Short communication

Analytical solution of the flow of a Newtonian fluid with pressure-dependent viscosity in a rectangular duct

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ABSTRACT

We derive a fully analytical solution for the steady flow of an isothermal Newtonian fluid with pressure-dependent viscosity in a rectangular duct. The analytical solution for the governing equations is exact (based on the work by Akyildiz and Siginer, Int. J. Eng. Sc., 104, 2016), while the total mass balance constraint is satisfied with a high-order asymptotic expression in terms of the dimensionless pressure-dependent coefficient ε, and an excellent improved solution derived with Shanks' nonlinear transformation. Numerical calculations confirm the correctness, accuracy and consistency of the asymptotic expression, even for large values of ε. Results for the average pressure difference required to drive the flow are also presented and discussed, revealing the significance of the pressure-dependent viscosity even for steady, unidirectional, Newtonian flow.

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1. Introduction and problem definition

During the last few decades, the flow of liquids with pressure-dependent viscosity has become an active area of research by many investigators [1–11]. This effect is important in cases for which a substantial range of the total pressure is developed in the flow domain, or when steep pressure gradients are observed. The nonlinear coupling of the pressure with the velocity field makes flows with pressure-dependent viscosity more difficult to solve than their constant-viscosity counterparts.

In this short note, we study the 2D unidirectional and pressure-driven steady flow of an isothermal Newtonian fluid in a rectangular duct with dimensions $L \times H \times l$. The length of the duct l is assumed much larger than its height H so that the flow is fully developed and entrance/end effects can be neglected. A sketch of the geometry of the flow and the Cartesian coordinate system xyz (used to describe the flow field) with unit vectors $\mathbf{e_x}$, $\mathbf{e_y}$ and $\mathbf{e_z}$ is given in Fig. 1. It is assumed that the mass density ρ^* of the fluid is constant while its viscosity η^* varies linearly with the total pressure p^* [2]:

$$\eta^* = \eta_0^*(1 + \beta^*(p^* - p_0^*)) \tag{1}$$

where η_0^* is the viscosity at the reference pressure p_0^*, and β^* is the coefficient of proportionality between the pressure difference and the viscosity (a star denotes dimensional quantity). Typical values for β^* are $10 - 50\,GPa^{-1}$ for polymer melts [12,13], $10 - 70\,GPa^{-1}$ for lubricants [14] and $10 - 20\,GPa^{-1}$ for mineral oils [15].

At steady state, neglecting any external forces and torques, and assuming unidirectional two-dimensional velocity profile $\mathbf{u}^* = w^*(x^*, y^*)\mathbf{e_z}$ and three-dimensional pressure p^*, the continuity equation is automatically satisfied, and the three

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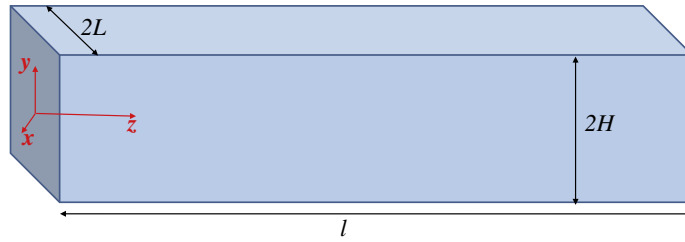*E-mail addresses:* housiada@aegean.gr (K.D. Housiadas), georgios@ucy.ac.cy (G.C. Georgiou).

Fig. 1. Sketch of the geometry and coordinate system ($H \ll l$, $H < L$). The flow is from left to right.

components of the momentum equation read:

$$-\frac{\partial p^*}{\partial x^*} + \eta_0^* \beta^* \frac{\partial p^*}{\partial z^*} \frac{\partial w^*}{\partial x^*} = 0 \tag{2}$$

$$-\frac{\partial p^*}{\partial y^*} + \eta_0^* \beta^* \frac{\partial p^*}{\partial z^*} \frac{\partial w^*}{\partial y^*} = 0 \tag{3}$$

$$-\frac{\partial p^*}{\partial z^*} + \eta_0^* \beta^* \left(\frac{\partial p^*}{\partial x^*} \frac{\partial w}{\partial x^*} + \frac{\partial p^*}{\partial y^*} \frac{\partial w^*}{\partial y^*} \right) + \eta_0^* (1 + \beta^* (p^* - p_0^*)) \left(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}} \right) = 0 \tag{4}$$

