Chapter 6

UNIDIRECTIONAL FLOWS

Isothermal, laminar, incompressible Newtonian flow is governed by a system of four scalar *partial differential equations* (PDEs); these are the continuity equation and the three components of the Navier-Stokes equation. The pressure and the three velocity components are the *primary unknowns*, which are, in general, functions of time and of spatial coordinates. This system of PDEs is amenable to analytical solution for limited classes of flow. Even in the case of relatively simple flows in regular geometries, the nonlinearities introduced by the convective terms rule out the possibility of finding analytical solutions. This explains the extensive use of numerical methods in Fluid Mechanics [1]. *Computational Fluid Dynamics (CFD)* is certainly the fastest growing branch of fluid mechanics, largely as a result of the increasing availability and power of computers, and the parallel advancement of versatile numerical techniques.

In this chapter, we study certain classes of incompressible flows, in which the Navier-Stokes equations are simplified significantly to lead to analytical solutions. These classes concern *unidirectional* flows, that is, flows which have only one nonzero velocity component, u_i. Hence, the number of the primary unknowns is reduced to two: the velocity component, u_i, and pressure, p. In many flows of interest, the PDEs corresponding to the two unknown fields are decoupled. As a result, one can first find u_i, by solving the corresponding component of the Navier-Stokes equation, and then calculate the pressure. Another consequence of the unidirectionality assumption, is that u_i is a function of at most two spatial variables and time. Therefore, in the worst case scenario of incompressible, unidirectional flow one has to solve a PDE with three independent variables, one of which is time.

The number of independent variables is reduced to two in

(a) *transient one-dimensional* (1D) unidirectional flows in which u_i is a function of one spatial independent variable and time; and

(b) *steady two-dimensional* (2D) unidirectional flows in which u_i is a function of two spatial independent variables.

The resulting PDEs in the above two cases can often be solved using various techniques, such as the *separation of variables* [2] and *similarity methods* [3].

In *steady, one-dimensional unidirectional flows*, the number of independent variables is reduced to one. In these flows, the governing equation for the nonzero velocity component is just a *linear*, second-order *ordinary differential equation* (ODE) which can be solved easily using well-known formulas and techniques. Such flows are studied in the first three sections of this chapter. In particular, in Sections 1 and 2, we study flows in which the streamlines are straight lines, i.e., one-dimensional *rectilinear flows* with $u_x = u_x(y)$ and $u_y = u_z = 0$ (Section 6.1), and *axisymmetric rectilinear flows* with $u_z = u_z(r)$ and $u_r = u_\theta = 0$ (Section 6.2). In Section 6.3, we study *axisymmetric torsional* (or *swirling*) *flows*, with $u_\theta = u_\theta(r)$ and $u_z = u_r = 0$. In this case, the streamlines are circles centered at the axis of symmetry.

In Sections 6.4 and 6.5, we discuss briefly steady *radial flows*, with *axial* and *spherical symmetry*, respectively. An interesting feature of radial flows is that the nonzero radial velocity component, $u_r = u_r(r)$, is determined from the continuity equation rather than from the radial component of the Navier-Stokes equation. In Section 6.6, we study transient, one-dimensional unidirectional flows. Finally, in Section 6.7, we consider examples of steady, two-dimensional unidirectional flows.

Unidirectional flows, although simple, are important in a diversity of fluid transferring and processing applications. As demonstrated in examples in the following sections, once the velocity and the pressure are known, the nonzero components of the stress tensor, such as the shear stress, as well as other useful macroscopic quantities, such as the volumetric flow rate and the shear force (or *drag*) on solid boundaries in contact with the fluid, can be easily determined.

Let us point out that analytical solutions can also be found for a limited class of two-dimensional *almost unidirectional* or *bidirectional* flows by means of the *potential function* and/or the *stream function*, as demonstrated in Chapters 8 to 10. Approximate solutions for limiting values of the involved parameters can be constructed by *asymptotic* and *perturbation analyses*, which are the topics of Chapters 7 and 9, with the most profound examples being the *lubrication*, *thin-film*, and *boundary-layer* approximations.

6.1 Steady, One-Dimensional Rectilinear Flows

Rectilinear flows, i.e., flows in which the streamlines are straight lines, are usually described in Cartesian coordinates, with one of the axes being parallel to the flow direction. If the flow is axisymmetric, a cylindrical coordinate system with the z-axis coinciding with the axis of symmetry of the flow is usually used.

Let us assume that a Cartesian coordinate system is chosen to describe a rectilinear flow, with the x-axis being parallel to the flow direction, as in Fig. 6.1, where the geometry of the flow in a channel of rectangular cross section is shown. Therefore, u_x is the only nonzero velocity component and

$$u_y = u_z = 0\,. \tag{6.1}$$

From the continuity equation for incompressible flow,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0\,,$$

we find that

$$\frac{\partial u_x}{\partial x} = 0\,,$$

which indicates that u_x does not change in the flow direction, i.e., u_x is independent of x:

$$u_x = u_x(y, z, t)\,. \tag{6.2}$$

Flows satisfying Eqs. (6.1) and (6.2) are called *fully developed*. Flows in tubes of constant cross section, such as the one shown in Fig. 6.1, can be considered fully developed if the tube is *sufficiently long* so that entry and exit effects can be neglected.

Due to Eqs. (6.1) and (6.2), the x-momentum equation,

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \rho g_x\,,$$

is reduced to

$$\rho\frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \rho g_x\,. \tag{6.3}$$

If now the flow is steady, then the time derivative in the x-momentum equation is zero, and Eq. (6.3) becomes

$$-\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \rho g_x = 0\,. \tag{6.4}$$

The last equation which describes any steady, two-dimensional rectilinear flow in the x-direction is studied in Section 6.5. In many unidirectional flows, it can be assumed that

$$\frac{\partial^2 u_x}{\partial y^2} \gg \frac{\partial^2 u_x}{\partial z^2}\,,$$

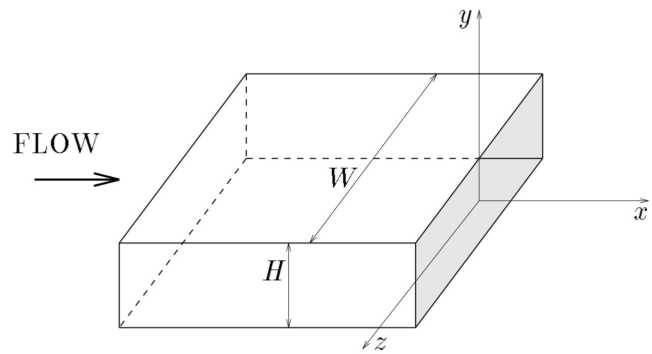


Figure 6.1. *Geometry of flow in a channel of rectangular cross section.*

and u_x can be treated as a function of y alone, i.e.,

$$u_x = u_x(y)\,. \tag{6.5}$$

With the latter assumption, the x-momentum equation is reduced to:

$$-\frac{\partial p}{\partial x} + \eta\,\frac{d^2 u_x}{dy^2} + \rho g_x = 0\,. \tag{6.6}$$

The only nonzero component of the stress tensor is the shear stress τ_{yx},

$$\tau_{yx} = \eta\,\frac{du_x}{dy}\,, \tag{6.7}$$

in terms of which the x-momentum equation takes the form

$$-\frac{\partial p}{\partial x} + \frac{d\tau_{yx}}{dy} + \rho g_x = 0\,. \tag{6.8}$$

Equation (6.6) is a linear second-order ordinary differential equation and can be integrated directly if

$$\frac{\partial p}{\partial x} = \text{const}\,. \tag{6.9}$$

Its general solution is given by

$$u_x(y) = \frac{1}{2\eta}\left(\frac{\partial p}{\partial x} - \rho g_x\right) y^2 + c_1 y + c_2\,. \tag{6.10}$$

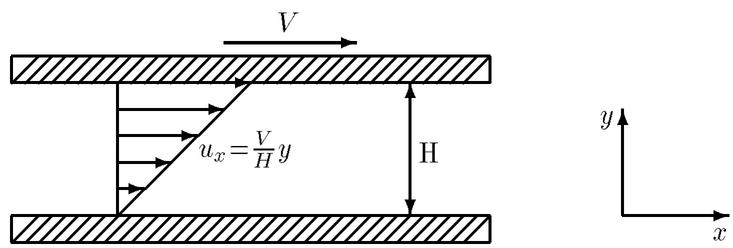


Figure 6.2. *Plane Couette flow.*

Therefore, the velocity profile is a parabola and involves two constants, c_1 and c_2, which are determined by applying appropriate boundary conditions for the particular flow. The shear stress, $\tau_{yx}=\tau_{xy}$, is linear, i.e.,

$$\tau_{yx} \;=\; \eta \, \frac{du_x}{dy} \;=\; \left(\frac{\partial p}{\partial x} - \rho g_x \right) \, y \;+\; \eta c_1 \, . \tag{6.11}$$

Note that the y- and z-momentum components do not involve the velocity u_x; since $u_y=u_z=0$, they degenerate to the hydrostatic pressure expressions

$$-\,\frac{\partial p}{\partial y} \;+\; \rho g_y \;=\; 0 \quad \text{and} \quad -\,\frac{\partial p}{\partial z} \;+\; \rho g_z \;=\; 0 \, . \tag{6.12}$$

Integrating Eqs. (6.9) and (6.12), we obtain the following expression for the pressure:

$$p \;=\; \frac{\partial p}{\partial x}\, x + \rho g_y y + \rho g_z z + c \, , \tag{6.13}$$

where c is a constant of integration which may be evaluated in any particular flow problem by specifying the value of the pressure at a point.

In Table 6.1, we tabulate the assumptions, the governing equations, and the general solution for steady, one-dimensional rectilinear flows in Cartesian coordinates. Important flows in this category are:

1. *Plane Couette flow*, i.e., fully-developed flow between parallel flat plates of infinite dimensions, driven by the steady motion of one of the plates. (Such a flow is called *shear-driven flow*.) The geometry of this flow is depicted in Fig. 6.2, where the upper wall is moving with constant speed V (so that it remains in the same plane) while the lower one is fixed. The pressure gradient is zero everywhere and the gravity term is neglected. This flow is studied in Example 1.6.1.

Assumptions:	$u_y = u_z = 0, \quad \dfrac{\partial u_x}{\partial z}=0, \quad \dfrac{\partial p}{\partial x}=\text{const.}$
Continuity:	$\dfrac{\partial u_x}{\partial x} = 0 \quad \Longrightarrow \quad u_x = u_x(y)$
x-momentum:	$-\dfrac{\partial p}{\partial x} + \eta \dfrac{d^2 u_x}{dy^2} + \rho g_x = 0$
y-momentum:	$-\dfrac{\partial p}{\partial y} + \rho g_y = 0$
z-momentum:	$-\dfrac{\partial p}{\partial z} + \rho g_z = 0$
General solution:	$u_x = \dfrac{1}{2\eta}\left(\dfrac{\partial p}{\partial x} - \rho g_x\right) y^2 + c_1 y + c_2$
	$\tau_{yx} = \tau_{xy} = \left(\dfrac{\partial p}{\partial x} - \rho g_x\right) y + \eta c_1$
	$p = \dfrac{\partial p}{\partial x}\, x + \rho g_y y + \rho g_z z + c$

Table 6.1. *Governing equations and general solution for steady, one-dimensional rectilinear flows in Cartesian coordinates.*

2. *Fully-developed plane Poiseuille flow*, i.e., flow between parallel plates of infinite width and length, driven by a constant pressure gradient, imposed by a pushing or pulling device (a pump or vacuum, respectively), and/or gravity. This flow is an idealization of the flow in a channel of rectangular cross section, with the width W being much greater than the height H of the channel (see Fig. 6.1). Obviously, this idealization does not hold near the two lateral walls, where the flow is two-dimensional. The geometry of the plane Poiseuille flow is depicted in Fig. 6.4. This flow is studied in Examples 6.1.2 to 6.1.5, for

different boundary conditions.

3. *Thin film flow* down an inclined plane, driven by gravity (i.e., elevation differences), under the absence of surface tension. The pressure gradient is usually assumed to be everywhere zero. Such a flow is illustrated in Fig. 6.8, and is studied in Example 1.6.6.

All the above flows are rotational, with vorticity generation at the solid boundaries,

$$\boldsymbol{\omega} \;=\; \nabla \times \mathbf{u}|_w \;=\; \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ u_x & 0 & 0 \end{array} \right|_w \;=\; -\left(\frac{\partial u_x}{\partial y}\right)_w \mathbf{k} \neq \mathbf{0}\,.$$

The vorticity diffuses away from the wall, and penetrates the main flow at a rate $\nu(d^2 u_x/dy^2)$. The extensional stretching or compression along streamlines is zero, i.e.,

$$\dot{\epsilon} \;=\; \frac{\partial u_x}{\partial x} = 0$$

Material lines connecting two moving fluid particles traveling along different streamlines both rotate and stretch, where stretching is induced by rotation. However, the principal directions of strain rotate with respect to those of vorticity. Therefore, strain is relaxed, and the flow is weak.

Example 6.1.1. Plane Couette flow

Plane Couette flow,[1] named after Couette who introduced it in 1890 to measure viscosity, is fully-developed flow induced between two infinite parallel plates, placed at a distance H apart, when one of them, say the upper one, is moving steadily with speed V relative to the other (Fig. 6.2). Assuming that the pressure gradient and the gravity in the x-direction are zero, the general solution for u_x is:

$$u_x \;=\; c_1 y \;+\; c_2\,.$$

For the geometry depicted in Fig. 6.2, the boundary conditions are:

$$\begin{array}{llll} u_x = 0 & \text{at} & y = 0 & \text{(lower plate is stationary);} \\ u_x = V & \text{at} & y = H & \text{(upper plate is moving).} \end{array}$$

By means of the above two conditions, we find that c_2=0 and c_1=V/H. Substituting the two constants into the general solution, yields

$$u_x \;=\; \frac{V}{H}\,y\,. \tag{6.14}$$

[1] Plane Couette flow is also known as *simple shear flow*.

The velocity u_x then varies linearly across the gap. The corresponding shear stress is constant,

$$\tau_{yx} \;=\; \eta\,\frac{V}{H}\,. \tag{6.15}$$

A number of macroscopic quantities, such as the volumetric flow rate and the shear stress at the wall, can be calculated. The volumetric flow rate per unit width is calculated by integrating u_x along the gap:

$$\frac{Q}{W} \;=\; \int_0^H u_x\,dy \;=\; \int_0^H \frac{V}{H}y\,dy \qquad \Longrightarrow$$

$$\frac{Q}{W} \;=\; \frac{1}{2}\,HV\,. \tag{6.16}$$

The shear stress τ_w exerted by the fluid on the upper plate is

$$\tau_w \;=\; -\tau_{yx}|_{y=H} \;=\; -\eta\,\frac{V}{H}\,. \tag{6.17}$$

The minus sign accounts for the upper wall facing the negative y-direction of the chosen system of coordinates. The shear force per unit width required to move the upper plate is then

$$\frac{F}{W} \;=\; -\int_0^L \tau_w\,dx \;=\; \eta\,\frac{V}{H}L\,,$$

where L is the length of the plate.

