta) \\ &\quad\times\int_0^1 R^2\log R f_0(\mathbf{p},R,\theta)\,\mathrm{d}R\,\mathrm{d}\theta \\ &\quad-\frac{1}{8\pi}\int_0^{2\pi}\int_0^1\left[[b_x(\theta)-p_x]\left(\frac{\partial p_x}{\partial n}\right)\right. \\ &\quad\left.+[b_y(\theta)-p_y]\left(\frac{\partial p_y}{\partial n}\right)\right] \\ &\quad\times\left[1+\log\left([b_x(\theta)-p_x]^2+[b_y(\theta)-p_y]^2\right)\right] \\ &\quad\times R^2H(\mathbf{p},\theta)f_0(\mathbf{p},R,\theta)\,\mathrm{d}R\,\mathrm{d}\theta \\ &\equiv I_1'+I_2' \end{aligned} \tag{32}$$

and

$$\begin{aligned} \nabla^2u_0(\mathbf{p}) &= \frac{1}{2\pi}\int_0^{2\pi}H(\mathbf{p},\theta)\int_0^1 R\log R f_0(\mathbf{p},R,\theta)\,\mathrm{d}R\,\mathrm{d}\theta \\ &\quad+\frac{1}{4\pi}\int_0^{2\pi}\int_0^1\left[2+\log\left([b_x(\theta)-p_x]^2\right.\right. \\ &\quad\left.\left.+[b_y(\theta)-p_y]^2\right)\right]RH(\mathbf{p},\theta)f_0(\mathbf{p},R,\theta)\,\mathrm{d}R\,\mathrm{d}\theta \\ &\equiv I_1''+I_2'' \ . \end{aligned} \tag{33}$$

The inner integrals of I_1' and I_1'' contain R^2 log R and R log R (mild) singularities, respectively. These integrals are evaluated using the combination of the three NAG integration routines used for the Poisson problems in Section 2.1, so that the singularities are taken into account.

Having evaluated the particular solution u_0 and its normal derivatives or its Laplacian at the specified boundary points from (31), (32) or (33), one can then solve the homogeneous problem (25)–(27) using the biharmonic MFS (Karageorghis and Fairweather 1987). The solution v at a point $\mathbf{p}_i$ is approximated by a linear combination of fundamental solutions of both the Laplace and biharmonic equations as follows:

$$\bar{v}_i = \bar{v}(\mathbf{c},\mathbf{d},\mathbf{t},\mathbf{p}_i) = \sum_{j=1}^{N}[c_jk_1(\mathbf{t}_j,\mathbf{p}_i)+d_jk_2(\mathbf{t}_j,\mathbf{p}_i)] \ , \tag{34}$$

where $\mathbf{d}=[d_1,d_2,\ldots,d_N]^{\mathrm{T}}$ is the vector of the unknown coefficients of the biharmonic fundamental solution $k_2(\mathbf{t}_j,\mathbf{p}_i)=r_{ij}^2\log r_{ij}$. The approximation $\bar{v}_i$ of the solution satisfies Eq. (25) and, therefore, the coefficients c_j, d_j and the positions of sources $\mathbf{t}_j$ are determined so that the boundary conditions are satisfied. To achieve this, we minimize one of the following two functionals

$$F(\mathbf{c},\mathbf{d},\mathbf{t}) = \sum_{i=1}^{M}\left[(\bar{v}_i-g_{1_i}+u_{o_i})^2+\left(\frac{\partial\bar{v}_i}{\partial n}-h_{1_i}+\frac{\partial u_{o_i}}{\partial n}\right)^2\right] \tag{35}$$

and

$$F(\mathbf{c},\mathbf{d},\mathbf{t}) = \sum_{i=1}^{M}\left[(\bar{v}_i-g_{2_i}+u_{o_i})^2 + \left(\nabla^2\bar{v}_i-h_{2_i}+\nabla^2u_{o_i}\right)^2\right] \ , \tag{36}$$

depending on the type of the boundary conditions. For more details see Karageorghis and Fairweather (1987) and Karageorghis (1992).

3.2
Numerical examples

3.2.1
Example 1

We consider the inhomogeneous biharmonic problem on a square domain,

$$\nabla^4u=f \quad \text{in } \Omega=(-1,1)\times(-1,1) \ , \tag{37}$$

where

$$f = 24(\mathrm{e}^x+\mathrm{e}^y)+(y^2-1)^2\mathrm{e}^x+(x^2-1)^2\mathrm{e}^y + 8[(3y^2-1)\mathrm{e}^x+(3x^2-1)\mathrm{e}^y] \ ,$$

subject to the boundary conditions

$$u=(y^2-1)^2\mathrm{e}^x+(x^2-1)^2\mathrm{e}^y \quad \text{on } \partial\Omega$$

and

$$\frac{\partial u}{\partial n} = [4x(x^2-1)\mathrm{e}^y+(y^2-1)^2\mathrm{e}^x]\left(\frac{\partial p_x}{\partial n}\right) + [4y(y^2-1)\mathrm{e}^x+(x^2-1)^2\mathrm{e}^y]\left(\frac{\partial p_y}{\partial n}\right) \quad \text{on } \partial\Omega \ .$$

The exact solution of this problem is $u=(y^2-1)^2\mathrm{e}^x+(x^2-1)^2\mathrm{e}^y$.