The equations are rendered dimensionless by scaling x^* by L, y^* by H, z^* by l, η^* by η_0^*, w^* by the average entrance velocity U^*, and $p^* - p_0^*$ by $3\eta_0^* U^* l/H^2$. The scaling for the pressure difference is such that in the case of a constant shear viscosity (i.e. for $\eta^* = \eta_0^*$) the required average pressure drop to drive the flow is unity. With these scales, the dimensionless forms of Eqs. (1–4) become:

$$\eta = 1 + \varepsilon p \tag{5}$$

$$-3\frac{\partial p}{\partial x} + \varepsilon a^2 \frac{\partial p}{\partial z} \frac{\partial w}{\partial x} = 0 \tag{6}$$

$$-3\frac{\partial p}{\partial y} + \varepsilon a^2 \frac{\partial p}{\partial z} \frac{\partial w}{\partial y} = 0 \tag{7}$$

$$-3\frac{\partial p}{\partial z} + \varepsilon c^2 \frac{\partial p}{\partial x} \frac{\partial w}{\partial x} + \varepsilon \frac{\partial p}{\partial y} \frac{\partial w}{\partial y} + (1 + \varepsilon p) \left(c^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0 \tag{8}$$

Three dimensionless numbers appear. The pressure-dependent coefficient ε and the two aspect ratios, respectively, defined by:

$$\varepsilon \equiv 3\beta^* \eta_0^* U^* l/H^2, \quad a \equiv H/l, \quad c \equiv H/L \tag{9}$$

By setting $H = l$ (i.e. $a = 1$) Eqs. (5–8) and ε reduce to those reported by Akyildiz and Signer [1]. Regarding the magnitude of the dimensionless parameters, ε and a are generally small, while c can take any positive value. Without any loss of generality, it can be assumed, however, that $0 < c \le 1$.

Eqs. (5)–(8) are closed with the usual no-slip boundary condition at the walls, i.e. $w(x = \pm 1, y) = w(x, y \pm 1) = 0$, and a reference value for the pressure, $p(x = 1, y = 1, z = 1) = 0$. Substituting (6) and (7) into (8) one gets:

$$-3\frac{\partial p}{\partial z}\left(1 - \frac{\varepsilon^2 c^2 a^2}{9} \left(\frac{\partial w}{\partial x} \right)^2 - \frac{\varepsilon^2 a^2}{9} \left(\frac{\partial w}{\partial y} \right)^2 \right) + (1 + \varepsilon p) \left(c^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0 \tag{10}$$

The analytical solution of Eq. (10) has been derived recently by Akyildiz & Signer [1]. First, Eq. (10) is rearranged as follows:

$$\left(c^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \bigg/ \left(1 - \frac{\varepsilon^2 c^2 a^2}{9} \left(\frac{\partial w}{\partial x} \right)^2 - \frac{\varepsilon^2 a^2}{9} \left(\frac{\partial w}{\partial y} \right)^2 \right) = \frac{3\frac{\partial p}{\partial z}}{1 + \varepsilon p} = -A \tag{11}$$

where A is a constant that must be determined as part of the solution. The solution of the first equation of (11) is given as:

$$w(x, y) = -\frac{1}{A A_1} \ln\left(u(x, y) + 1 \right), \quad u(x, y) = \sum_{n}^{\infty} v_n(x) G_n(y) \tag{12a}$$

where

$$\nu_n(x) = \frac{A_{3,n}}{A_{2,n}}\left(\frac{\cosh\left(x\sqrt{A_{2,n}/c}\right)}{\cosh\left(\sqrt{A_{2,n}/c}\right)} - 1\right), \quad G_n(y) = \cos\left[\left(n + \tfrac{1}{2}\right)\pi y\right]$$

$$A_1 = (\varepsilon a/3)^2, \quad A_{2,n} = (n + 1/2)^2\pi^2 + A^2 A_1, \quad A_{3,n} = \frac{4A^2 A_1 (-1)^n}{(1 + 2n)\pi} \tag{12b}$$

Notice that $\{G_n\}_{n=0}^{\infty}$ is a set of orthogonal functions defined in $[-1, 1]$.