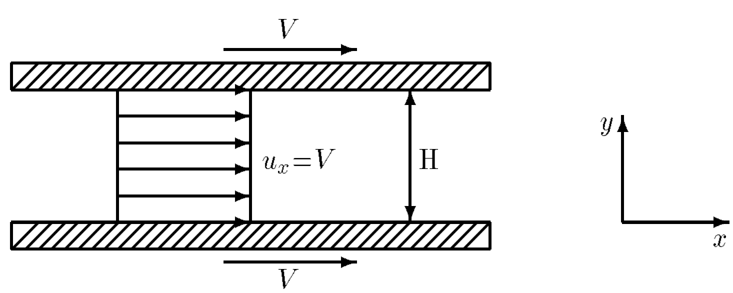


Figure 6.3. *Plug flow.*

Finally, let us consider the case where both plates move with the same speed V, as in Fig. 6.3. By invoking the boundary conditions

$$u_x(0) = u_x(H) = V\ ,$$

we find that c_1=0 and c_2=V, and, therefore,

$$u_x = V \ .$$

Thus, in this case, plane Couette flow degenerates into *plug flow*. □

Example 6.1.2. Fully-developed plane Poiseuille flow

Plane Poiseuille flow, named after the channel experiments by Poiseuille in 1840, occurs when a liquid is forced between two stationary infinite flat plates, under constant pressure gradient $\partial p/\partial x$ and zero gravity. The general steady-state solution is

$$u_x(y) \;=\; \frac{1}{2\eta}\,\frac{\partial p}{\partial x}\,y^2 \;+\; c_1 y \;+\; c_2 \qquad (6.18)$$

and

$$\tau_{yx} \;=\; \frac{\partial p}{\partial x}\,y \;+\; \eta c_1 \,. \qquad (6.19)$$

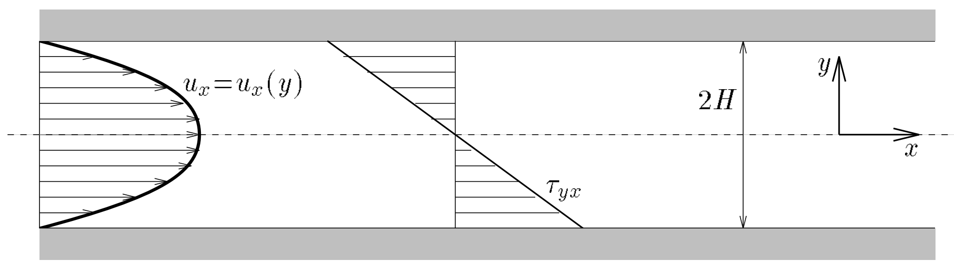


Figure 6.4. *Plane Poiseuille flow.*

By taking the origin of the Cartesian coordinates to be on the plane of symmetry of the flow, as in Fig. 6.4, and by assuming that the distance between the two plates is $2H$, the boundary conditions are:

$$\tau_{yx} = \eta\,\frac{du_x}{dy} = 0 \quad \text{at} \quad y = 0 \quad \text{(symmetry)}\,;$$
$$u_x = 0 \quad \text{at} \quad y = H \quad \text{(stationary plate)}\,.$$

Note that the condition u_x=0 at y=$-H$ may be used instead of any of the above conditions. By invoking the boundary conditions at y=0 and H, we find that c_1=0 and

$$c_2 \;=\; -\frac{1}{2\eta}\,\frac{\partial p}{\partial x}\,H^2 \,.$$

The two constants are substituted into the general solution to obtain the following parabolic velocity profile,

$$u_x = -\frac{1}{2\eta}\frac{\partial p}{\partial x}(H^2 - y^2)\,. \tag{6.20}$$

If the pressure gradient is negative, then the flow is in the positive direction, as in Fig. 6.4. Obviously, the velocity u_x attains its maximum value at the centerline (y=0):

$$u_{x,max} = -\frac{1}{2\eta}\frac{\partial p}{\partial x}\,H^2\,.$$

The volumetric flow rate per unit width is

$$\frac{Q}{W} = \int_{-H}^{H} u_x\,dy = 2\int_0^H -\frac{1}{2\eta}\frac{\partial p}{\partial x}(H^2 - y^2)\,dy \quad \Longrightarrow$$

$$Q = -\frac{2}{3\eta}\frac{\partial p}{\partial x}\,H^3 W\,. \tag{6.21}$$

As expected, Eq. (6.21) indicates that the volumetric flow rate Q is proportional to the pressure gradient, $\partial p/\partial x$, and inversely proportional to the viscosity η. Note also that, since $\partial p/\partial x$ is negative, Q is positive. The average velocity, $\bar{u}_x$, in the channel is:

$$\bar{u}_x = \frac{Q}{WH} = -\frac{2}{3\eta}\frac{\partial p}{\partial x}\,H^2\,.$$

The shear stress distribution is given by

$$\tau_{yx} = \frac{\partial p}{\partial x}\,y\,, \tag{6.22}$$

i.e., τ_{yx} varies linearly from y=0 to H, being zero at the centerline and attaining its maximum absolute value at the wall. The shear stress exerted by the fluid on the wall at y=H is

$$\tau_w = -\tau_{yx}|_{y=H} = -\frac{\partial p}{\partial x}\,H\,.$$

□

Example 6.1.3. Plane Poiseuille flow with slip

Consider again the fully-developed plane Poiseuille flow of the previous example, and assume that slip occurs along the two plates according to the slip law

$$\tau_w = \beta\, u_w \quad \text{at} \quad y = H\,,$$

where β is a material slip parameter, τ_w is the shear stress exerted by the fluid on the plate,

$$\tau_w = -\tau_{yx}|_{y=H} ,$$

and u_w is the *slip velocity*. Calculate the velocity distribution and the volume flow rate per unit width.

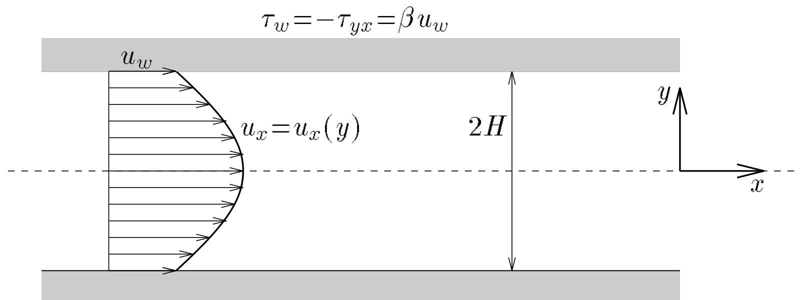


Figure 6.5. *Plane Poiseuille flow with slip.*

Solution:
We first note that the flow is still symmetric with respect to the centerline. In this case, the boundary conditions are:

$$\begin{aligned} \tau_{yx} &= \eta \frac{du_x}{dy} = 0 \quad \text{at} \quad y = 0 , \\ \tau_w &= \beta\, u_w \quad \text{at} \quad y = H . \end{aligned}$$

The condition at $y=0$ yields $c_1=0$. Consequently,

$$u_x = \frac{1}{2\eta} \frac{\partial p}{\partial x} y^2 + c_2 ,$$

and

$$\tau_{yx} = \frac{\partial p}{\partial x} y \quad \Longrightarrow \quad \tau_w = -\frac{\partial p}{\partial x} H .$$

Applying the condition at $y=H$, we obtain

$$u_w = \frac{1}{\beta}\tau_w \quad \Longrightarrow \quad u_x(H) = -\frac{1}{\beta} \frac{\partial p}{\partial x} H \quad \Longrightarrow \quad \frac{1}{2\eta} \frac{\partial p}{\partial x} H^2 + c_2 = -\frac{1}{\beta} \frac{\partial p}{\partial x} H .$$

Consequently,

$$c_2 = -\frac{1}{2\eta}\frac{\partial p}{\partial x}\left(H^2+\frac{2\eta H}{\beta}\right),$$

and

$$u_x = -\frac{1}{2\eta}\frac{\partial p}{\partial x}\left(H^2+\frac{2\eta H}{\beta}-y^2\right). \tag{6.23}$$

Note that this expression reduces to the standard Poiseuille flow profile when $\beta\rightarrow\infty$. Since the slip velocity is inversely proportional to the slip coefficient β, the standard no-slip condition is recovered.

An alternative expression of the velocity distribution is

$$u_x = u_w - \frac{1}{2\eta}\frac{\partial p}{\partial x}\left(H^2-y^2\right),$$

which indicates that u_x is just the superposition of the slip velocity u_w to the velocity distribution of the previous example.

For the volumetric flow rate per unit width, we obtain:

$$\frac{Q}{W} = 2\int_0^H u_x\,dy = 2u_wH - \frac{2}{3\eta}\frac{\partial p}{\partial x}H^3 \quad\Longrightarrow$$

$$Q = -\frac{2}{3\eta}\frac{\partial p}{\partial x}H^3\left(1+\frac{3\eta}{\beta H}\right)W. \tag{6.24}$$

□

Example 6.1.4. Plane Couette-Poiseuille flow

Consider again fully-developed plane Poiseuille flow with the upper plate moving with constant speed, V (Fig. 6.6). This flow is called *plane Couette-Poiseuille flow* or *general Couette flow*. In contrast to the previous two examples, this flow is not symmetric with respect to the centerline of the channel, and, therefore, having the origin of the Cartesian coordinates on the centerline is not convenient. Therefore, the origin is moved to the lower plate.

The boundary conditions for this flow are:

$$u_x = 0 \quad \text{at} \quad y=0,$$
$$u_x = V \quad \text{at} \quad y=a,$$

where a is the distance between the two plates. Applying the two conditions, we get c_2=0 and

$$V = \frac{1}{2\eta}\frac{\partial p}{\partial x}a^2 + c_1 a \quad\Longrightarrow\quad c_1 = \frac{V}{a} - \frac{1}{2\eta}\frac{\partial p}{\partial x}a,$$

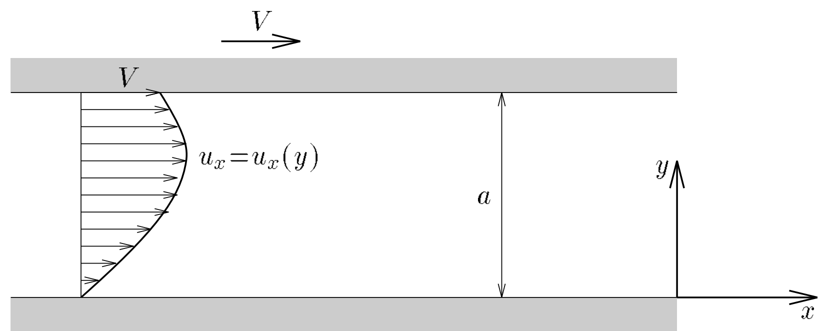


Figure 6.6. *Plane Poiseuille flow with the upper plate moving with constant speed.*

respectively. Therefore,

$$u_x = \frac{V}{a} y - \frac{1}{2\eta} \frac{\partial p}{\partial x} (ay - y^2) . \tag{6.25}$$

The shear stress distribution is given by

$$\tau_{yx} = \eta \frac{V}{a} - \frac{1}{2} \frac{\partial p}{\partial x} (a - 2y) . \tag{6.26}$$

It is a simple exercise to show that Eq. (6.25) reduces to the standard Poiseuille velocity profile for stationary plates, given by Eq. (6.20). (Keep in mind that $a=2H$ and that the y-axis has been translated by a distance H.) If instead, the pressure gradient is zero, the flow degenerates to the plane Couette flow studied in Example 1.6.1, and the velocity distribution is linear. Hence, the solution in Eq. (6.25) is the sum of the solutions to the above two separate flow problems. This superposition of solutions is a result of the linearity of the governing equation (6.6) and boundary conditions. Note also that Eq. (6.25) is valid not only when both the pressure gradient and the wall motion drive the fluid in the same direction, as in the present example, but also when they oppose each other. In the latter case, some reverse flow –in the negative x direction– can occur when $\partial p/\partial x > 0$.

Finally, let us find the point y^* where the velocity attains its maximum value. This point is a zero of the shear stress (or, equivalently, of the velocity derivative, du_x/dy):

$$0 = \eta \frac{V}{a} - \frac{1}{2} \frac{\partial p}{\partial x} (a - 2y^*) \quad \Longrightarrow \quad y^* = \frac{a}{2} + \frac{\eta V}{a \left(\frac{\partial p}{\partial x} \right)} .$$

The flow is symmetric with respect to the centerline, if $y^*=a/2$, i.e., when $V=0$. The maximum velocity $u_{x,max}$ is determined by substituting y^* into Eq. (6.25).

□

Example 6.1.5. Poiseuille flow between inclined plates

Consider steady flow between two parallel inclined plates, driven by both constant pressure gradient and gravity. The distance between the two plates is $2H$ and the chosen system of coordinates is shown in Fig. 6.7. The angle formed by the two plates and the horizontal direction is θ.

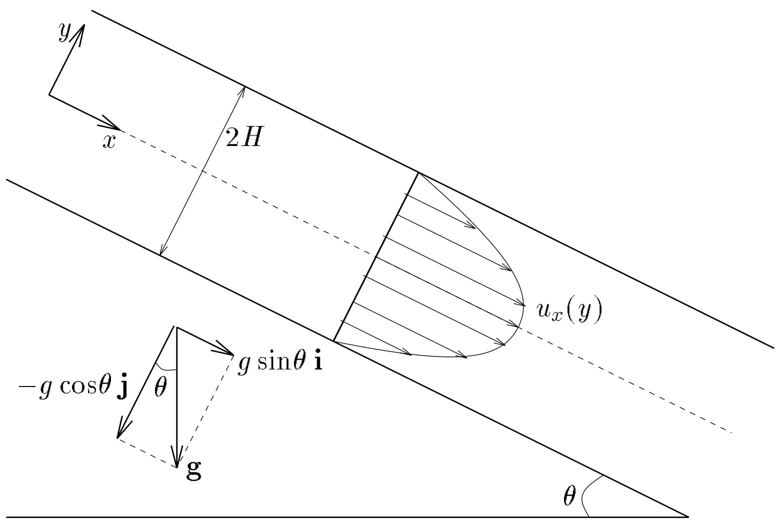


Figure 6.7. *Poiseuille flow between inclined plates.*

The general solution for u_x is given by Eq. (6.10):

$$u_x(y) = \frac{1}{2\eta}\left(\frac{\partial p}{\partial x} - \rho g_x\right) y^2 + c_1 y + c_2 .$$

Since,

$$g_x = g \sin\theta ,$$

we get

$$u_x(y) = \frac{1}{2\eta}\left(\frac{\partial p}{\partial x} - \rho g \sin\theta\right) y^2 + c_1 y + c_2 .$$

Integration of this equation with respect to y and application of the boundary conditions, $du_x/dy{=}0$ at $y{=}0$ and $u_x{=}0$ at $y{=}H$, give

$$u_x(y) \;=\; \frac{1}{2\eta}\left(-\frac{\partial p}{\partial x}+\rho g\sin\theta\right)\,(H^2-y^2)\,. \tag{6.27}$$

The pressure is obtained from Eq. (6.13) as

$$p \;=\; \frac{\partial p}{\partial x}\,x \;+\; \rho g_y\,y \;+\; c \qquad \Longrightarrow$$

$$p \;=\; \frac{\partial p}{\partial x}\,x \;+\; \rho g\cos\theta\,y \;+\; c \tag{6.28}$$

□

Example 6.1.6. Thin film flow

Consider a thin film of an incompressible Newtonian liquid flowing down an inclined plane (Fig. 6.8). The ambient air is assumed to be stationary, and, therefore, the flow is driven by gravity alone. Assuming that the surface tension of the liquid is negligible, and that the film is of uniform thickness δ, calculate the velocity and the volumetric flow rate per unit width.

Solution:

The governing equation of the flow is

$$\eta\,\frac{d^2u_x}{dy^2} \;+\; \rho g_x \;=\; 0 \qquad\Longrightarrow\qquad \eta\,\frac{d^2u_x}{dy^2} \;=\; -\rho g\sin\theta\,,$$

with general solution

$$u_x \;=\; -\frac{\rho g\sin\theta}{\eta}\,\frac{y^2}{2} \;+\; c_1 y \;+\; c_2\,.$$

As for the boundary conditions, we have no slip along the solid boundary,

$$u_x = 0 \quad \text{at} \quad y = 0\,,$$

and no shearing at the free surface (the ambient air is stationary),

$$\tau_{yx} = \eta\,\frac{du_x}{dy} = 0 \quad \text{at} \quad y = \delta\,.$$

Applying the above two conditions, we find that $c_2{=}0$ and $c_1{=}\rho g\sin\theta/(\eta\delta)$, and thus

$$u_x \;=\; \frac{\rho g\sin\theta}{\eta}\left(\delta y-\frac{y^2}{2}\right)\,. \tag{6.29}$$

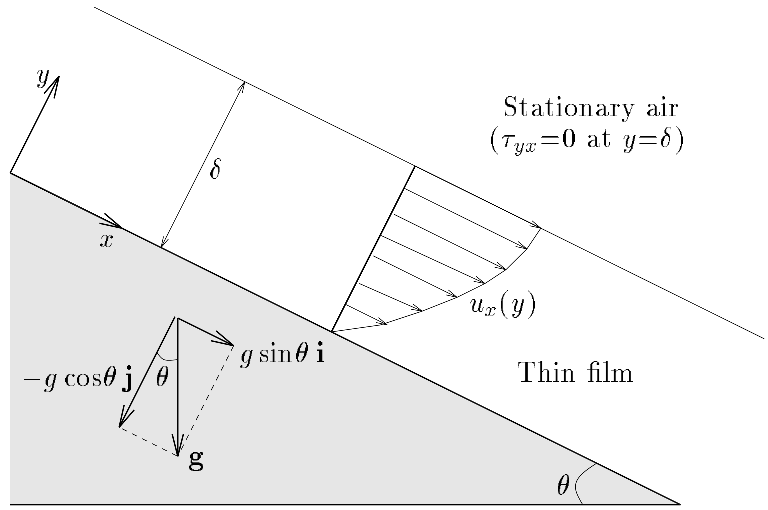


Figure 6.8. *Film flow down an inclined plane.*

The velocity profile is semiparabolic, and attains its maximum value at the free surface,

$$u_{x,max} \;=\; u_x(\delta) \;=\; \frac{\rho g \sin\theta \, \delta^2}{2\eta} \, .$$

The volume flow rate per unit width is

$$\frac{Q}{W} \;=\; \int_0^\delta u_x \, dy \;=\; \frac{\rho g \sin\theta \, \delta^3}{3\eta} \, , \tag{6.30}$$

and the average velocity, $\bar{u}_x$, over a cross section of the film is given by

$$\bar{u}_x \;=\; \frac{Q}{W\delta} \;=\; \frac{\rho g \sin\theta \, \delta^2}{3\eta} \, .$$

Note that if the film is horizontal, then $\sin\theta$=0 and u_x is zero, i.e., no flow occurs. If the film is vertical, then $\sin\theta$=1, and

$$u_x \;=\; \frac{\rho g}{\eta} \left(\delta y - \frac{y^2}{2} \right) \tag{6.31}$$

and

$$\frac{Q}{W} = \frac{\rho g \delta^3}{3\eta} . \tag{6.32}$$

By virtue of Eq. (6.13), the pressure is given by

$$p = \rho g_y \, y + c = -\rho g \cos\theta \, y + c .$$

At the free surface, the pressure must be equal to the atmospheric pressure, p_0, so

$$p_0 = -\rho g \cos\theta \, \delta + c$$

and

$$p = p_0 + \rho g \, (\delta - y) \, \cos\theta . \tag{6.33}$$

□

Example 6.1.7. Two-layer plane Couette flow

Two immiscible incompressible liquids A and B of densities ρ_A and ρ_B ($\rho_A > \rho_B$) and viscosities η_A and η_B flow between two parallel plates. The flow is induced by the motion of the upper plate which moves with speed V, while the lower plate is stationary (Fig. 6.9).