Table 5. Maximum absolute errors in the solution for various values of N; Inhomogeneous biharmonic test problems, 0.25×0.25 grid, $R_o = 2.0$, $NFEV = 20\,000$

N	Problem 1	Problem 2
28	0.11D-1	0.45D-2
32	0.11D-3	0.98D-3
36	0.10D-3	0.43D-4

Table 6. Maximum absolute errors in the solution for various values of R_o; Inhomogeneous biharmonic test problems, 0.25×0.25 grid, $N = 36$, $NFEV = 20\,000$

R_o	Problem 1	Problem 2
1.8	0.92D-3	0.10D-3
2.0	0.10D-3	0.43D-4
3.0	0.13D-2	0.25D-2
4.0	0.11D-1	0.19D-1

The particular solution (31) and its normal derivative (32) were evaluated on a circle of radius R_0. As with the Poisson problems, the accuracy of the approximation improves as N is increased (Table 5) and R_0 is reduced (Table 6), provided Ω_0 is not too close to Ω.

3.2.2 Example 2

Consider Eq. (37) subject to the boundary conditions

$$u = \left(y^2 - 1\right)^2 e^x + \left(x^2 - 1\right)^2 e^y \quad \text{on } \partial\Omega$$

and

$$\nabla^2 u = \left(y^2 - 1\right)^2 e^x + \left(x^2 - 1\right)^2 e^y + 4\left[\left(3x^2 - 1\right)e^y + \left(3y^2 - 1\right)e^x\right)] \quad \text{on } \partial\Omega \ .$$

Again, we evaluated $u_0(\mathbf{p})$ (31) and its Laplacian (33) on a circle of radius R_0. The results, shown in Tables 5 and 6, are similar to those obtained in example 1.

4 Poisson problems with a harmonic inhomogeneous term

In this section, a special class of inhomogeneous harmonic problems is considered, in which the inhomogeneous term is harmonic. In this case, Atkinson (1985) showed that the construction of a particular solution involves only the calculation of single integrals for each point. We apply Atkinson's method and also illustrate that problems with a harmonic inhomogeneous term can be solved by direct application of the MFS.

4.1 The method of Atkinson

In problem (1)–(3), suppose that f is harmonic, i.e. $\nabla^2 f(x, y) = 0$. Following Atkinson (1985), the particular solution at a point $\mathbf{p} = (x, y)$ is assumed to be of the form:

$$u_0(\mathbf{p}) = \tfrac{1}{2} x H(x, y) \ ,$$

where $H(x, y)$ is harmonic and, therefore,

$$\frac{\partial H}{\partial x}(x, y) = f \ .$$

Integrating gives

$$H(x, y) = \int_{x_0}^{x} f(s, y)\, ds + h(y) \ , \tag{38}$$

where the point x_0 and the function $h(y)$ are arbitrary. Since, $H(x, y)$ is harmonic, it follows that

$$h''(y) = -\frac{\partial f}{\partial x}(x_0, y) \ ,$$

and, therefore,

$$h(y) = -\int_{y_0}^{y} (y - t) \frac{\partial f}{\partial x}(x_0, t)\, dt \ . \tag{39}$$

Combining (38) and (39), the following particular solution can be obtained:

$$u_0(\mathbf{p}) = \frac{1}{2} x \left(\int_{x_0}^{x} f(s, y)\, ds - \int_{y_0}^{y} (y - t) \frac{\partial f}{\partial x}(x_0, t)\, dt \right) \ . \tag{40}$$

The single integrals involved can be easily evaluated, e.g., by using the NAG subroutine D01AHF. The reference point (x_0, y_0) is usually chosen to be at the center of the domain. The evaluation of the particular solution from Eq. (40), using a subroutine such as D01AHF, can be incorporated in the MFS which is used to solve the resulting homogeneous problem.

4.2 Direct MFS application

In the case of harmonic f the Poisson problem (1)–(3) can be transformed into the homogeneous biharmonic problem

$$\nabla^4 u = 0 \quad \text{in } \Omega \ , \tag{41}$$

subject to

$$u = g(x, y) \quad \text{on } \partial\Omega \ . \tag{42}$$

The solution u at a point $\mathbf{p}_i$ is approximated by (34)

$$\bar{u}_i = \bar{u}(\mathbf{c}, \mathbf{d}, \mathbf{t}, \mathbf{p}_i) = \sum_{j=1}^{N} \left[c_j\, k_1\left(\mathbf{t}_j, \mathbf{p}_i\right) + d_j\, k_2\left(\mathbf{t}_j, \mathbf{p}_i\right)\right] \ . \tag{43}$$