The total pressure p is found by solving the second differential equation of Eq. (11) with the aid of Eqs. (6) and (7) and using that $p(1, 1, 1) = 0$:

$$p(x, y, z) = \frac{1}{\varepsilon}\left\{-1 + \exp\left(\frac{A\varepsilon}{3}(1 - z) - \frac{A\varepsilon^2 a^2}{9}w(x, y)\right)\right\} \tag{13a}$$

Substituting w in terms of u, one finally gets:

$$p(x, y, z) = \frac{1}{\varepsilon}\left\{-1 + (1 + u(x, y))e^{A\varepsilon(1-z)/3}\right\} \tag{13b}$$

The solution of Eqs. (12) and (13), is completed with the determination of A through the total mass balance:

$$\int_{x=-1}^{x=1}\int_{y=-1}^{y=1} w(x, y)\,dx\,dy = 1 \tag{14}$$

Eq. (14) cannot be solved analytically, and thus Akyildiz and Siginer [1] solved it numerically. Hence, the solution of the problem under investigation is analytic except for the constant A.

2. Asymptotic solution for a

In order to obtain a fully analytical solution, we solve Eq. (14) asymptotically for small values of the pressure-dependent parameter ε. The latter approach has been implemented successfully for similar type of 1D internal steady flows with pressure-dependent viscosity such as the weakly compressible Newtonian flow in channel and tubes [4], and the unidi-rectional viscoelastic flow of a Maxwell fluid [3].

Noting that in the solution for w, ε appears only as a product with the aspect ratio a, we define the modified pressure-dependent parameter $\hat{\varepsilon} \equiv \varepsilon a$ and assume that the solution for A is given as:

$$A = \Lambda_0 + \hat{\varepsilon}^2 \Lambda_2 + \hat{\varepsilon}^4 \Lambda_4 + O(\hat{\varepsilon}^6) \tag{15}$$

Also, suitable asymptotic expressions for u and w are:

$$u = \hat{\varepsilon}^2 u_2 + \hat{\varepsilon}^4 u_4 + O(\hat{\varepsilon}^6), \quad w = w_0 + \hat{\varepsilon}^2 w_2 + \hat{\varepsilon}^4 w_4 + O(\hat{\varepsilon}^6) \tag{16}$$

It is worth noting that only even powers of $\hat{\varepsilon}$ appear in Eq. (16), and also that the first term in the expression for u is $O(\hat{\varepsilon}^2)$ for consistency with the expression for w. Using Eq. (12a) the relation between u_j and w_j can be found. For the lowest-order profiles (u_2 and w_0) the expression $w_0 = -9u_2/\Lambda_0$ holds, where w_0 is simply the velocity distribution for a Newtonian fluid with constant viscosity, and the expression for u_2 is given by:

$$u_2(x, y) = \Lambda_0^2 \sum_{n=0}^{\infty} F_n(x) G_n(y), \quad F_n(x) = \frac{2(-1)^n}{9\pi^3(n + 1/2)^3}\left(\frac{\cosh(\omega_n x)}{\cosh(\omega_n)} - 1\right), \quad \omega_n := \frac{(n + 1/2)\pi}{c} \tag{17}$$

Substituting Eqs. (15) and (16) into Eqs. (12) and (14), expanding all quantities as power series in terms of $\hat{\varepsilon}$, and after many tedious algebraic manipulations one gets:

$$\Lambda_0 = \frac{c^4}{8}\left\{\sum_{n=0}^{\infty}\omega_n^{-4}\left(1 - \frac{\tanh(\omega_n)}{\omega_n}\right)\right\}^{-1} \tag{18a}$$

$$\Lambda_2 = \frac{2\Lambda_0^4}{9c^6}\left\{\sum_{n=0}^{\infty}\omega_n^{-6}\left(2 + \frac{1}{\cosh^2(\omega_n)} - \frac{3\tanh(\omega_n)}{\omega_n}\right)\right\} \tag{18b}$$

$$\Lambda_4 = 3\left(I + \frac{\Lambda_2^2}{\Lambda_0}\right) \tag{18c}$$

In Eq. (18c) $I \equiv \int_{-1}^{1}\int_{-1}^{1} u_2^3(x, y)\,dx\,dy$ has been found analytically, however the solution is too long to be given here. Using the least-squares method and low order polynomial fitting functions, an approximate expression for I is given by:

$$I_a = 10^{-4}\left(-0.56561 - 1.11957c - 4.7854c^2 + 4.51844c^3 - 8.43169c^4\right) \tag{19}$$