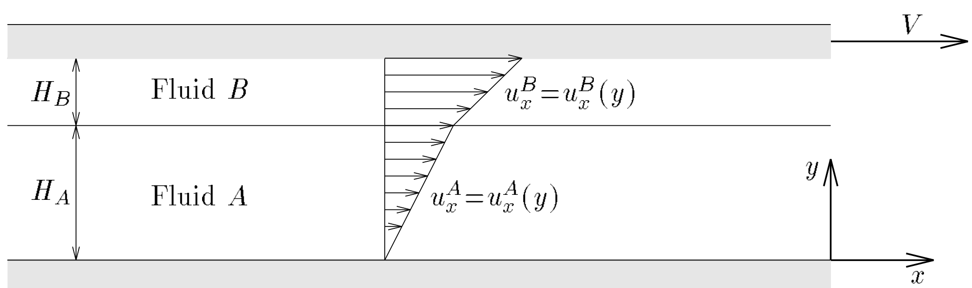


Figure 6.9. *Two-layer plane Couette flow.*

The velocity distributions in both layers obey Eq. (6.6) and are given by Eq. (6.10). Since the pressure gradient and gravity are both zero,

$$\begin{aligned} u_x^A &= c_1^A y + c_2^A , \quad 0 \le y \le H_A , \\ u_x^B &= c_1^B y + c_2^B , \quad H_A \le y \le H_A + H_B , \end{aligned}$$

where c_1^A, c_2^A, c_1^B and c_2^B are integration constants determined by conditions at the solid boundaries and the interface of the two layers. The no-slip boundary conditions at the two plates are applied first. At y=0, u_x^A=0; therefore,

$$c_2^A = 0 \,.$$

At y=$H_A + H_B$, u_x^B=V; therefore,

$$c_2^B = V - C_1^B\,(H_A + H_B)\,.$$

The two velocity distributions become

$$\begin{aligned} u_x^A &= c_1^A y\,, \qquad 0 \leq y \leq H_A\,, \\ u_x^B &= V \;-\; c_1^B\,(H_A + H_B - y)\,, \qquad H_A \leq y \leq H_A + H_B\,. \end{aligned}$$

At the interface (y=H_A), we have two additional conditions:
(a) the velocity distribution is continuous, i.e.,

$$u_x^A \;=\; u_x^B \qquad \text{at} \qquad y = H_A\,;$$

(b) momentum transfer through the interface is continuous, i.e.,

$$\tau_{yx}^A \;=\; \tau_{yx}^B \qquad \text{at} \qquad y = H_A \qquad \Longrightarrow$$

$$\eta_A \frac{du_x^A}{dy} \;=\; \eta_B \frac{du_x^B}{dy} \qquad \text{at} \qquad y = H_A\,.$$

From the interface conditions, we find that

$$c_1^A \;=\; \frac{\eta_B V}{\eta_A H_B + \eta_B H_A} \qquad \text{and} \qquad c_1^B \;=\; \frac{\eta_A V}{\eta_A H_B + \eta_B H_A}\,.$$

Hence, the velocity profiles in the two layers are

$$u_x^A \;=\; \frac{\eta_B V}{\eta_A H_B + \eta_B H_A}\, y\,, \quad 0 \leq y \leq H_A\,, \tag{6.34}$$

$$u_x^B \;=\; V \;-\; \frac{\eta_A V}{\eta_A H_B + \eta_B H_A}\,(H_A + H_B - y)\,, \quad H_A \leq y \leq H_A + H_B\,. \tag{6.35}$$

If the two liquids are of the same viscosity, η_A=η_B=η, then the two velocity profiles are the same, and the results simplify to the linear velocity profile for one-layer Couette flow,

$$u_x^A \;=\; u_x^B \;=\; \frac{V}{H_A + H_B}\, y\,.$$

□

6.2 Steady, Axisymmetric Rectilinear Flows

Axisymmetric flows are conveniently studied in a cylindrical coordinate system, (r, θ, z), with the z-axis coinciding with the axis of symmetry of the flow. *Axisymmetry* means that there is no variation of the velocity with the angle θ,

$$\frac{\partial \mathbf{u}}{\partial \theta} = \mathbf{0} . \tag{6.36}$$

There are three important classes of axisymmetric *unidirectional* flows (i.e., flows in which only one of the three velocity components, u_r, u_θ and u_z, is nonzero):

1. *Axisymmetric rectilinear flows*, in which only the axial velocity component, u_z, is nonzero. The streamlines are straight lines. Typical flows are fully-developed pressure-driven flows in cylindrical tubes and annuli, and open film flows down cylinders or conical pipes.

2. *Axisymmetric torsional flows*, in which only the azimuthal velocity component, u_θ, is nonzero. The streamlines are circles centered on the axis of symmetry. These flows, studied in Section 6.3, are good prototypes of rigid-body rotation, flow in rotating mixing devices, and swirling flows, such as tornados.

3. *Axisymmetric radial flows*, in which only the radial velocity component, u_r, is nonzero. These flows, studied in Section 6.4, are typical models for radial flows through porous media, migration of oil towards drilling wells, and suction flows from porous pipes and annuli.

As already mentioned, in axisymmetric rectilinear flows,

$$u_r = u_\theta = 0 . \tag{6.37}$$

The continuity equation for incompressible flow,

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 ,$$

becomes

$$\frac{\partial u_z}{\partial z} = 0 .$$

From the above equation and the axisymmetry condition (6.36), we deduce that

$$u_z = u_z(r, t) . \tag{6.38}$$

Due to Eqs. (6.36)-(6.38), the z-momentum equation,

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z\,,$$

is simplified to

$$\rho\frac{\partial u_z}{\partial t} \;=\; -\frac{\partial p}{\partial z}\;+\;\eta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\;+\;\rho g_z\,. \tag{6.39}$$

For steady flow, $u_z{=}u_z(r)$ and Eq. (6.39) becomes an ordinary differential equation,

$$-\frac{\partial p}{\partial z}\;+\;\eta\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)\;+\;\rho g_z\;=\;0\,. \tag{6.40}$$

The only nonzero components of the stress tensor are the shear stresses τ_{rz} and τ_{zr},

$$\tau_{rz}\;=\tau_{zr}\;=\;\eta\,\frac{du_z}{dr}\,, \tag{6.41}$$

for which we have

$$-\frac{\partial p}{\partial z}\;+\;\frac{1}{r}\frac{d}{dr}\left(r\tau_{rz}\right)\;+\;\rho g_z\;=\;0\,. \tag{6.42}$$

When the pressure gradient $\partial p/\partial z$ is constant, the general solution of Eq. (6.39) is

$$u_z\;=\;\frac{1}{4\eta}\left(\frac{\partial p}{\partial z}-\rho g_z\right)\,r^2\;+\;c_1\ln r\;+\;c_2\,. \tag{6.43}$$

For τ_{rz}, we get

$$\tau_{rz}\;=\;\frac{1}{2}\left(\frac{\partial p}{\partial z}-\rho g_z\right)\,r\;+\;\eta\frac{c_1}{r}\,. \tag{6.44}$$

The constants c_1 and c_2 are determined from the boundary conditions of the flow. The assumptions, the governing equations and the general solution for steady, axisymmetric rectilinear flows are summarized in Table 6.2.

Example 6.2.1. Hagen-Poiseuille flow

Fully-developed axisymmetric Poiseuille flow, or *Hagen-Poiseuille flow*, studied experimentally by Hagen in 1839 and Poiseuille in 1840, is the pressure-driven flow in infinitely long cylindrical tubes. The geometry of the flow is shown in Fig. 6.10.

Assuming that gravity is zero, the general solution for u_z is

$$u_z\;=\;\frac{1}{4\eta}\frac{\partial p}{\partial z}\,r^2\;+\;c_1\ln r\;+\;c_2\,.$$

Assumptions:	$u_r = u_\theta = 0, \quad \frac{\partial u_z}{\partial \theta}=0, \quad \frac{\partial p}{\partial z}=\text{const.}$
Continuity:	$\frac{\partial u_z}{\partial z} = 0 \quad \Longrightarrow \quad u_z = u_z(r)$
z-momentum:	$-\frac{\partial p}{\partial z} + \eta \frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) + \rho g_z = 0$
r-momentum:	$-\frac{\partial p}{\partial r} + \rho g_r = 0$
θ-momentum:	$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta = 0$
General solution:	$u_z = \frac{1}{4\eta}\left(\frac{\partial p}{\partial z} - \rho g_z\right) r^2 + c_1 \ln r + c_2$
	$\tau_{rz} = \tau_{zr} = \frac{1}{2}\left(\frac{\partial p}{\partial z} - \rho g_z\right) r + \eta\frac{c_1}{r}$
	$p = \frac{\partial p}{\partial z}\, z + c(r,\theta)$
	[$c(r,\theta)$=const. when $g_r=g_\theta=0$]

Table 6.2. *Governing equations and general solution for steady, axisymmetric rectilinear flows.*

The constants c_1 and c_2 are determined by the boundary conditions of the flow. Along the axis of symmetry, the velocity u_z must be finite,

$$u_z \text{ finite} \quad \text{at} \quad r = 0\,.$$

Since the wall of the tube is stationary,

$$u_z = 0 \quad \text{at} \quad r = R\,.$$

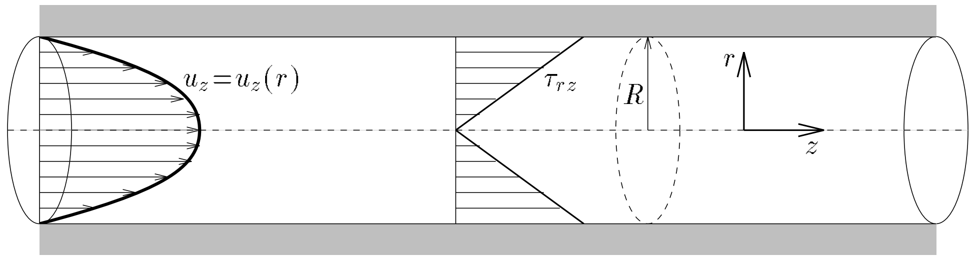


Figure 6.10. *Axisymmetric Poiseuille flow.*

By applying the two conditions, we get c_1=0 and

$$c_2 \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\,R^2\;,$$

and, therefore,

$$u_z \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\,\left(R^2 - r^2\right)\;, \tag{6.45}$$

which represents a parabolic velocity profile (Fig. 6.10). The shear stress varies linearly with r,

$$\tau_{rz} \;=\; \frac{1}{2}\frac{\partial p}{\partial z}\,r\;,$$

and the shear stress exerted by the fluid on the wall is

$$\tau_w \;=\; -\tau_{rz}|_{r=R} \;=\; -\frac{1}{2}\frac{\partial p}{\partial z}\,R\;.$$

(Note that the contact area faces the negative r-direction.)

The maximum velocity occurs at r=0,

$$u_{z,max} \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\,R^2\;.$$

For the volume flow rate, we get:

$$Q \;=\; \int_0^R u_z\,2\pi r\,dr \;=\; -\frac{\pi}{2\eta}\frac{\partial p}{\partial z}\int_0^R (R^2 - r^2) r\,dr \qquad \Longrightarrow$$

$$Q \;=\; -\frac{\pi}{8\eta}\frac{\partial p}{\partial z}\,R^4\;. \tag{6.46}$$

Note that, since the pressure gradient $\partial p/\partial z$ is negative, Q is positive. Equation (6.46) is the famous experimental result of Hagen and Poiseuille, also known as the *fourth-power law.* This basic equation is used to determine the viscosity from *capillary viscometer* data after taking into account the so-called *Bagley correction* for the inlet and exit pressure losses.

The average velocity, $\bar{u}_z$, in the tube is

$$\bar{u}_z = \frac{Q}{\pi R^2} = -\frac{1}{8\eta}\frac{\partial p}{\partial z} R^2 .$$

□

Example 6.2.2. Fully-developed flow in an annulus

Consider fully-developed pressure-driven flow of a Newtonian liquid in a sufficiently long annulus of radii R and κR, where κ <1 (Fig. 6.11). For zero gravity, the general solution for the axial velocity u_z is

$$u_z = \frac{1}{4\eta}\frac{\partial p}{\partial z} r^2 + c_1 \ln r + c_2 .$$

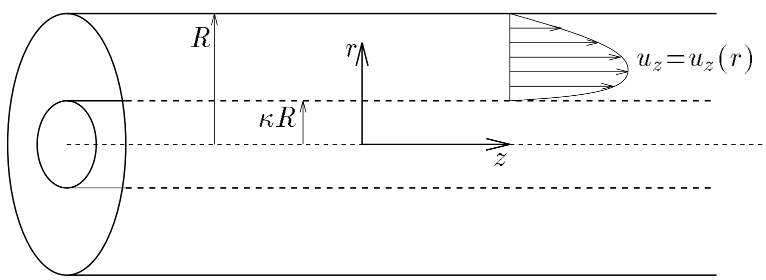


Figure 6.11. *Fully-developed flow in an annulus.*

Applying the boundary conditions,

$$u_z = 0 \quad \text{at} \quad r = \kappa R ,$$
$$u_z = 0 \quad \text{at} \quad r = R ,$$

we find that

$$c_1 = -\frac{1}{4\eta}\frac{\partial p}{\partial z} R^2 \frac{1-\kappa^2}{\ln(1/\kappa)}$$

and

$$c_2 \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\,R^2 - c_1 \ln R\,.$$

Substituting c_1 and c_2 into the general solution we obtain:

$$u_z \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\,R^2\left[1-\left(\frac{r}{R}\right)^2+\frac{1-\kappa^2}{\ln(1/\kappa)}\,\ln\frac{r}{R}\right]\,. \tag{6.47}$$

The shear stress is given by

$$\tau_{rz} \;=\; \frac{1}{4}\frac{\partial p}{\partial z}\,R\left[2\left(\frac{r}{R}\right) \;-\; \frac{1-\kappa^2}{\ln(1/\kappa)}\,\left(\frac{R}{r}\right)\right]\,. \tag{6.48}$$

The maximum velocity occurs at the point where τ_{rz}=0 (which is equivalent to du_z/dr=0), i.e., at

$$r^* \;=\; R\,\left[\frac{1-\kappa^2}{2\ln(1/\kappa)}\right]^{1/2}\,.$$

Substituting into Eq. (6.47), we get

$$u_{z,max} \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\,R^2\left\{1 \;-\; \frac{1-\kappa^2}{2\ln(1/\kappa)}\left[1-\ln\frac{1-\kappa^2}{2\ln(1/\kappa)}\right]\right\}\,.$$

For the volume flow rate, we have

$$Q \;=\; \int_0^R u_z\,2\pi r\,dr \;=\; -\frac{\pi}{2\eta}\frac{\partial p}{\partial z}R^2\int_0^R\left[1-\left(\frac{r}{R}\right)^2+\frac{1-\kappa^2}{\ln(1/\kappa)}\,\ln\frac{r}{R}\right]r\,dr \qquad \Longrightarrow$$

$$Q \;=\; -\frac{\pi}{8\eta}\frac{\partial p}{\partial z}R^4\,\left[\left(1-\kappa^4\right)-\frac{\left(1-\kappa^2\right)^2}{\ln(1/\kappa)}\right]\,. \tag{6.49}$$

The average velocity, $\bar{u}_z$, in the annulus is

$$\bar{u}_z \;=\; \frac{Q}{\pi R^2-\pi(\kappa R)^2} \;=\; -\frac{1}{8\eta}\frac{\partial p}{\partial z}R^2\,\left[\left(1+\kappa^2\right)-\frac{(1-\kappa^2)}{\ln(1/\kappa)}\right]\,.$$

□

Example 6.2.3. Film flow down a vertical cylinder

A Newtonian liquid is falling vertically on the outside surface of an infinitely long cylinder of radius R, in the form of a thin uniform axisymmetric film, in contact

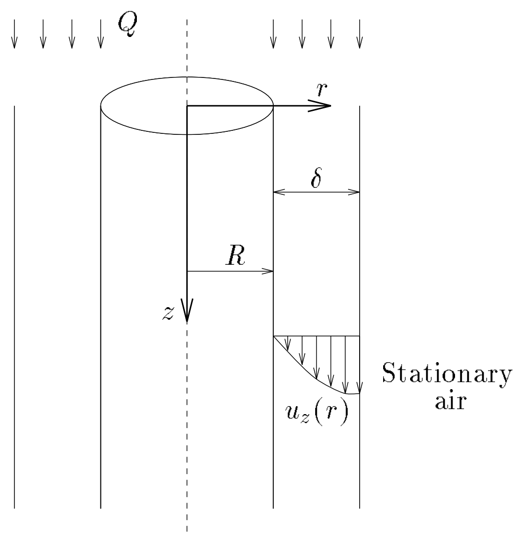


Figure 6.12. *Thin film flow down a vertical cylinder.*

with stationary air (Fig. 6.12). If the volumetric flow rate of the film is Q, calculate its thickness δ. Assume that the flow is steady, and that surface tension is zero.