Since $\nabla^2 u = f$, it follows that f can be approximated by

$$\bar{f}_i = \bar{f}(\mathbf{d}, \mathbf{t}, \mathbf{p}_i) = \nabla^2 \bar{u}_i = \sum_{j=1}^{N} 4\, d_j\, (1 + \log r_{ij}) \ . \tag{44}$$

The functional to be minimized in the MFS is therefore

$$F(\mathbf{c}, \mathbf{d}, \mathbf{t}) = \sum_{i=1}^{M} \left[(\bar{u}_i - u_i)^2 + (\nabla^2 \bar{u}_i - f_i)^2\right] \ . \tag{45}$$

4.3
Numerical examples

4.3.1
Example 1

In this example, we solve the Poisson problem with a harmonic right hand side on a square domain

$$\nabla^2 u = f \quad \text{in } \Omega = (-1,1) \times (-1,1) \ , \tag{46}$$

with

$$f = -\tfrac{26}{3}xy \ ,$$

subject to the Dirichlet boundary condition

$$u = xy\left(1 - \tfrac{4}{9}x^2 - y^2\right) \quad \text{on } \partial\Omega \ .$$

The exact solution of this problem is $u = xy\left(1 - \frac{4}{9}x^2 - y^2\right)$. To solve the above problem we applied two different methods. We first applied the method of Atkinson (1985) presented in Section 4.1, using Eq. (40) to evaluate the particular solution and solving the resulting homogeneous problem with the MFS. We also solved the problem with the direct MFS formulation presented in Section 4.2. The maximum absolute errors in the solution obtained with the two methods on a 0.25×0.25 grid with $N = 12$ and $NFEV = 2000$ are illustrated in Table 7. We observed that Atkinson's method performed slightly better than the direct MFS.

4.3.2
Example 2

Similar results were obtained for the Poisson problem (46) with harmonic $f(x,y)$ on a square domain $(-1,1) \times (-1,1)$ with

$$f = x^2 - y^2 + x + y \ ,$$

subject to the Dirichlet boundary condition

$$u = \frac{x^4 - y^4}{12} + \frac{x^3 - y^3}{6} \quad \text{on } \partial\Omega \ .$$

The exact solution of the problem is

$$u = \frac{x^4 - y^4}{12} + \frac{x^3 - y^3}{6} \ .$$

The maximum absolute errors in the solution obtained on a 0.25×0.25 grid with $N = 12$ and $NFEV = 2000$ are tabulated in Table 7.

4.3.3
Example 3

Finally, we solved the Poisson problem (46) on a square domain $(-1,1) \times (-1,1)$ with

$$f = e^x \cos y \ ,$$

subject to the Dirichlet boundary condition

$$u = \tfrac{1}{2}x\, e^x \cos y \quad \text{on } \partial\Omega \ .$$

The exact solution of this problem is $u = \frac{1}{2}\, x\, e^x \cos y$. The maximum absolute errors obtained in the solution with the two methods on a 0.25×0.25 grid with $N = 12$ and $NFEV = 2000$ are presented in Table 7. Also, the CPU times required by the two methods for different tolerances with $N = 12$ are tabulated in Table 8. We observe that Atkinson's method is considerably faster. This is to be expected since in the case of the direct MFS application, we are essentially transforming a second order problem into a fourth order problem, which is then solved with the MFS. The direct MFS application is, however, much easier to implement.

Table 7. Maximum absolute errors in the solution for Poisson test problems with harmonic inhomogeneous term; 0.25×0.25 grid, $N = 12$, $NFEV = 2000$

	Atkinson (1985)	Direct MFS
Problem 1	0.12D-4	0.11D-3
Problem 2	0.19D-4	0.29D-3
Problem 3	0.43D-5	0.57D-4

Table 8. CPU times (in s) required by Atkinson's method and by direct application of the MFS; $N = 12$

Tolerance	Atkinson (1985)	Direct MFS
10^{-2}	2.21	16.41
10^{-3}	14.81	137.48
10^{-4}	131.06	1114.58
10^{-5}	503.16	2269.46

5
Conclusions

In this work, we present extensions of the MFS for solving inhomogeneous harmonic and biharmonic problems. The problems are transformed into homogeneous ones after constructing particular solutions, based on the Newton potential, and subtracting them from the solution. The resulting homogeneous problems are solved using the MFS. The numerical calculations indicate that accurate results can be obtained with relatively few degrees of freedom. Two methods for the special case of Poisson problems where the inhomogeneous term is harmonic are also examined. The first is based on the particular solution approach and the second on the direct application of the MFS. The latter is computationally more costly but easier to implement.

The methods described in this paper can be extended to any elliptic boundary value problem governed by a linear inhomogeneous equation of the form $\mathscr{L}u = f$, if the fundamental solution of $\mathscr{L}$ is known. This is extremely important as the method could then be applied to nonlinear problems, the solution of which could be obtained by solving a sequence of inhomogeneous linear elliptic boundary value problems.

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