Comparison between I and I_a is performed in Fig. 2 where excellent agreement is observed.
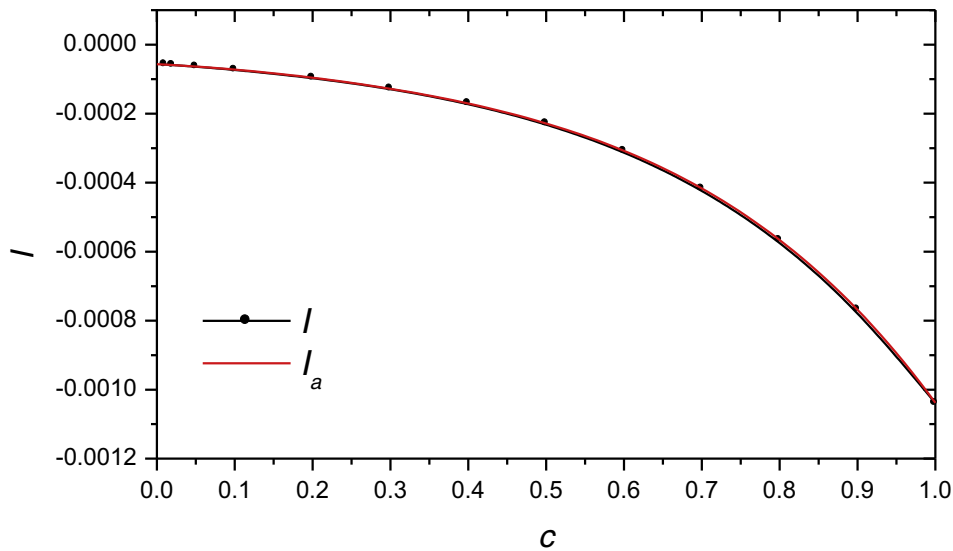
Fig. 2. The integral *I* which appears in Eq. (16) along with its approximate expression I_a given by Eq. (17) as functions of the aspect ratio c.
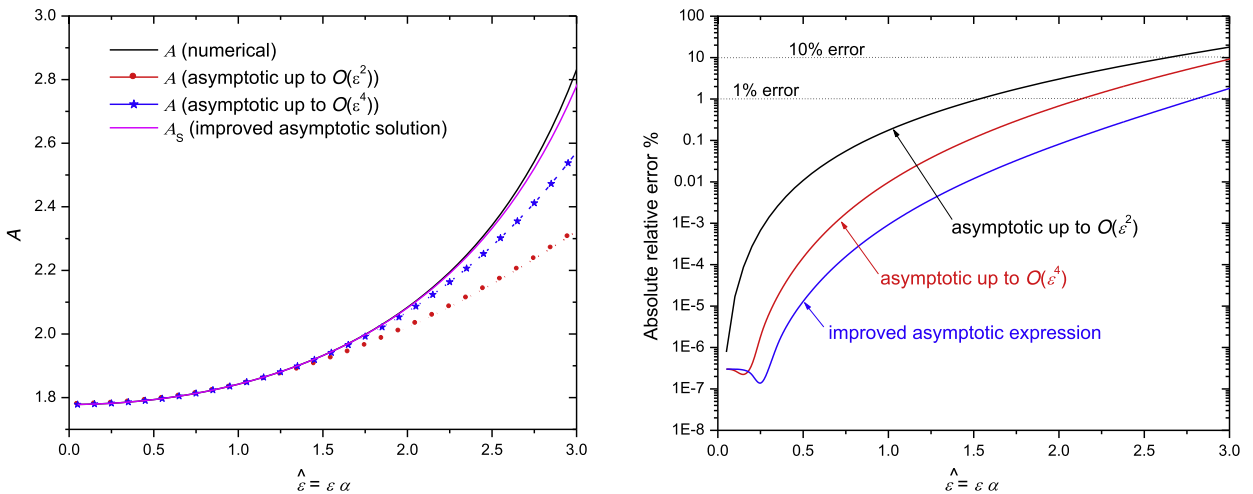

Fig. 3. The parameter *A* versus $\hat{\varepsilon}$ and the corresponding relative absolute errors for a square duct (c = 1).