Solution:
Equation (6.43) applies with $\frac{\partial p}{\partial z}$=0:

$$u_z \;=\; -\frac{1}{4\eta}\rho g_z\, r^2 + c_1 \ln r + c_2$$

Since the air is stationary, the shear stress on the free surface of the film is zero,

$$\tau_{rz} = \eta \frac{du_z}{dr} = 0 \quad \text{at} \quad r = R + \delta \quad \Longrightarrow \quad c_1 \;=\; \rho g \frac{(R+\delta)^2}{2\eta}\,.$$

At r=R, u_z=0; consequently,

$$c_2 \;=\; \frac{1}{4\eta}\rho g R^2 - c_1 \ln R\,.$$

Substituting into the general solution, we get

$$u_z \;=\; \frac{1}{4\eta}\rho g \left[R^2 - r^2 + 2(R+\delta)^2 \ln \frac{r}{R}\right]\,. \tag{6.50}$$

For the volume flow rate, Q, we have:

$$Q \;=\; \int_R^{R+\delta} u_z \, 2\pi r \, dr \;=\; \frac{\pi}{2\eta}\rho g \int_R^{R+\delta} \left[R^2 - r^2 + 2(R+\delta)^2 \, \ln \frac{r}{R} \right] r \, dr \, .$$

After integration and some algebraic manipulations, we find that

$$Q = \frac{\pi}{8\eta}\rho g R^4 \left\{ 4\left(1+\frac{\delta}{R}\right)^4 \ln\left(1+\frac{\delta}{R}\right) - \frac{\delta}{R}\left(2+\frac{\delta}{R}\right)\left[3\left(1+\frac{\delta}{R}\right)^2 - 1\right] \right\} . \quad (6.51)$$

When the annular film is very thin, it can be approximated as a thin planar film. We will show that this is indeed the case, by proving that for

$$\frac{\delta}{R} \ll 1 \, ,$$

Eq. (6.51) reduces to the expression found in Example 6.1.6 for a thin vertical planar film. Letting

$$\epsilon = \frac{\delta}{R}$$

leads to the following expression for Q,

$$Q \;=\; \frac{\pi}{8\eta}\rho g R^4 \, \left\{ 4\,(1+\epsilon)^4 \, \ln(1+\epsilon) \;-\; \epsilon\,(2+\epsilon)\left[3\,(1+\epsilon)^2 - 1\right] \right\} \, .$$

Expanding $\ln(1+\epsilon)$ into Taylor series, we get

$$\ln(1+\epsilon) \;=\; \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \frac{\epsilon^4}{4} + O(\epsilon^5) \, .$$

Thus

$$\begin{aligned} (1+\epsilon)^4 \, \ln(1+\epsilon) \;&=\; (1 + 4\epsilon + 6\epsilon^2 + 4\epsilon^3 + \epsilon^4) \left[\epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \frac{\epsilon^4}{4} + O(\epsilon^5) \right] \\ &=\; \epsilon + \frac{7}{2}\epsilon^2 + \frac{13}{3}\epsilon^3 + \frac{25}{12}\epsilon^4 + O(\epsilon^5) \end{aligned}$$

Consequently,

$$Q \;=\; \frac{\pi}{8\eta}\rho g R^4 \, \left\{ 4\left[\epsilon + \frac{7}{2}\epsilon^2 + \frac{13}{3}\epsilon^3 + \frac{25}{12}\epsilon^4 + O(\epsilon^5) \right] - (4\epsilon + 14\epsilon^2 + 12\epsilon^3 + 3\epsilon^4) \right\} \, ,$$

or

$$Q \;=\; \frac{\pi}{8\eta}\rho g R^4 \, \left[\frac{16}{3}\epsilon^3 - \frac{11}{12}\epsilon^4 + O(\epsilon^5) \right] \, .$$

Keeping only the third-order term, we get

$$Q \;=\; \frac{\pi}{8\eta}\rho g R^4\,\frac{16}{3}\left(\frac{\delta}{R}\right)^3 \quad\Longrightarrow\quad \frac{Q}{2\pi R} \;=\; \frac{\rho g\delta^3}{3\eta}\,.$$

By setting $2\pi R$ equal to W, the last equation becomes identical to Eq. (6.32). □

Example 6.2.4. Annular flow with the outer cylinder moving

Consider fully-developed flow of a Newtonian liquid between two coaxial cylinders of infinite length and radii R and κR, where $\kappa<1$. The outer cylinder is steadily translated parallel to its axis with speed V, whereas the inner cylinder is fixed (Fig. 6.13). For this problem, the pressure gradient and gravity are assumed to be negligible.

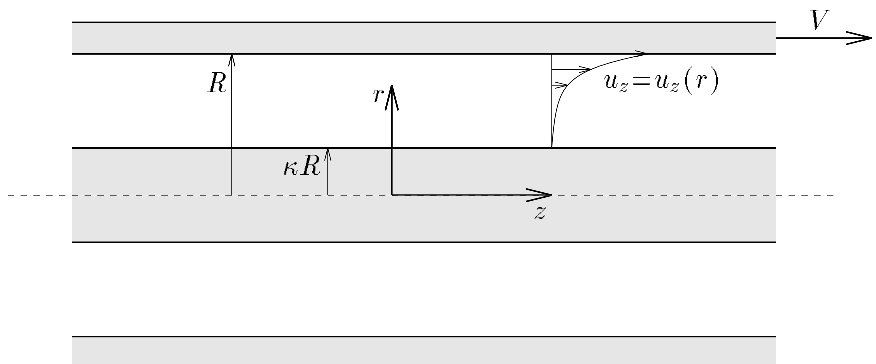


Figure 6.13. *Flow in an annulus driven by the motion of the outer cylinder.*

The general solution for the axial velocity u_z takes the form

$$u_z \;=\; c_1 \ln r + c_2\,.$$

For $r=\kappa R$, $u_z=0$, and for $r=R$, $u_z=V$. Consequently,

$$c_1 \;=\; \frac{V}{\ln(1/\kappa)} \quad \text{and} \quad c_2 \;=\; -V\frac{\ln(\kappa R)}{\ln(1/\kappa)}\,.$$

Therefore, the velocity distribution is given by

$$u_z \;=\; V\,\frac{\ln\left(\frac{r}{\kappa R}\right)}{\ln(1/\kappa)}\,. \tag{6.52}$$

Let us now examine two limiting cases of this flow.
(a) For $\kappa \to 0$, the annular flow degenerates to flow in a tube. From Eq. (6.52), we have

$$u_z = \lim_{\kappa \to 0} V \frac{\ln\left(\frac{r}{\kappa R}\right)}{\ln(1/\kappa)} = V \lim_{\kappa \to 0} \left[1 + \frac{\ln \frac{r}{R}}{\ln(1/\kappa)}\right] = V .$$

In other words, we have plug flow (solid-body translation) in a tube.
(b) For $\kappa \to 1$, the annular flow is approximately a plane Couette flow. To demonstrate this, let

$$\epsilon = \frac{1}{\kappa} - 1 = \frac{1-\kappa}{\kappa}$$

and

$$\Delta R = R - \kappa R = (1-\kappa) R \quad \Longrightarrow \quad \kappa R = \frac{\Delta R}{\epsilon} .$$

Introducing Cartesian coordinates, (y, z), with the origin on the surface of the inner cylinder, we have

$$y = r - \kappa R \quad \Longrightarrow \quad \frac{r}{\kappa R} = 1 + \epsilon \frac{y}{\Delta R} .$$

Substituting into Eq. (6.52), we get

$$u_z = V \frac{\ln\left(1 + \epsilon \frac{y}{\Delta R}\right)}{\ln(1+\epsilon)} . \tag{6.53}$$

Using L'Hôpital's rule, we find that

$$\lim_{\epsilon \to 0} V \frac{\ln\left(1 + \epsilon \frac{y}{\Delta R}\right)}{\ln(1+\epsilon)} = \lim_{\epsilon \to 0} V \frac{y}{\Delta R} \frac{1+\epsilon}{1 + \epsilon \frac{y}{\Delta R}} = V \frac{y}{\Delta R} .$$

Therefore, for small values of ϵ, that is for $\kappa \to 1$, we obtain a linear velocity distribution which corresponds to plane Couette flow between plates separated by a distance ΔR. □

6.3 Steady, Axisymmetric Torsional Flows

In axisymmetric torsional flows, also referred to as *swirling flows*,

$$u_r = u_z = 0 , \tag{6.54}$$

and the streamlines are circles centered at the axis of symmetry. Such flows usually occur when rigid cylindrical boundaries (concentric to the symmetry axis of the flow) are rotating about their axis. Due to the axisymmetry condition, $\partial u_\theta/\partial\theta=0$, the continuity equation for incompressible flow,

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 ,$$

is automatically satisfied.

Assuming that the gravitational acceleration is parallel to the symmetry axis of the flow,

$$\mathbf{g} = -g\,\mathbf{e}_z , \tag{6.55}$$

the r- and z-momentum equations are simplified as follows,

$$\rho\,\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r} , \tag{6.56}$$

$$\frac{\partial p}{\partial z} + \rho g = 0 . \tag{6.57}$$

Equation (6.56) suggests that the centrifugal force on an element of fluid balances the force produced by the radial pressure gradient. Equation (6.57) represents the standard hydrostatic expression. Note also that Eq. (6.56) provides an example in which the nonlinear convective terms are not vanishing. In the present case, however, this nonlinearity poses no difficulties in obtaining the analytical solution for u_θ. As explained below, u_θ is determined from the θ-momentum equation which is decoupled from Eq. (6.56).

By assuming that

$$\frac{\partial p}{\partial \theta} = 0$$

and by integrating Eq. (6.57), we get

$$p = -\rho g\, z + c(r,t) ;$$

consequently, $\partial p/\partial r$ is not a function of z. Then, from Eq. (6.56) we deduce that

$$u_\theta = u_\theta(r,t) . \tag{6.58}$$

Due to the above assumptions, the θ-momentum equation reduces to

$$\rho\frac{\partial u_\theta}{\partial t} = \eta\,\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) . \tag{6.59}$$

For steady flow, we obtain the linear ordinary differential equation

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r u_\theta)\right) = 0 , \tag{6.60}$$

the general solution of which is

$$u_\theta \;=\; c_1\, r \;+\; \frac{c_2}{r}\,. \tag{6.61}$$

The constants c_1 and c_2 are determined from the boundary conditions of the flow.

Assumptions:	$u_r = u_z = 0, \quad \frac{\partial u_\theta}{\partial\theta}=0, \quad \frac{\partial p}{\partial\theta}=0, \quad \mathbf{g} = -g\,\mathbf{e}_z$
Continuity:	Satisfied identically
θ-momentum:	$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r u_\theta)\right) \;=\; 0$
z-momentum:	$\frac{\partial p}{\partial z} \;+\; \rho g = 0$
r-momentum:	$\rho\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r} \quad \Longrightarrow \quad u_\theta = u_\theta(r)$
General solution:	$u_\theta \;=\; c_1\, r \;+\; \frac{c_2}{r}$
	$\tau_{r\theta} \;=\; \tau_{\theta r} \;=\; -2\eta\, \frac{c_2}{r^2}$
	$p \;=\; \rho\left(\frac{c_1^2 r^2}{2} \;+\; 2c_1 c_2 \ln r \;-\; \frac{c_2^2}{2r^2}\right) \;-\; \rho g\, z \;+\; c$

Table 6.3. *Governing equations and general solution for steady, axisymmetric torsional flows.*

The pressure distribution is determined by integrating Eqs. (6.56) and (6.57):

$$p \;=\; \int \frac{u_\theta^2}{r}\, dr \;-\; \rho g\, z \quad \Longrightarrow$$

$$p \;=\; \rho \left(\frac{c_1^2 r^2}{2} \;+\; 2c_1c_2 \ln r \;-\; \frac{c_2^2}{2r^2} \right) \;-\; \rho g\, z \;+\; c\,, \tag{6.62}$$

where c is a constant of integration, evaluated in any particular problem by specifying the value of the pressure at a reference point.

Note that, under the above assumptions, the only nonzero components of the stress tensor are the shear stresses,

$$\tau_{r\theta} \;=\; \tau_{\theta r} \;=\; \eta\, r\, \frac{d}{dr}\left(\frac{u_\theta}{r}\right)\,, \tag{6.63}$$

in terms of which the θ-momentum equation takes the form

$$\frac{d}{dr}(r^2 \tau_{r\theta}) \;=\; 0\,. \tag{6.64}$$

The general solution for $\tau_{r\theta}$ is

$$\tau_{r\theta} \;=\; -2\,\eta\, \frac{c_2}{r^2}\,. \tag{6.65}$$

The assumptions, the governing equations and the general solution for steady, axisymmetric torsional flows are summarized in Table 6.3.

Example 6.3.1. Steady flow between rotating cylinders

The flow between rotating coaxial cylinders is known as the *circular Couette flow*, and is the basis for Couette rotational-type viscometers. Consider the steady flow of an incompressible Newtonian liquid between two vertical coaxial cylinders of infinite length and radii R_1 and R_2, respectively, occurring when the two cylinders are rotating about their common axis with angular velocities Ω_1 and Ω_2, in the absence of gravity (Fig. 6.14).[2]

The general form of the angular velocity u_θ is given by Eq. (6.61),

$$u_\theta \;=\; c_1\, r \;+\; \frac{c_2}{r}\,.$$

The boundary conditions,

$$\begin{aligned} u_\theta = \Omega_1 R_1 \quad &\text{at} \quad r = R_1\,, \\ u_\theta = \Omega_2 R_2 \quad &\text{at} \quad r = R_2\,, \end{aligned}$$

[2] The time-dependent flow between rotating cylinders is much more interesting, especially the manner in which it destabilizes for large values of Ω_1, leading to the generation of axisymmetric *Taylor vortices* [4].

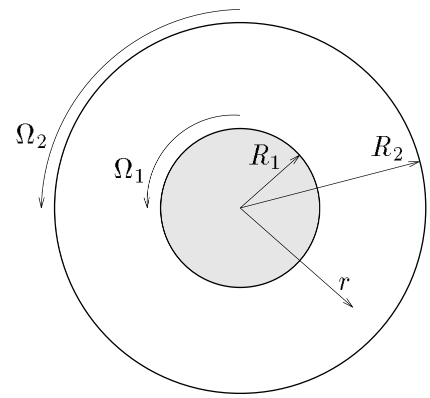


Figure 6.14. *Geometry of circular Couette flow.*

result in

$$c_1 \;=\; \frac{R_2^2\Omega_2 - R_1^2\Omega_1}{R_2^2 - R_1^2} \quad \text{and} \quad c_2 \;=\; -\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\,(\Omega_2 - \Omega_1)\,.$$

Therefore,

$$u_\theta \;=\; \frac{1}{R_2^2 - R_1^2}\,\left[(R_2^2\Omega_2 - R_1^2\Omega_1)\,r \;-\; R_1^2R_2^2(\Omega_2 - \Omega_1)\,\frac{1}{r}\right]\,. \tag{6.66}$$

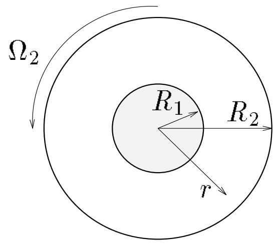
Note that the viscosity does not appear in Eq. (6.66), because shearing between adjacent cylindrical shells of fluid is zero. This observation is analogous to that made for the plane Couette flow [Eq. (6.14)]. Also, from Eqs. (6.62) and (6.65), we get

$$p = \rho\frac{1}{(R_2^2 - R_1^2)^2}\left[\frac{1}{2}(R_2^2\Omega_2 - R_1^2\Omega_1)^2\,r^2 + 2R_1^2R_2^2(R_2^2\Omega_2 - R_1^2\Omega_1)(\Omega_2 - \Omega_1)\ln r \right.$$
$$\left. -\,\frac{1}{2}R_1^4R_2^4(\Omega_2 - \Omega_1)^2\,\frac{1}{r^2}\right] \;+\; c\,, \tag{6.67}$$

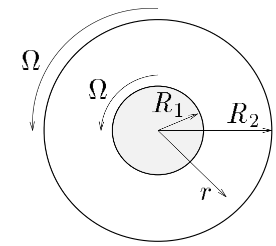
and

$$\tau_{r\theta} \;=\; 2\eta\,\frac{R_1^2R_2^2}{(R_2^2 - R_1^2)^2}\,(\Omega_2 - \Omega_1)\,\frac{1}{r^2}\,. \tag{6.68}$$

Let us now examine the four special cases of flow between rotating cylinders, illustrated in Fig. 6.15.

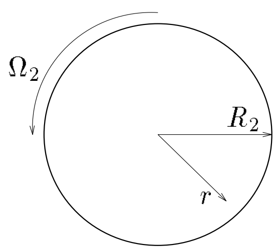
<table>
<tr><td>
</td><td>(a) The inner cylinder is fixed (Ω_1=0)

$u_\theta = \dfrac{R_2^2\Omega_2}{R_2^2 - R_1^2}\left(r - \dfrac{R_1^2}{r}\right)$

For p, see Eq. (6.70)</td></tr>
<tr><td>
</td><td>(b) $\Omega_1 = \Omega_2 = \Omega$

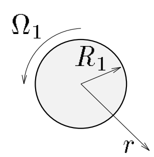
$u_\theta = \Omega\, r$

$p = \frac{1}{2}\rho\Omega^2\, r^2 + c$

(Rigid-body rotation)</td></tr>
<tr><td>
</td><td>(c) No inner cylinder

$u_\theta = \Omega_2\, r$

$p = \frac{1}{2}\rho\Omega_2^2\, r^2 + c$

(Rigid-body rotation)</td></tr>
<tr><td>
</td><td>(d) No outer cylinder

$u_\theta = R_1^2\Omega_1\, \dfrac{1}{r}$

$p = -\dfrac{1}{2}\rho R_1^4\Omega_1^2\, \dfrac{1}{r^2} + c$</td></tr>
</table>

Figure 6.15. *Different cases of flow between rotating vertical coaxial cylinders of infinite height.*

(a) The inner cylinder is fixed, i.e., Ω_1=0. In this case,

$$u_\theta \;=\; \frac{R_2^2\Omega_2}{R_2^2-R_1^2}\left(r-\frac{R_1^2}{r}\right) \tag{6.69}$$

and

$$p \;=\; \rho\frac{R_2^4\Omega_2^2}{(R_2^2-R_1^2)^2}\left(\frac{r^2}{2}+2R_1^2\ln r-\frac{R_1^4}{2r^2}\right)+c\,. \tag{6.70}$$

The constant c can be determined by setting p=p_0 at r=R_1; accordingly,

$$p \;=\; \rho\frac{R_2^4\Omega_2^2}{(R_2^2-R_1^2)^2}\left[\frac{r^2-R_1^2}{2}+2R_1^2\ln\frac{r}{R_1}-\frac{R_1^4}{2}\left(\frac{1}{r^2}-\frac{1}{R_1^2}\right)\right]\;+\;p_0\,. \tag{6.71}$$

For the shear stress, $\tau_{r\theta}$, we get

$$\tau_{r\theta} \;=\; 2\eta\,\frac{R_1^2R_2^2}{R_2^2-R_1^2}\,\Omega_2\,\frac{1}{r^2}\,. \tag{6.72}$$

The shear stress exerted by the liquid to the outer cylinder is

$$\tau_w \;=\; -\tau_{r\theta}|_{r=R_2} \;=\; -2\eta\,\frac{R_1^2}{R_2^2-R_1^2}\,\Omega_2\,. \tag{6.73}$$

In viscosity measurements, one measures the torque T per unit height L, at the outer cylinder,

$$\frac{T}{L} \;=\; 2\pi\,R_2^2\,(-\tau_w) \qquad \Longrightarrow$$

$$\frac{T}{L} \;=\; 4\pi\eta\,\frac{R_1^2R_2^2}{R_2^2-R_1^2}\,\Omega_2\,. \tag{6.74}$$

The unknown viscosity of a liquid can be determined using the above relation.

When the gap between the two cylinders is very small, circular Couette flow can be approximated as a plane Couette flow. Indeed, letting r=R_1+Δr, we get from Eq. (6.69)

$$u_\theta \;=\; \frac{R_2^2\Omega_2}{R_2^2-R_1^2}\,\frac{2+\frac{\Delta r}{R_1}}{1+\frac{\Delta r}{R_1}}\,\Delta r\,.$$

When $R_1\rightarrow R_2$, $\Delta r/R_1 \ll 1$ and, therefore,

$$u_\theta \;=\; \frac{R_2\Omega_2}{2(R_2-R_1)}\,2\Delta r \;=\; \frac{R_2\Omega_2}{R_2-R_1}\,\Delta r\,,$$

which is a linear velocity distribution corresponding to plane Couette flow between plates separated by a distance R_2-R_1, with the upper plate moving with velocity $R_2\Omega_2$.

(b) The two cylinders rotate with the same angular velocity, i.e.,

$$\Omega_1 = \Omega_2 = \Omega \, .$$

In thic case, c_1=Ω and c_2=0. Consequently,

$$u_\theta \;=\; \Omega\, r\,, \tag{6.75}$$

which corresponds to rigid-body rotation. This is also indicated by the zero tangential stress,

$$\tau_{r\theta} \;=\; -2\eta\,\frac{c_2}{r^2} \;=\; 0\,.$$

For the pressure, we get

$$p \;=\; \frac{1}{2}\rho\Omega^2\, r^2 \;+\; c\,. \tag{6.76}$$

(c) The inner cylinder is removed. In thic case, c_1=Ω_2 and c_2=0, since u_θ (and $\tau_{r\theta}$) are finite at r=0. This flow is the limiting case of the previous one for $R_1\rightarrow 0$,

$$u_\theta \;=\; \Omega_2\, r\,, \qquad \tau_{r\theta} \;=\; 0 \qquad \text{and} \qquad p \;=\; \frac{1}{2}\rho\Omega_2^2\, r^2 \;+\; c\,.$$

(d) The outer cylinder is removed, i.e., the inner cylinder is rotating in an infinite pool of liquid. In this case, $u_\theta\rightarrow 0$ as $r\rightarrow\infty$, and, therefore, c_1=0. At r=R_1, u_θ=$\Omega_1 R_1$ which gives

$$c_2 \;=\; R_1^2\Omega_1\,.$$

Consequently,

$$u_\theta \;=\; R_1^2\Omega_1\,\frac{1}{r}\,, \tag{6.77}$$

$$\tau_{r\theta} \;=\; -2\eta\, R_1^2\Omega_1\,\frac{1}{r^2}\,, \tag{6.78}$$

and

$$p \;=\; -\frac{1}{2}\rho\, R_1^4\Omega_1^2\,\frac{1}{r^2} \;+\; c\,. \tag{6.79}$$

The shear stress exerted by the liquid to the cylinder is

$$\tau_w \;=\; \tau_{r\theta}|_{r=R_1} \;=\; -2\eta\,\Omega_1\,. \tag{6.80}$$

The torque per unit height required to rotate the cylinder is

$$\frac{T}{L} = 2\pi R_1^2 (-\tau_w) = 4\pi\eta R_1^2 \Omega_1 . \tag{6.81}$$

□

In the previous example, we studied flows between vertical coaxial cylinders of infinite height ignoring the gravitational acceleration. As indicated by Eq. (6.62), gravity has no influence on the velocity and affects only the pressure. In case of rotating liquids with a free surface, the gravity term should be included if the top part of the flow and the shape of the free surface were of interest. If surface tension effects are neglected, the pressure on the free surface is constant. Therefore, the locus of the free surface can be determined using Eq. (6.62).

Example 6.3.2. Shape of free surface in torsional flows

In this example, we study two different torsional flows with a free surface. First, we consider steady flow of a liquid contained in a large cylindrical container and agitated by a vertical rod of radius R that is coaxial to the container and rotates at angular velocity Ω. If the radius of the container is much larger than R, one may assume that the rod rotates in an infinite pool of liquid (Fig. 6.16).

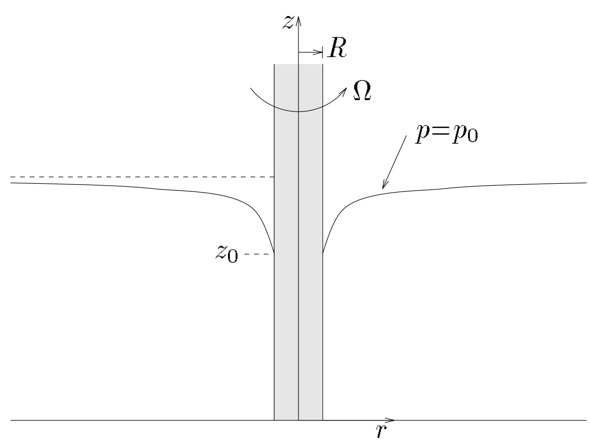


Figure 6.16. *Rotating rod in a pool of liquid.*

From the results of Example 6.3.1, we have $c_1=0$ and $c_2=\Omega R$. Therefore,

$$u_\theta = R^2\Omega \frac{1}{r}$$

and

$$p = -\frac{1}{2}\,\rho R^4 \Omega^2\,\frac{1}{r^2} - \rho g\, z + c\,.$$

With the surface tension effects neglected, the pressure on the free surface is equal to the atmospheric pressure, p_0. To determine the constant c, we assume that the free surface contacts the rod at $z{=}z_0$. Thus, we obtain

$$c = p_0 + \frac{1}{2}\,\rho R^4 \Omega^2\,\frac{1}{R^2} + \rho g\, z_0$$

and

$$p = \frac{1}{2}\,\rho R^4 \Omega^2 \left(\frac{1}{R^2} - \frac{1}{r^2}\right) - \rho g\,(z - z_0) + p_0\,. \tag{6.82}$$

Since the pressure is constant along the free surface, the equation of the latter is

$$0 = p - p_0 = \frac{1}{2}\,\rho R^4 \Omega^2 \left(\frac{1}{R^2} - \frac{1}{r^2}\right) - \rho g\,(z - z_0) \quad \Longrightarrow$$

$$z = z_0 + \frac{R^2\Omega^2}{2g}\left(1 - \frac{R^2}{r^2}\right)\,. \tag{6.83}$$

The elevation of the free surface increases with the radial distance r and approaches asymptotically the value

$$z_\infty = z_0 + \frac{R^2\Omega^2}{2g}\,.$$

This flow behavior, known as *rod dipping*, is a characteristic of generalized-Newtonian liquids, whereas viscoelastic liquids exhibit *rod climbing* (i.e., they climb the rotating rod) [5].

Consider now steady flow of a liquid contained in a cylindrical container of radius R rotating at angular velocity Ω (Fig. 6.17). From Example 6.3.1, we know that this flow corresponds to rigid-body rotation, i.e.,

$$u_\theta = \Omega\, r\,.$$

The pressure is given by

$$p = \frac{1}{2}\rho\Omega^2\, r^2 - \rho g\, z + c\,.$$

Letting z_0 be the elevation of the free surface at $r{=}0$, and p_0 be the atmospheric pressure, we get

$$c = p_0 + \rho g\, z_0\,,$$

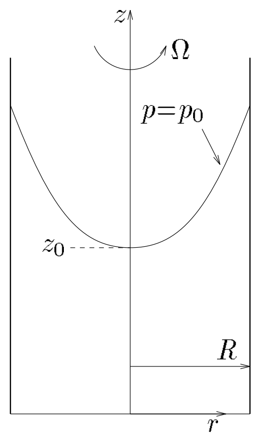


Figure 6.17. *Free surface of liquid in a rotating cylindrical container.*

and thus

$$p \;=\; \frac{1}{2}\rho\Omega^2\, r^2 \;-\; \rho g\,(z - z_0) \;+\; p_0\,. \tag{6.84}$$

The equation of the free surface is

$$0 \;=\; p - p_0 \;=\; \frac{1}{2}\rho\Omega^2\, r^2 \;-\; \rho g\,(z - z_0) \qquad \Longrightarrow$$

$$z \;=\; z_0 \;+\; \frac{\Omega^2}{2g}\, r^2\,, \tag{6.85}$$

i.e., the free surface is a parabola. □

Example 6.3.3. Superposition of Poiseuille and Couette flows

Consider steady flow of a liquid in a cylindrical tube occurring when a constant pressure gradient $\partial p/\partial z$ is applied, while the tube is rotating about its axis with constant angular velocity Ω (Fig. 6.18). This is obviously a *bidirectional* flow, since the axial and azimuthal velocity components, u_z and u_θ, are nonzero.

The flow can be considered as a superposition of axisymmetric Poiseuille and circular Couette flows, for which we have:

$$u_z \,=\, u_z(r) =\, -\frac{1}{4\eta}\,\frac{\partial p}{\partial z}\,(R^2 - r^2) \quad \text{and} \quad u_\theta \,=\, u_\theta(r) =\, \Omega\, r\,.$$

This superposition is *dynamically admissible*, since it does not violate the continuity equation, which is automatically satisfied.

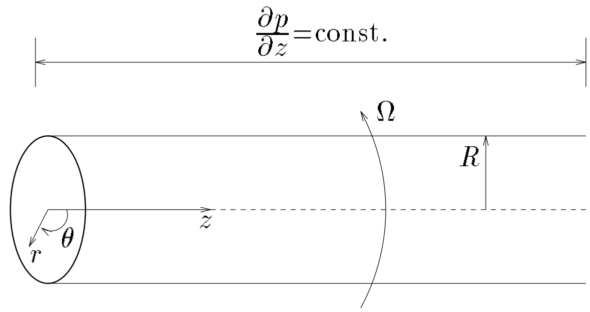


Figure 6.18. *Flow in a rotating tube under constant pressure gradient.*

Moreover, the governing equations of the flow, i.e., the z- and θ-momentum equations,

$$-\frac{\partial p}{\partial z} + \eta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = 0 \quad \text{and} \quad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) = 0\,,$$

are linear and uncoupled. Hence, the velocity for this flow is given by

$$\mathbf{u} = u_z\,\mathbf{e}_z + u_\theta\,\mathbf{e}_\theta = -\frac{1}{4\eta}\frac{\partial p}{\partial z}(R^2 - r^2)\,\mathbf{e}_z + \Omega\, r\,\mathbf{e}_\theta\,, \tag{6.86}$$

which describes a *helical flow.*

The pressure is obtained by integrating the r-momentum equation,

$$\rho\,\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r}\,,$$

taking into account that $\partial p/\partial z$ is constant. It turns out that

$$p = \frac{\partial p}{\partial z}\,z + \frac{1}{2}\rho\Omega^2\,r^2 + c\,, \tag{6.87}$$

which is simply the sum of the pressure distributions of the two superposed flows. It should be noted, however, that this might not be the case in superposition of other unidirectional flows. □

6.4 Steady, Axisymmetric Radial Flows

In axisymmetric radial flows,

$$u_z = u_\theta = 0\,. \tag{6.88}$$

Evidently, the streamlines are straight lines perpendicular to the axis of symmetry (Fig. 6.19).

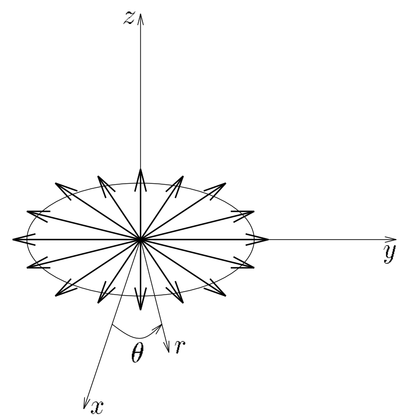


Figure 6.19. *Streamlines in axisymmetric radial flow.*

For the sake of simplicity, we will assume that u_r, in addition to being axisymmetric, does not depend on z. In other words, we assume that, in steady-state, u_r is only a function of r:

$$u_r = u_r(r)\,. \tag{6.89}$$

A characteristic of radial flows is that the non-vanishing radial velocity component is determined by the conservation of mass rather than by the r-component of the conservation of momentum equation. This implies that u_r is independent of the viscosity of the liquid. (More precisely, u_r is independent of the constitutive equation of the fluid.) Due to Eq. (6.88), the continuity equation is simplified to

$$\frac{\partial}{\partial r}(r u_r) \;=\; 0\,, \tag{6.90}$$

which gives

$$u_r \;=\; \frac{c_1}{r}\,, \tag{6.91}$$

where c_1 is a constant. The velocity u_r can also be obtained from a macroscopic mass balance. If Q is the volumetric flow rate per unit height, L, then

$$Q \;=\; u_r\,(2\pi r L) \quad\Longrightarrow$$

$$u_r = \frac{Q}{2\pi L\, r}\,, \tag{6.92}$$

which is identical to Eq. (6.91) for $c_1 = Q/(2\pi L)$.

Assumptions:	$u_z = u_\theta = 0, \quad u_r = u_r(r), \quad \mathbf{g} = -g\,\mathbf{e}_z$
Continuity:	$\frac{d}{dr}(r u_r) = 0 \quad \Longrightarrow \quad u_r = \frac{c_1}{r}$
r-momentum:	$\rho\, u_r \frac{du_r}{dr} = -\frac{\partial p}{\partial r}$
z-momentum:	$\frac{\partial p}{\partial z} + \rho g = 0$
θ-momentum:	$\frac{\partial p}{\partial \theta} = 0 \quad \Longrightarrow \quad p = p(r, z)$
General solution:	$u_r = \frac{c_1}{r}$
	$\tau_{rr} = -2\eta\, \frac{c_1}{r^2}, \quad \tau_{\theta\theta} = 2\eta\, \frac{c_1}{r^2}$
	$p = -\rho \frac{c_1^2}{2r^2} - \rho g\, z + c$

Table 6.4. *Governing equations and general solution for steady, axisymmetric radial flows.*

Letting

$$\mathbf{g} = -g\,\mathbf{e}_z\,, \tag{6.93}$$

the r-component of the Navier-Stokes equation is simplified to

$$\rho\, u_r \frac{du_r}{dr} = -\frac{\partial p}{\partial r}\,. \tag{6.94}$$

Note that the above equation contains a non-vanishing nonlinear convective term. The z- and θ-components of the Navier-Stokes equation are reduced to the standard hydrostatic expression,

$$\frac{\partial p}{\partial z} + \rho g = 0 , \tag{6.95}$$

and to

$$\frac{\partial p}{\partial \theta} = 0 , \tag{6.96}$$

respectively. The latter equation dictates that p=$p(r, z)$. Integration of Eqs. (6.94) and (6.95) gives

$$\begin{aligned} p(r,z) &= -\rho \int u_r \frac{du_r}{dr}\, dr - \rho g\, z + c \\ &= \rho\, c_1^2 \int \frac{1}{r^3}\, dr - \rho g\, z + c \quad \Longrightarrow \end{aligned}$$

$$p(r,z) = -\rho\, \frac{c_1^2}{2r^2} - \rho g\, z + c , \tag{6.97}$$

where the integration constant c is determined by specifying the value of the pressure at a point.

In axisymmetric radial flows, there are two non-vanishing stress components:

$$\tau_{rr} = 2\eta\, \frac{du_r}{dr} = -2\eta\, \frac{c_1}{r^2} ; \tag{6.98}$$

$$\tau_{\theta\theta} = 2\eta\, \frac{u_r}{r} = 2\eta\, \frac{c_1}{r^2} . \tag{6.99}$$

The assumptions, the governing equations and the general solution for steady, axisymmetric radial flows are summarized in Table 6.4.

6.5 Steady, Spherically Symmetric Radial Flows

In spherically symmetric radial flows, the fluid particles move towards or away from the center of solid, liquid or gas spheres. Examples of such flows are flow around a gas bubble which grows or collapses in a liquid bath, flow towards a spherical sink, and flow away from a point source.