Since three terms in the asymptotic expression for *A* have been found, Shanks' non-linear transformation [16] can be applied to find a more accurate formula for *A*:

$$A_S = \frac{\Lambda_0 \Lambda_2 - \hat{\varepsilon}^2 \left(\Lambda_0 \Lambda_4 - \Lambda_2^2 \right)}{\Lambda_2 - \hat{\varepsilon}^2 \Lambda_4} = \Lambda_0 + \frac{\hat{\varepsilon}^2 \Lambda_2^2}{\Lambda_2 - \hat{\varepsilon}^2 \Lambda_4} \qquad (20)$$

The denominator of Eq. (20) also provides a rough estimate for the critical pressure-dependent parameter at which the solution diverges, i.e. $\hat{\varepsilon}_c \approx \sqrt{\Lambda_2 / \Lambda_4}$.

In order to verify the correctness, accuracy and consistency of the asymptotic expressions for *A*, as well as to check the validity of A_S, we have also solved Eq. (14) numerically. The numerical results for *A*, the asymptotic solution up to $\hat{\varepsilon}^2$, the asymptotic solution up to $\hat{\varepsilon}^4$, and A_S are shown in Fig. 3a, while the corresponding percentage absolute relative errors $|1 - A_a/A_n| \times 100$ are shown in Fig. 3b, where A_a stands for the approximate value of *A* and A_n for its numerical value. These results clearly show the importance of the fourth-order term and also that A_S is even more accurate than the second- and fourth-order formulas over the whole range of $\hat{\varepsilon}$. For instance, the relative absolute error of A_S at $\hat{\varepsilon} = 1$ is about 0.0009%. For relative absolute error less than 1%, expression (20) can be used up to $\hat{\varepsilon} \approx 2.7$ (a rather high value for the modified pressure-dependent parameter), and for error less than 10% Eq. (20) can be used up to $\hat{\varepsilon} \approx 3.6$. As far as $\hat{\varepsilon}_c$ is concerned, we find numerically that for c = 1 the solution diverges at $\hat{\varepsilon}_c \approx 3.9$ while the expression $\sqrt{\Lambda_2 / \Lambda_4}$ gives $\hat{\varepsilon}_c \approx 4.4$. It should be noted that our numerical results (calculated using the 50 first terms of the infinite series involved in the solution) are different from those reported by Akyildiz and Siginer [1].
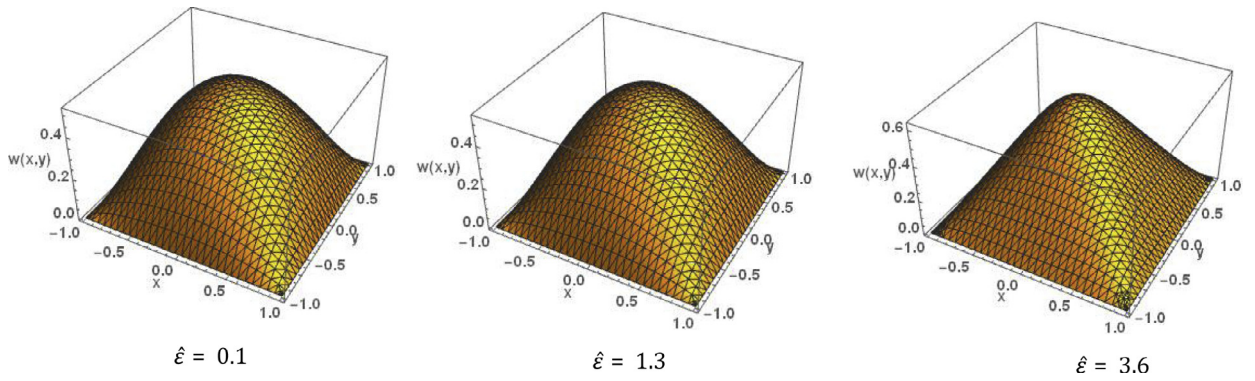
$\hat{\varepsilon} = 0.1$ $\hat{\varepsilon} = 1.3$ $\hat{\varepsilon} = 3.6$