The analysis of spherically symmetric radial flows is similar to that of the axisymmetric ones. The assumptions and the results are tabulated in Table 6.5. Obviously,

Assumptions:	$u_\theta = u_\phi = 0, \quad u_r = u_r(r), \quad \mathbf{g} = \mathbf{0}$
Continuity:	$\frac{d}{dr}(r^2 u_r) = 0 \quad \Longrightarrow \quad u_r = \frac{c_1}{r^2}$
r-momentum:	$\rho\, u_r \frac{du_r}{dr} = -\frac{\partial p}{\partial r}$
θ-momentum:	$\frac{\partial p}{\partial \theta} = 0$
ϕ-momentum:	$\frac{\partial p}{\partial \phi} = 0$
General solution:	$u_r = \frac{c_1}{r^2}$ $\tau_{rr} = -4\eta\, \frac{c_1}{r^3}, \quad \tau_{\theta\theta} = \tau_{\phi\phi} = 2\eta\, \frac{c_1}{r^3}$ $p = -\rho \frac{c_1^2}{2r^4} + c$

Table 6.5. *Governing equations and general solution for steady, spherically symmetric radial flows.*

spherical coordinates are the natural choice for the analysis. In steady-state, the radial velocity component is a function of the radial distance,

$$u_r = u_r(r) , \tag{6.100}$$

while the other two velocity components are zero:

$$u_\theta = u_\phi = 0 . \tag{6.101}$$

As in axisymmetric radial flows, u_r is determined from the continuity equation as

$$u_r \;=\; \frac{c_1}{r^2}\,, \qquad (6.102)$$

or

$$u_r \;=\; \frac{Q}{4\pi\, r^2}\,, \qquad (6.103)$$

where Q is the volumetric flow rate.

The pressure is given by

$$p(r) \;=\; -\rho\,\frac{c_1^2}{2r^4} \;+\; c\,. \qquad (6.104)$$

(Note that, in spherically symmetric flows, gravity is neglected.) Finally, there are now three non-vanishing stress components:

$$\tau_{rr} \;=\; 2\eta\,\frac{du_r}{dr} \;=\; -4\eta\,\frac{c_1}{r^3}\,; \qquad (6.105)$$

$$\tau_{\theta\theta} \;=\; \tau_{\phi\phi} \;=\; 2\eta\,\frac{u_r}{r} \;=\; 2\eta\,\frac{c_1}{r^3}\,. \qquad (6.106)$$

Example 6.5.1. Bubble growth in a Newtonian liquid

Boiling of a liquid often originates from small air bubbles which grow radially in the liquid. Consider a spherical bubble of radius $R(t)$ in a pool of liquid, growing at a rate

$$\frac{dR}{dt} \;=\; k\,.$$

The velocity, u_r, and the pressure, p, can be calculated using Eqs. (6.102) and (6.104), respectively. At first, we calculate the constant c_1. At $r{=}R$, $u_r{=}dR/dt{=}k$ or

$$\frac{c_1}{R^2} \;=\; k \quad\Longrightarrow\quad c_1 \;=\; kR^2\,.$$

Substituting c_1 into Eqs. (6.102) and (6.104), we get

$$u_r \;=\; k\frac{R^2}{r^2}$$

and

$$p \;=\; -\rho k^2\,\frac{R^4}{2r^4} \;+\; c\,.$$

Note that the pressure near the surface of the bubble may attain small or even negative values, which favor evaporation of the liquid and expansion of the bubble.

□

6.6 Transient One-Dimensional Unidirectional Flows

In Sections 6.1 to 6.3, we studied three classes of steady-state unidirectional flows, where the dependent variable, i.e., the nonzero velocity component, was assumed to be a function of a single spatial independent variable. The governing equation for such a flow is a linear second-order ordinary differential equation which is integrated to arrive at a general solution. The general solution contains two integration constants which are determined by the boundary conditions at the endpoints of the one-dimensional domain over which the analytical solution is sought.

In the present section, we consider one-dimensional, *transient* unidirectional flows. Hence, the dependent variable is now a function of two independent variables, one of which is time, t. The governing equations for these flows are partial differential equations. In fact, we have already encountered some of these PDEs in Sections 6.1-6.3, while simplifying the corresponding components of the Navier-Stokes equation. For the sake of convenience, these are listed below.

(a) For transient one-dimensional rectilinear flow in Cartesian coordinates with $u_y = u_z = 0$ and $u_x = u_x(y, t)$,

$$\rho \frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u_x}{\partial y^2} + \rho g_x \,. \tag{6.107}$$

(b) For transient axisymmetric rectilinear flow with $u_r = u_\theta = 0$ and $u_z = u_z(r, t)$,

$$\rho \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} + \eta \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \rho g_z \,,$$

or

$$\rho \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} + \eta \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right) + \rho g_z \,. \tag{6.108}$$

(c) For transient axisymmetric torsional flow with $u_z = u_r = 0$ and $u_\theta = u_\theta(r, t)$,

$$\rho \frac{\partial u_\theta}{\partial t} = \eta \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) \right) \,,$$

or

$$\rho \frac{\partial u_\theta}{\partial t} = \eta \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{1}{r^2} u_\theta \right) \,. \tag{6.109}$$

The above equations are all *parabolic* PDEs. For any particular flow, they are supplemented by appropriate boundary conditions at the two endpoints of the one-dimensional flow domain, and by an *initial condition* for the entire flow domain. Note that the pressure gradients in Eqs. (6.107) and (6.108) may be functions of time. These two equations are *inhomogeneous* due to the presence of the pressure gradient and gravity terms. The inhomogeneous terms can be eliminated by decomposing the dependent variable into a properly chosen steady-state component (satisfying the corresponding steady-state problem and the boundary conditions) and a transient one which satisfies the *homogeneous* problem. A similar decomposition is often used for transforming inhomogeneous boundary conditions into homogeneous ones. *Separation of variables* [2] and the *similarity solution* method [3,6] are the standard methods for solving Eq. (6.109) and the homogeneous counterparts of Eqs. (6.107) and (6.108).

In homogeneous problems admitting separable solutions, the dependent variable $u(x_i, t)$ is expressed in the form

$$u(x_i, t) \;=\; X(x_i)\, T(t)\,. \tag{6.110}$$

Substitution of the above expression into the governing equation leads to the equivalent problem of solving two ordinary differential equations with X and T as the dependent variables.

In similarity methods, the two independent variables, x_i and t, are combined into the *similarity variable*

$$\xi \;=\; \xi(x_i, t)\,. \tag{6.111}$$

If a similarity solution does exist, then the original partial differential equation for $u(x_i, t)$ is reduced to an ordinary differential equation for $u(\xi)$.

Similarity solutions exist for problems involving parabolic PDEs in two independent variables where external length and time scales are absent. A typical problem is flow of a semi-infinite fluid above a plate suddenly set in motion with a constant velocity (Example 6.6.1). Length and time scales do exist in transient plane Couette flow, and in flow of a semi-infinite fluid above a plate oscillating along its own plane. In the former flow, the length scale is the distance between the two plates, whereas in the latter case, the length scale is the period of oscillations. These two flows are governed by Eq. (6.107), with the pressure-gradient and gravity terms neglected; they are solved in Examples 6.6.2 and 6.6.3, using separation of variables. In Example 6.6.4, we solve the problem of transient plane Poiseuille flow, due to the sudden application of a constant pressure gradient.

Finally, in the last two examples, we solve transient axisymmetric rectilinear and torsional flow problems, governed, respectively, by Eqs. (6.108) and (6.109). In

Example 6.6.5, we consider transient axisymmetric Poiseuille flow, and in Example 6.6.6, we consider flow inside an infinite long cylinder which is suddenly rotated.

Example 6.6.1. Flow near a plate suddenly set in motion

Consider a semi-infinite incompressible Newtonian liquid of viscosity η and density ρ, bounded below by a plate at y=0 (Fig. 6.20). Initially, both the plate and the liquid are at rest. At time t=0$^+$, the plate starts moving in the x direction (i.e., along its plane) with constant speed V. Pressure gradient and gravity in the direction of the flow are zero. This flow problem was studied by Stokes in 1851, and is called *Rayleigh's problem* or *Stokes' first problem.*

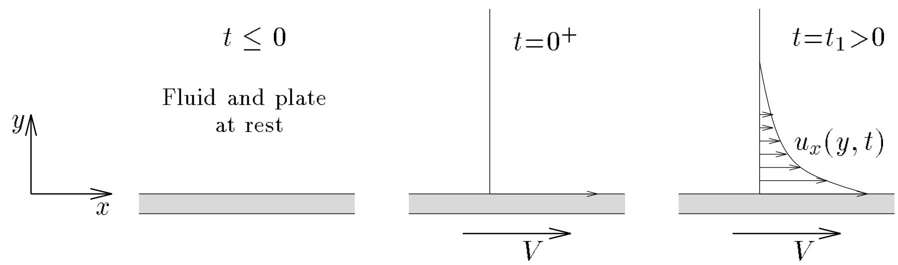


Figure 6.20. *Flow near a plate suddenly set in motion.*

The governing equation for $u_x(y,t)$ is homogeneous:

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}, \tag{6.112}$$

where $\nu \equiv \eta/\rho$ is the kinematic viscosity. Mathematically, Eq. (6.112) is called the *heat* or *diffusion equation.* The boundary and initial conditions are:

$$\left.\begin{array}{lll} u_x = V & \text{at} & y = 0,\ t > 0 \\ u_x = 0 & \text{at} & y \to \infty,\ t \geq 0 \\ u_x = 0 & \text{at} & t = 0,\ 0 \leq y < \infty \end{array}\right\} . \tag{6.113}$$

The problem described by Eqs. (6.112) and (6.113) can be solved by Laplace transforms and by the similarity method. Here, we employ the latter which is useful in solving some nonlinear problems arising in boundary layer theory (see Chapter 8). A solution with Laplace transforms can be found in Ref. [7].

Examining Eq. (6.112), we observe that if y and t are magnified k and k^2 times, respectively, Eq. (6.112) along with the boundary and initial conditions (6.113) will still be satisfied. This clearly suggests that u_x depends on a combination of y and t of the form $y/\sqrt{t}$. The same conclusion is reached by noting that the dimensionless velocity u_x/V must be a function of the remaining kinematic quantities of this flow problem: ν, t and y. From these three quantities, only one dimensionless group can be formed, $\xi{=}y/\sqrt{\nu t}$.

Let us, however, assume that the existence of a similarity solution and the proper combination of y and t are not known a priori, and assume that the solution is of the form

$$u_x(y,t) \;=\; V\, f(\xi)\,, \tag{6.114}$$

where

$$\xi \;=\; a\,\frac{y}{t^n}\,, \quad \text{with} \quad n>0\,. \tag{6.115}$$

Here $\xi(y,t)$ is the similarity variable, a is a constant to be determined later so that ξ is dimensionless, and n is a positive number to be chosen so that the original partial differential equation (6.112) can be transformed into an ordinary differential equation with f as the dependent variable and ξ as the independent one. Note that a precondition for the existence of a similarity solution is that ξ is of such a form that the original boundary and initial conditions are combined into two boundary conditions for the new dependent variable f. This is easily verified in the present flow. The boundary condition at $y{=}0$ is equivalent to

$$f=1 \;\;\text{at}\;\; \xi=0\,, \tag{6.116}$$

whereas the boundary condition at $y{\to}\infty$ and the initial condition collapse to a single boundary condition for f,

$$f=0 \;\;\text{at}\;\; \xi\to\infty\,. \tag{6.117}$$

Differentiation of Eq. (6.114) using the chain rule gives

$$\frac{\partial u_x}{\partial t} \;=\; -V\,n\frac{ay}{t^{n+1}}\,f' \;=\; -V\,n\frac{\xi}{t}\,f'\,,$$

$$\frac{\partial u_x}{\partial y} \;=\; V\,\frac{a}{t^n}\,f' \quad \text{and} \quad \frac{\partial^2 u_x}{\partial y^2} \;=\; V\,\frac{a^2}{t^{2n}}\,f''\,,$$

where primes denote differentiation with respect to ξ. Substitution of the above derivatives into Eq. (6.112) gives the following equation:

$$f'' \;+\; \frac{n\xi}{\nu a^2}\,t^{2n-1}\,f' \;=\; 0\,.$$

By setting n=1/2, time is eliminated and the above expression becomes a second-order ordinary differential equation,

$$f'' + \frac{\xi}{2\nu a^2} f' = 0 \quad \text{with} \quad \xi = a\,\frac{y}{\sqrt{t}} .$$

Taking a equal to $1/\sqrt{\nu}$ makes the similarity variable dimensionless. For convenience in the solution of the differential equation, we set a=$1/(2\sqrt{\nu})$. Hence,

$$\xi = \frac{y}{2\sqrt{\nu\, t}} , \tag{6.118}$$

whereas the resulting ordinary differential equation is

$$f'' + 2\xi\, f' = 0 . \tag{6.119}$$

This equation is subject to the boundary conditions (6.116) and (6.117). By straightforward integration, we obtain

$$f(\xi) = c_1 \int_0^{\xi} e^{-z^2}\, dz + c_2 ,$$

where z is a dummy variable of integration. At ξ=0, f=1; consequently, c_2=1. At $\xi \rightarrow \infty$, f=0; therefore,

$$c_1 \int_0^{\infty} e^{-z^2}\, dz + 1 = 0 \quad \text{or} \quad c_1 = -\frac{2}{\sqrt{\pi}} ,$$

and

$$f(\xi) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-z^2}\, dz = 1 - \text{erf}(\xi) , \tag{6.120}$$

where erf is the *error function*, defined as

$$\text{erf}(\xi) \equiv \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-z^2}\, dz . \tag{6.121}$$

Values of the error function are tabulated in several math textbooks. It is a monotone increasing function with

$$erf(0) = 0 \quad \text{and} \quad \lim_{\xi\rightarrow\infty} \text{erf}(\xi) = 1 .$$

Note that the second expression was used when calculating the constant c_1. Substituting into Eq. (6.114), we obtain the solution

$$u_x(y,t) = V \left[1 - \text{erf}\left(\frac{y}{2\sqrt{\nu\, t}} \right) \right] . \tag{6.122}$$

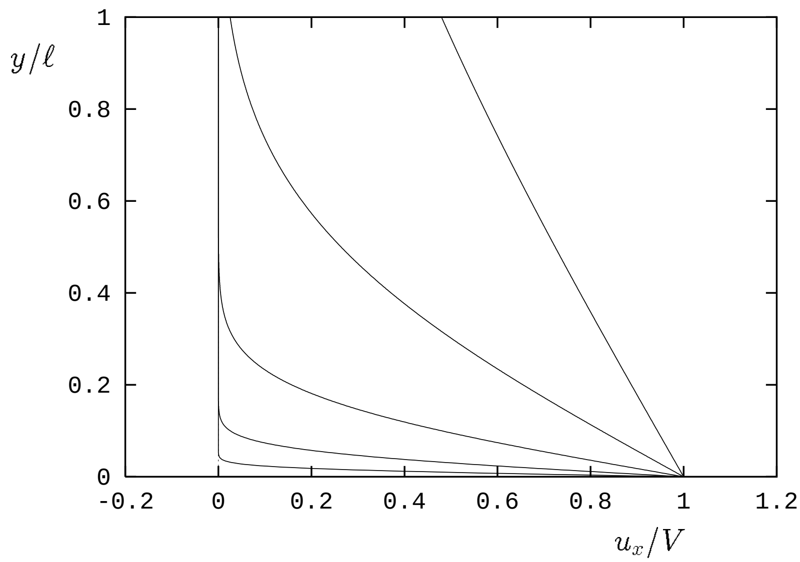


Figure 6.21. *Transient flow due to the sudden motion of a plate. Velocity profiles at* $\nu t/\ell^2$*=0.0001, 0.001, 0.01, 0.1 and 1, where* ℓ *is an arbitrary length scale.*

The evolution of $u_x(y,t)$ is illustrated in Fig. 6.21, where the velocity profiles are plotted at different values of $\nu t/\ell^2$, ℓ being an arbitrary length scale.

From Eq. (6.122), we observe that, for a fixed value of u_x/V, y varies as $2\sqrt{\nu t}$. A *boundary-layer thickness*, $\delta(t)$, can be defined as the distance from the moving plate at which u_x/V=0.01. This happens when ξ is about 1.8, and thus

$$\delta(t) \;=\; 3.6\,\sqrt{\nu t}\;.$$

The sudden motion of the plate generates vorticity, since the velocity profile is discontinuous at the initial distance. The thickness $\delta(t)$ is the penetration of vorticity distance into regions of uniform velocity after a time t. Note that Eq. (6.112) can also be viewed as a vorticity diffusion equation. Indeed, since $\mathbf{u}$=$u_x(y,t)\mathbf{i}$,

$$\omega(y,t) \;=\; |\boldsymbol{\omega}| \;=\; |\nabla\times\mathbf{u}| \;=\; \frac{\partial u_x}{\partial y}\;,$$

and Eq. (6.112) can be cast in the form

$$\frac{\partial}{\partial t}\int_0^y \omega\,dy \;=\; \nu\frac{\partial\omega}{\partial y}\;,$$

or, equivalently,

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}. \tag{6.123}$$

The above expression is a vorticity conservation equation and highlights the role of kinematic viscosity, which acts as a *vorticity diffusion* coefficient, in a manner analogous to that of thermal diffusivity in heat diffusion.

The shear stress on the plate is given by

$$\tau_w = \tau_{yx}|_{y=0} = \eta \left.\frac{\partial u_x}{\partial y}\right|_{y=0} = -\eta V \left.\frac{\partial \mathrm{erf}(\xi)}{\partial \xi}\right|_{\xi=0} \left.\frac{\partial \xi}{\partial y}\right|_{y=0} = -\frac{\eta V}{\sqrt{\pi \nu t}}, \tag{6.124}$$

which suggests that the stress is singular at the instant the plate starts moving, and decreases as $1/\sqrt{t}$.

The physics of this example are similar to those of boundary layer flow, which is examined in detail in Chapter 8. In fact, the same similarity variable was invoked by Rayleigh to calculate skin-friction over a plate moving with velocity V through a stationary liquid which leads to [8]

$$\tau_w = \frac{\eta V}{\sqrt{\pi \nu}} \sqrt{\frac{V}{x}},$$

by simply replacing t by x/V in Eq. (6.124). This situation arises in free stream flows overtaking submerged bodies, giving rise to boundary layers [9].

□

In the following example, we demonstrate the use of separation of variables by solving a transient plane Couette flow problem.

Example 6.6.2. Transient plane Couette flow

Consider a Newtonian liquid of density ρ and viscosity η bounded by two infinite parallel plates separated by a distance H, as shown in Fig. 6.22. The liquid and the two plates are initially at rest. At time t=0^+, the lower plate is suddenly brought to a steady velocity V in its own plane, while the upper plate is held stationary.

The governing equation is the same as in the previous example,

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}, \tag{6.125}$$

with the following boundary and initial conditions:

$$\left.\begin{array}{lll} u_x = V & \text{at} & y = 0,\ t > 0 \\ u_x = 0 & \text{at} & y = H,\ t \geq 0 \\ u_x = 0 & \text{at} & t = 0,\ 0 \leq y \leq H \end{array}\right\} \tag{6.126}$$

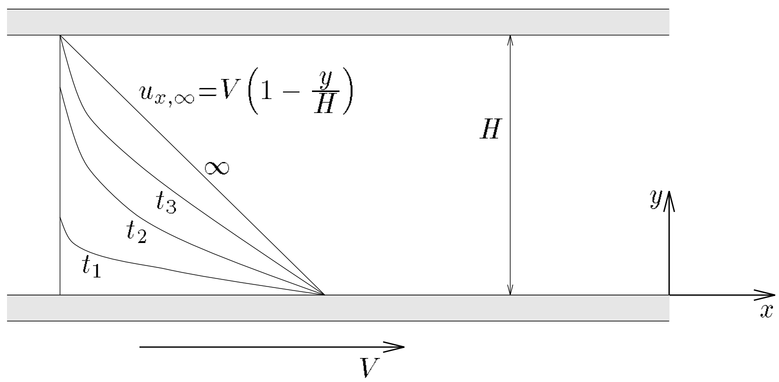


Figure 6.22. *Schematic of the evolution of the velocity in start-up plane Couette flow.*

Note that, while the governing equation is homogeneous, the boundary conditions are inhomogeneous. Therefore, separation of variables cannot be applied directly. We first have to transform the problem so that the governing equation and the two boundary conditions are homogeneous. This can be achieved by decomposing $u_x(y,t)$ into the steady plane Couette velocity profile, which is expected to prevail at large times, and a transient component:

$$u_x(y,t) \;=\; V\left(1-\frac{y}{H}\right) \;-\; u'_x(y,t)\,. \tag{6.127}$$

Substituting into Eqs. (6.125) and (6.126), we obtain the following problem

$$\frac{\partial u'_x}{\partial t} \;=\; \nu\,\frac{\partial^2 u'_x}{\partial y^2}\,, \tag{6.128}$$

with

$$\left.\begin{array}{lll} u'_x = 0 & \text{at} & y=0,\; t>0 \\ u'_x = 0 & \text{at} & y=H,\; t\geq 0 \\ u'_x = V\left(1-\frac{y}{H}\right) & \text{at} & t=0,\; 0\leq y\leq H \end{array}\right\} \tag{6.129}$$

Note that the new boundary conditions are homogeneous, while the governing equation remains unchanged. Therefore, separation of variables can now be used. The first step is to express $u'_x(y,t)$ in the form

$$u'_x(y,t) \;=\; Y(y)\,T(t)\,. \tag{6.130}$$

Substituting into Eq. (6.128) and separating the functions Y and T, we get

$$\frac{1}{\nu T}\,\frac{dT}{dt} \;=\; \frac{1}{Y}\,\frac{d^2Y}{dY^2}\,.$$

The only way a function of t can be equal to a function of y is for both functions to be equal to the same constant. For convenience, we choose this constant to be $-\alpha^2/H^2$. (One advantage of this choice is that α is dimensionless.) We thus obtain two ordinary differential equations:

$$\frac{dT}{dt} \;+\; \frac{\nu\alpha^2}{H^2}\,T \;=\; 0\,, \tag{6.131}$$

$$\frac{d^2Y}{dy^2} \;+\; \frac{\alpha^2}{H^2}\,Y \;=\; 0\,. \tag{6.132}$$

The solution to Eq. (6.131) is

$$T \;=\; c_0\, e^{-\frac{\nu\alpha^2}{H^2}t}\,, \tag{6.133}$$

where c_0 is an integration constant to be determined.

Equation (6.132) is a homogeneous second-order ODE with constant coefficients, and its general solution is

$$Y(y) \;=\; c_1\,\sin(\frac{\alpha y}{H}) \;+\; c_2\,\cos(\frac{\alpha y}{H})\,. \tag{6.134}$$

The form of the general solution justifies the choice we made earlier for the constant $-\alpha^2/H^2$. The constants c_1 and c_2 are determined by the boundary conditions. Applying Eq. (6.130) to the boundary conditions at y=0 and H, we obtain

$$Y(0)\,T(t) \;=\; 0 \quad \text{and} \quad Y(H)\,T(t) \;=\; 0\,.$$

The case of $T(t)$=0 is excluded, since this corresponds to the steady-state problem. Hence, we get the following boundary conditions for Y:

$$Y(0) = 0 \quad \text{and} \quad Y(H) = 0\,. \tag{6.135}$$

Note that in order to get the boundary conditions on Y, it is essential that the boundary conditions are homogeneous.

Applying the boundary condition at y=0, we get c_2=0. Thus,

$$Y(y) \;=\; c_1\,\sin(\frac{\alpha y}{H})\,. \tag{6.136}$$

Applying now the boundary condition at $y{=}H$, we get

$$\sin(\alpha) \;=\; 0\,, \tag{6.137}$$

which has infinitely many roots,

$$\alpha_k = k\pi\,, \qquad k = 1, 2, \cdots \tag{6.138}$$

To each of these roots correspond solutions Y_k and T_k. These infinitely many solutions are superimposed by defining

$$u_x'(y,t) \;=\; \sum_{k=1}^{\infty} B_k\,\sin(\frac{\alpha_k y}{H})\,e^{-\dfrac{\nu\alpha_k^2}{H^2}t} \;=\; \sum_{k=1}^{\infty} B_k\,\sin(\frac{k\pi\,y}{H})\,e^{-\dfrac{k^2\pi^2}{H^2}\nu t}\,, \tag{6.139}$$

where the constants $B_k{=}c_{0k}c_{1k}$ are determined from the initial condition. For $t{=}0$, we get

$$\sum_{k=1}^{\infty} B_k\,\sin(\frac{k\pi\,y}{H}) \;=\; V\left(1-\frac{y}{H}\right)\,. \tag{6.140}$$

To isolate B_k, we will take advantage of the orthogonality property

$$\int_0^1 \sin(k\pi x)\,\sin(n\pi x)\,dx \;=\; \begin{cases} \frac{1}{2}\,, & k = n \\ 0\,, & k \neq n \end{cases} \tag{6.141}$$

BY multiplying both sides of Eq. (6.140) by $\sin(n\pi y/H)\,dy$, and by integrating from 0 to H, we have:

$$\sum_{k=1}^{\infty} B_k \int_0^H \sin(\frac{k\pi\,y}{H})\,\sin(\frac{n\pi\,y}{H})\,dy \;=\; V\int_0^H \left(1-\frac{y}{H}\right)\,\sin(\frac{n\pi\,y}{H})\,dy\,.$$

Setting $\xi{=}y/H$, we get

$$\sum_{k=1}^{\infty} B_k \int_0^1 \sin(k\pi\xi)\,\sin(n\pi\xi)\,d\xi \;=\; V\int_0^1 (1-\xi)\,\sin(n\pi\xi)\,d\xi\,.$$

Due to the orthogonality property (6.141), the only nonzero term on the left hand side is that for $k{=}n$; hence,

$$B_k\,\frac{1}{2} \;=\; V\int_0^1 (1-\xi)\,\sin(k\pi\xi)\,d\xi \;=\; V\,\frac{1}{k\pi} \qquad \Longrightarrow$$

$$B_k \;=\; \frac{2V}{k\pi}\,. \qquad (6.142)$$

Substituting into Eq. (6.139) gives

$$u'_x(y,t) \;=\; \frac{2V}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\,\sin(\frac{k\pi\,y}{H})\,e^{-\frac{k^2\pi^2}{H^2}\nu t}\,. \qquad (6.143)$$

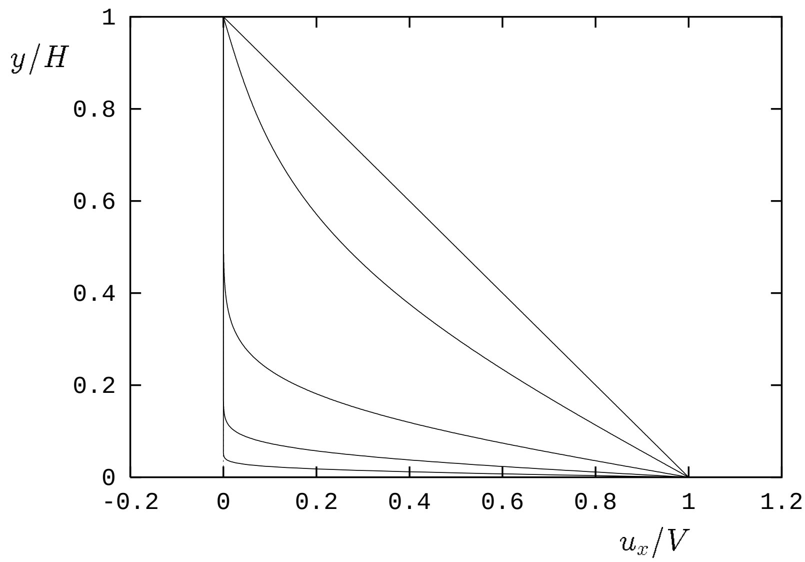


Figure 6.23. *Transient plane Couette flow. Velocity profiles at* $\nu t/H^2$*=0.0001, 0.001, 0.01, 0.1 and 1.*

Finally, for the original dependent variable $u_x(y,t)$ we get

$$u_x(y,t) \;=\; V\left(1-\frac{y}{H}\right) \;-\; \frac{2V}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\,\sin(\frac{k\pi\,y}{H})\,e^{-\frac{k^2\pi^2}{H^2}\nu t}\,. \qquad (6.144)$$

The evolution of the solution is illustrated in Fig. 6.23. Initially, the presence of the stationary plate does not affect the development of the flow, and thus the solution is similar to the one of the previous example. This is evident when comparing Figs. 6.21 and 6.23. □

Example 6.6.3. Flow due to an oscillating plate

Consider flow of a semi-infinite Newtonian liquid, set in motion by an oscillating plate of velocity

$$V = V_0 \cos\omega t , \quad t > 0 . \tag{6.145}$$

The governing equation, the initial condition and the boundary condition at $y \rightarrow \infty$ are the same as those of Example 6.6.1. At y=0, u_x is now equal to $V_0 \cos\omega t$. Hence, we have the following problem:

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2} , \tag{6.146}$$

with

$$\left. \begin{array}{lll} u_x = V_0 \cos\omega t & \text{at} & y = 0, \ t > 0 \\ u_x \rightarrow 0 & \text{at} & y \rightarrow \infty, \ t \geq 0 \\ u_x = 0 & \text{at} & t = 0, \ 0 \leq y \leq \infty \end{array} \right\} . \tag{6.147}$$

This is known as *Stokes problem* or *Stokes' second problem*, first studied by Stokes in 1845.

Since the period of the oscillations of the plate introduces a time scale, no similarity solution exists to this problem. By virtue of Eq. (6.145), it may be expected that u_x will also oscillate in time with the same frequency, but possibly with a phase shift relative to the oscillations of the plate. Thus, we separate the two independent variables by representing the velocity as

$$u_x(y,t) = \mathcal{R}e \left[Y(y)\, e^{i\omega t} \right] , \tag{6.148}$$

where $\mathcal{R}e$ denotes the real part of the expression within the brackets, i is the imaginary unit, and $Y(y)$ is a complex function. Substituting into the governing equation, we have

$$\frac{d^2 Y}{dy^2} - \frac{i\omega}{\nu} Y = 0 . \tag{6.149}$$

The general solution of the above equation is

$$Y(y) = c_1 \exp\left\{ -\sqrt{\frac{\omega}{2\nu}}\,(1+i)\,y \right\} + c_2 \exp\left\{ \sqrt{\frac{\omega}{2\nu}}\,(1+i)\,y \right\} .$$

The fact that u_x=0 at $y \rightarrow \infty$, dictates that c_2 be zero. Then, the boundary condition at y=0 requires that c_1=V_0. Thus,

$$u_x(y,t) = V_0\, \mathcal{R}e \left[\exp\left\{ -\sqrt{\frac{\omega}{2\nu}}\,(1+i)\,y \right\} e^{i\omega t} \right] , \tag{6.150}$$

The resulting solution,

$$u_x(y,t) = V_0 \exp\left(-\sqrt{\frac{\omega}{2\nu}}\, y \right) \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\, y \right) , \tag{6.151}$$

describes a damped transverse wave of wavelength $2\pi\sqrt{2\nu/\omega}$, propagating in the y-direction with phase velocity $\sqrt{2\nu\omega}$. The amplitude of the oscillations decays exponentially with y. The depth of penetration of vorticity is $\delta \sim \sqrt{2\nu/\omega}$, suggesting that the distance over which the fluid feels the motion of the plate gets smaller as the frequency of the oscillations increases. □

Example 6.6.4. Transient plane Poiseuille flow

Let us now consider a transient flow which is induced by a suddenly applied constant pressure gradient. A Newtonian liquid of density ρ and viscosity η, is contained between two horizontal plates separated by a distance $2H$ (Fig. 6.24). The liquid is initially at rest; at time t=0^+, a constant pressure gradient, $\partial p/\partial x$, is applied, setting the liquid into motion.

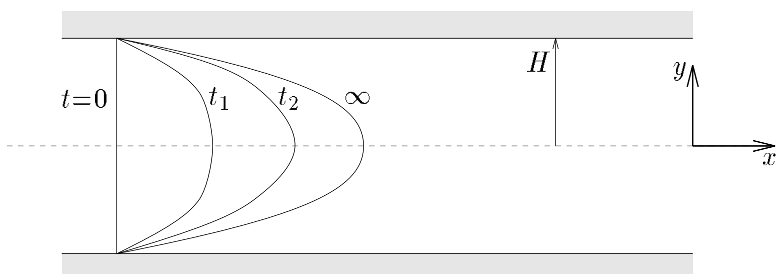


Figure 6.24. *Schematic of the evolution of the velocity in transient plane Poiseuille flow.*

The governing equation for this flow is

$$\rho \frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u_x}{\partial y^2} \,. \tag{6.152}$$

Positioning the x-axis on the symmetry plane of the flow (Fig. 6.24), the boundary and initial conditions become:

$$\left.\begin{array}{lll} u_x = 0 & \text{at} & y = H,\ t \geq 0 \\ \frac{\partial u_x}{\partial y} = 0 & \text{at} & y = 0,\ t \geq 0 \\ u_x = 0 & \text{at} & t = 0,\ 0 \leq y \leq H \end{array}\right\} \tag{6.153}$$

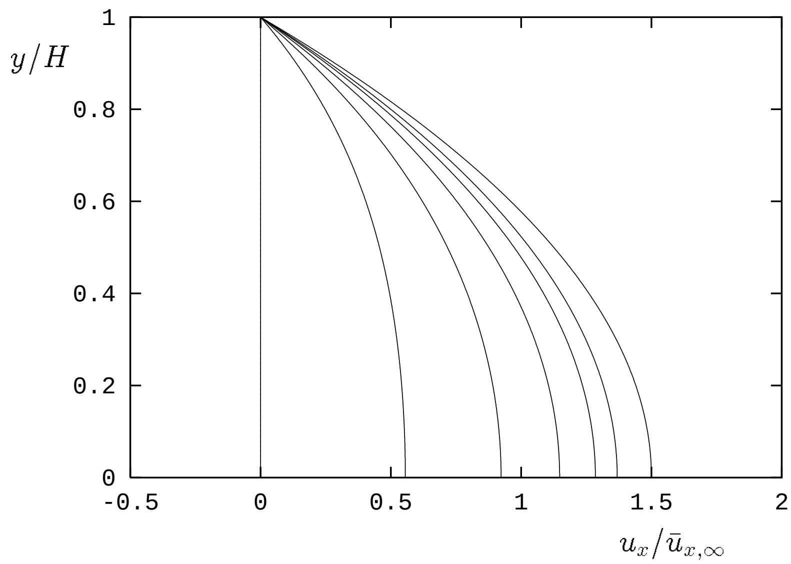


Figure 6.25. *Transient plane Poiseuille flow. Velocity profiles at $\nu t/H^2$=0.2, 0.4, 0.6, 0.8, 1 and ∞.*

The problem of Eqs. (6.152) and (6.153) is solved using separation of variables. Since the procedure is similar to that used in Example 6.6.2, it is left as an exercise for the reader (Problem 6.8) to show that

$$u_x(y,t) = -\frac{1}{2\eta}\frac{\partial p}{\partial x}H^2\left\{1-\left(\frac{y}{H}\right)^2\right.$$

$$\left. -\frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3}\cos\left[\frac{(2k-1)\pi}{2}\frac{y}{H}\right]\exp\left[-\frac{(2k-1)^2\pi^2}{4H^2}\nu t\right]\right\}. \tag{6.154}$$

The evolution of the velocity towards the parabolic steady-state profile is shown in Fig. 6.25. □

Example 6.6.5. Transient axisymmetric Poiseuille flow

Consider a Newtonian liquid of density ρ and viscosity η, initially at rest in an infinitely long horizontal cylindrical tube of radius R. At time t=0^+, a constant pressure gradient, $\partial p/\partial z$, is applied, setting the liquid into motion.