Fig. 4. Velocity profiles in a square duct ($c = 1$) for various values of $\hat{\varepsilon}$.
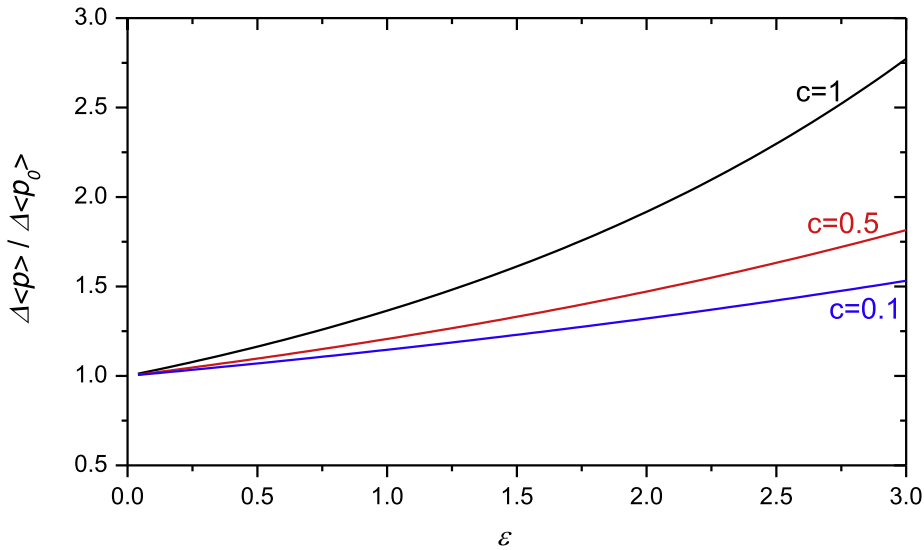

Fig. 5. The normalized average pressure difference required to drive the flow versus ε.

The velocity profiles for $c = 1$ (square duct) and $\hat{\varepsilon}_c = 0.1$, 1.3, and 3.6 are shown in Fig. 4. For these cases, we find that the numerical values for A are 1.77899, 1.89039, and 2.83127, respectively, while the corresponding approximate values A_S are 1.77899, 1.89030, and 2.78021. In agreement with previous results for the 1D unidirectional flow in a channel [5], as well as for the 2D unidirectional flow in a rectangular duct [1], transition from a paraboloidal to a pyramidal profile as the dimensionless pressure-dependent number ε increases, is observed.

3. Average pressure difference

An important feature for internal flows is the average pressure difference required to drive the flow. Defining the difference operator Δ as $\Delta f := f(z=0) - f(z=1)$, the cross-section average of any quantity f as $\langle f \rangle = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} f(x, y) dx dy$, and using Eq. (13a) or (13b), we find:

$$\Delta \langle p \rangle = \frac{1}{\varepsilon} \left(e^{A\varepsilon/3} - 1 \right) \left[1 + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{A_{3,n}(-1)^n}{A_{2,n}(1 + 2n)} \left(-1 + \frac{\tanh\left(\sqrt{A_{2,n}}/c\right)}{\sqrt{A_{2,n}}/c} \right) \right] \tag{21}$$

It is interesting to note that $\lim_{\varepsilon \to 0} \Delta \langle p \rangle \equiv \Delta \langle p_0 \rangle = \Lambda_0/3$. Results for the normalized pressure difference $\Delta \langle p \rangle / \Delta \langle p_0 \rangle$ as a function of ε are presented in Fig. 5 for various values of the aspect ratio c, i.e. for $c = 1$, $c = 0.5$, and $c = 0.1$ with the latter value corresponding essentially to a straight channel; notice that the effect of a on the results is negligible. It is clear that the effect of ε is significant; the normalized pressure difference to drive the flow increases monotonically with the increase of the pressure-dependent parameter. Similar behavior has been reported for pressure-driven flow in straight channels and circular tubes [3–5].

4. Conclusions

The steady-state flow of a Newtonian fluid in a rectangular duct has been solved fully analytically under the assumption that the viscosity varies linearly with pressure. To that end, the constant resulting from the separation of variables is expanded in terms of the pressure-dependence parameter and the resulting perturbation problem is solved up to fourth order. Shanks' nonlinear transformation is then applied to find an improved expression, the accuracy and validity of which is checked by direct comparison with the numerical solution. In agreement with the findings of Akyildiz and Siginer [1], the velocity distribution switches from a paraboloidal distribution to a pyramidal one, as the pressure-dependence number increases. It is also demonstrated that viscosity pressure-dependence may affect significantly the pressure distribution across the rectangular duct and the total pressure-drop required to drive the flow.

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