This is obviously a transient axisymmetric rectilinear flow. Since gravity is zero, the governing equation is

$$\rho \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} + \eta \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right) , \tag{6.155}$$

subject to the following boundary conditions:

$$\left. \begin{array}{lll} u_z = 0 & \text{at} & r = R, \ t \geq 0 \\ u_z \text{ finite} & \text{at} & r = 0, \ t \geq 0 \\ u_z = 0 & \text{at} & t = 0, \ 0 \leq r \leq R \end{array} \right\} \tag{6.156}$$

By decomposing $u_z(r,t)$ into the steady-state Poiseuille flow component (expected to prevail at large times) and a new dependent variable,

$$u_z(r,t) = -\frac{1}{4\eta} \frac{\partial p}{\partial z} (R^2 - r^2) - u'_z(r,t) , \tag{6.157}$$

the inhomogeneous pressure-gradient term in Eq. (6.155) is eliminated, and the following homogeneous problem is obtained:

$$\frac{\partial u'_z}{\partial t} = \nu \left(\frac{\partial^2 u'_z}{\partial r^2} + \frac{1}{r} \frac{\partial u'_z}{\partial r} \right) \tag{6.158}$$

with

$$\left. \begin{array}{lll} u'_z = 0 & \text{at} & r = R, \ t \geq 0 \\ u'_z \text{ finite} & \text{at} & r = 0, \ t \geq 0 \\ u'_z = -\frac{1}{4\eta} \frac{\partial p}{\partial z} (R^2 - r^2) & \text{at} & t = 0, \ 0 \leq r \leq R \end{array} \right\} \tag{6.159}$$

Using separation of variables, we express $u'_z(r,t)$ in the form

$$u'_z(r,t) = X(r) \, T(t) . \tag{6.160}$$

Substituting into Eq. (6.158) and separating the functions X and T, we get

$$\frac{1}{\nu T} \frac{dT}{dt} = \frac{1}{X} \left(\frac{d^2 X}{dr^2} + \frac{1}{r} \frac{dX}{dr} \right) .$$

Equating both sides of the above expression to $-\alpha^2/R^2$, where α is a dimensionless constant, we obtain two ordinary differential equations:

$$\frac{dT}{dt} + \frac{\nu \alpha^2}{R^2} T = 0 , \tag{6.161}$$

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \frac{\alpha^2}{R^2} X = 0 . \tag{6.162}$$

The solution to Eq. (6.161) is

$$T = c_0 \, e^{-\frac{\nu\alpha^2}{R^2}t} , \tag{6.163}$$

where c_0 is an integration constant.

Equation (6.162) is a Bessel's differential equation, whose general solution is given by

$$X(r) = c_1 \, J_0(\frac{\alpha r}{R}) + c_2 \, Y_0(\frac{\alpha r}{R}) , \tag{6.164}$$

where J_0 and Y_0 are the zeroth-order Bessel functions of the first and second kind, respectively. From the theory of Bessel functions, we know that $Y_0(x)$ and its first derivative are unbounded at x=0. Since u_z' and thus X must be finite at r=0, we get c_2=0.

Differentiating Eq. (6.164) and noting that

$$\frac{dJ_0}{dx}(x) = -J_1(x) ,$$

where J_1 is the first-order Bessel function of the first kind, we obtain:

$$\frac{dX}{dr}(r) = -c_1 \, \frac{\alpha}{R} \, J_1(\frac{\alpha r}{R}) + c_2 \, \frac{\alpha}{R} \, \frac{dY_0}{dr}(\frac{\alpha r}{R}) .$$

Given that $J_1(0)$=0, we find again that c_2 must be zero so that dX/dr=0 at r=0. Thus,

$$X(r) = c_1 J_0(\frac{\alpha r}{R}) . \tag{6.165}$$

Applying the boundary condition at r=R, we get

$$J_0(\alpha) = 0 . \tag{6.166}$$

Note that $J_0(x)$ is an oscillating function with infinitely many roots,

$$\alpha_k , \quad k = 1, 2, \cdots$$

Therefore, $u_z'(r,t)$ is expressed as an infinite sum of the form

$$u_z'(r,t) = \sum_{k=1}^{\infty} B_k \, J_0(\frac{\alpha_k r}{R}) \, e^{-\frac{\nu\alpha_k^2}{R^2}t} , \tag{6.167}$$

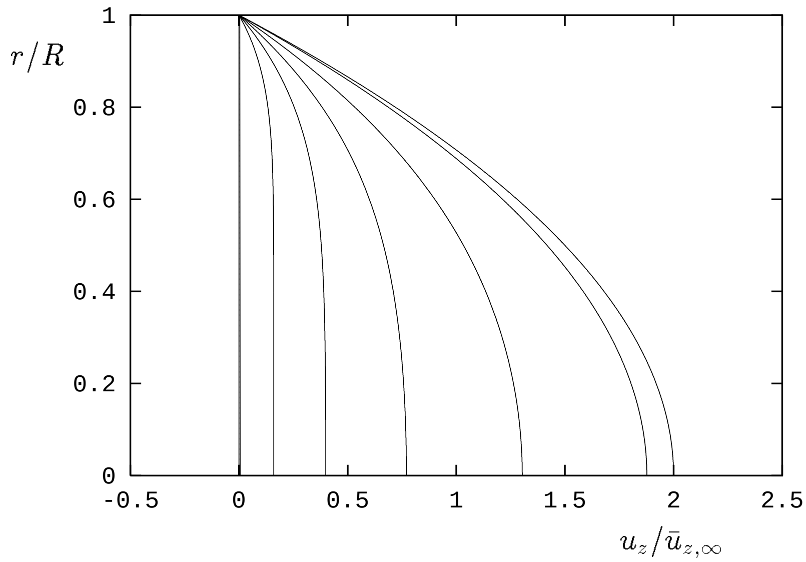


Figure 6.26. *Transient axisymmetric Poiseuille flow. Velocity profiles at* $\nu t/R^2$*=0.02, 0.05, 0.1, 0.2, 0.5 and* ∞.

where the constants B_k are to be determined from the initial condition. For t=0, we have

$$\sum_{k=1}^{\infty} B_k\, J_0(\frac{\alpha_k r}{R}) \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\, R^2 \left[1-\left(\frac{r}{R}\right)^2\right] . \tag{6.168}$$

In order to take advantage of the orthogonality property of Bessel functions,

$$\int_0^1 J_0(\alpha_k r)\, J_0(\alpha_n r)\, rdr \;=\; \begin{cases} \frac{1}{2}J_1^2(\alpha_k)\,, & k=n \\ \\ 0\,, & k\neq n \end{cases} \tag{6.169}$$

where both α_k and α_n are roots of J_0, we multiply both sides of Eq. (6.168) by $J_0(\alpha_n r/R)rdr$, and then integrate from 0 to R, to get

$$\sum_{k=1}^{\infty} B_k \int_0^R J_0(\frac{\alpha_k r}{R})\, J_0(\frac{\alpha_n r}{R})\, rdr \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\, R^2 \int_0^R \left[1-\left(\frac{r}{R}\right)^2\right] J_0(\frac{\alpha_n r}{R})\, rdr\,,$$

or

$$\sum_{k=1}^{\infty} B_k \int_0^1 J_0(\alpha_k \xi)\, J_0(\alpha_n \xi)\, \xi d\xi \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\, R^2 \int_0^1 (1-\xi^2)\, J_0(\alpha_n \xi)\, \xi d\xi\,,$$

where $\xi = r/R$. The only nonzero term on the left hand side corresponds to $k=n$. Hence,

$$B_k \, \frac{1}{2} J_1^2(\alpha_k) \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\, R^2 \int_0^1 (1-\xi^2)\, J_0(\alpha_k \xi)\, \xi d\xi \,. \tag{6.170}$$

Using standard relations for Bessel functions, we find that

$$\int_0^1 (1-\xi^2)\, J_0(\alpha_k \xi)\, \xi d\xi \;=\; \frac{4 J_1(\alpha_k)}{\alpha_k^3}\,.$$

Therefore,

$$B_k \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\, \frac{8}{\alpha_k^3\, J_1(\alpha_k)}\,,$$

and

$$u_z' \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\,(8R^2) \sum_{k=1}^{\infty} \frac{J_0\left(\frac{\alpha_k r}{R}\right)}{\alpha_k^3\, J_1(\alpha_k)}\, e^{-\frac{\nu \alpha_k^2}{R^2} t}\,. \tag{6.171}$$

Substituting into Eq. (6.167) gives

$$u_z(r,t) \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\, R^2 \left[1 - \left(\frac{r}{R}\right)^2 \;-\; 8 \sum_{k=1}^{\infty} \frac{J_0\left(\frac{\alpha_k r}{R}\right)}{\alpha_k^3\, J_1(\alpha_k)}\, e^{-\frac{\nu \alpha_k^2}{R^2} t} \right]. \tag{6.172}$$

The evolution of the velocity is shown in Fig. 6.26. □

Example 6.6.6. Flow inside a cylinder that is suddenly rotated

A Newtonian liquid of density ρ and viscosity η is initially at rest in a vertical, infinitely long cylinder of radius R. At time $t=0^+$, the cylinder starts rotating about its axis with constant angular velocity Ω, setting the liquid into motion.

This is a transient axisymmetric torsional flow, governed by

$$\frac{\partial u_\theta}{\partial t} \;=\; \nu \left(\frac{\partial^2 u_\theta}{\partial r^2} \;+\; \frac{1}{r}\frac{\partial u_\theta}{\partial r} \;-\; \frac{1}{r^2} u_\theta \right)\,, \tag{6.173}$$

subject to the following conditions:

$$\left.\begin{array}{lll} u_\theta = \Omega R & \text{at} & r=R,\; t>0 \\ u_\theta \text{ finite} & \text{at} & r=0,\; t\geq 0 \\ u_\theta = 0 & \text{at} & t=0,\; 0\leq r\leq R \end{array}\right\} \tag{6.174}$$

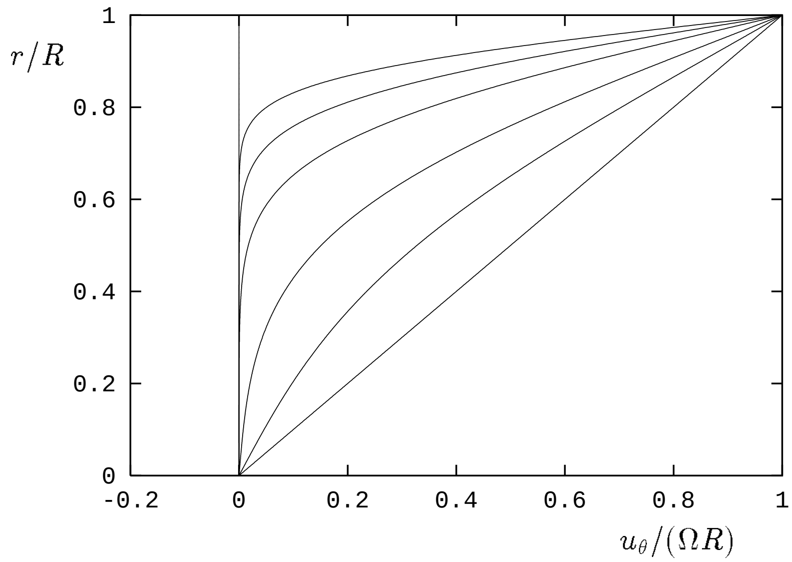


Figure 6.27. *Flow inside a cylinder that is suddenly rotated. Velocity profiles at* $\nu t/R^2$*=0.005, 0.01, 0.02, 0.05, 0.1 and* ∞*.*

The solution procedure for the problem described by Eqs. (6.173) and (6.174) is the same as in the previous example. The steady-state solution has been obtained in Example 6.3.1. Setting

$$u_\theta(r,t) \;=\; \Omega\, r \;-\; u'_\theta(r,t)\,, \tag{6.175}$$

we obtain the following homogeneous problem

$$\frac{\partial u'_\theta}{\partial t} \;=\; \nu\left(\frac{\partial^2 u'_\theta}{\partial r^2} \;+\; \frac{1}{r}\frac{\partial u'_\theta}{\partial r} \;-\; \frac{1}{r^2}u'_\theta\right)\,, \tag{6.176}$$

$$\left.\begin{array}{lll} u'_\theta = 0 & \text{at} & r=R,\; t>0 \\ u'_\theta \text{ finite} & \text{at} & r=0,\; t\geq 0 \\ u'_\theta = \Omega r & \text{at} & t=0,\; 0\leq r\leq R \end{array}\right\} \tag{6.177}$$

The independent variables are separated by setting

$$u'_\theta(r,t) \;=\; X(r)\,T(t)\,, \tag{6.178}$$

which leads to two ordinary differential equations:

$$\frac{dT}{dt} + \frac{\nu\alpha^2}{R^2}\,T = 0\,, \tag{6.179}$$

and

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \left(\frac{\alpha^2}{R^2} - \frac{1}{r^2}\right) X = 0\,. \tag{6.180}$$

Equation (6.179) is identical to Eq. (6.161) of the previous example, whose general solution is

$$T = c_0\, e^{-\frac{\nu\alpha^2}{R^2}t}\,. \tag{6.181}$$

The general solution of Eq. (6.180) is

$$X(r) = c_1\, J_1(\frac{\alpha r}{R}) + c_2\, Y_1(\frac{\alpha r}{R})\,, \tag{6.182}$$

where J_1 and Y_1 are the first-order Bessel functions of the first and second kind, respectively. Since $Y_1(x)$ is unbounded at x=0, c_2 must be zero. Therefore,

$$X(r) = c_1 J_1(\frac{\alpha r}{R})\,. \tag{6.183}$$

The boundary condition at r=R requires that

$$J_1(\alpha) = 0\,, \tag{6.184}$$

which has infinitely many roots. Therefore, $u'_\theta(r,t)$ is expressed as an infinite sum of the form

$$u'_\theta(r,t) = \sum_{k=1}^{\infty} B_k\, J_1(\frac{\alpha_k r}{R})\, e^{-\frac{\nu\alpha_k^2}{R^2}t}\,, \tag{6.185}$$

where the constants B_k are to be determined from the initial condition. For t=0, we have

$$\sum_{k=1}^{\infty} B_k\, J_1(\frac{\alpha_k r}{R}) = \Omega\, r\,. \tag{6.186}$$

The constants B_k are determined by using the orthogonality property of Bessel functions,

$$\int_0^1 J_1(\alpha_k r)\, J_1(\alpha_n r)\, r dr = \begin{cases} \frac{1}{2}J_0^2(\alpha_k)\,, & k = n \\ 0\,, & k \neq n \end{cases} \tag{6.187}$$

where both α_k and α_n are roots of J_1. Multiplying both sides of Eq. (6.186) by $J_1(\alpha_n r/R) r dr$, and integrating from 0 to R, we get

$$\sum_{k=1}^{\infty} B_k \int_0^R J_1(\frac{\alpha_k r}{R})\, J_1(\frac{\alpha_n r}{R})\, r dr \;=\; \Omega \int_0^R J_1(\frac{\alpha_n r}{R})\, r^2\, dr\,,$$

or

$$\sum_{k=1}^{\infty} B_k \int_0^1 J_1(\alpha_k \xi)\, J_1(\alpha_n \xi)\, \xi d\xi \;=\; \Omega\, R \int_0^1 J_1(\alpha_n \xi)\, \xi^2\, d\xi\,,$$

where $\xi = r/R$. Invoking Eq. (6.187), we get

$$B_k\, \frac{1}{2} J_0^2(\alpha_k) \;=\; \Omega\, R \int_0^1 J_1(\alpha_k \xi)\, \xi^2 d\xi \;=\; -\Omega\, R\, \frac{J_0(\alpha_k)}{\alpha_k} \qquad \Longrightarrow$$

$$B_k \;=\; -\frac{2\Omega R}{\alpha_k\, J_0(\alpha_k)}\,.$$

Therefore,

$$u'_\theta \;=\; -2\Omega R \sum_{k=1}^{\infty} \frac{J_1\left(\frac{\alpha_k r}{R}\right)}{\alpha_k\, J_0(\alpha_k)}\, e^{-\frac{\nu \alpha_k^2}{R^2} t} \tag{6.188}$$

and

$$u_\theta(r,t) \;=\; \Omega\, r \;+\; 2\Omega R \sum_{k=1}^{\infty} \frac{J_1\left(\frac{\alpha_k r}{R}\right)}{\alpha_k\, J_0(\alpha_k)}\, e^{-\frac{\nu \alpha_k^2}{R^2} t}\,. \tag{6.189}$$

The evolution of the u_θ is shown in Fig. 6.27. □

6.7 Steady Two-Dimensional Rectilinear Flows

As explained in Section 6.1, in steady, rectilinear flows in the x direction, $u_x = u_x(y, z)$ and the x-momentum equation is reduced to a *Poisson equation*,

$$\frac{\partial^2 u_x}{\partial y^2} \;+\; \frac{\partial^2 u_x}{\partial z^2} \;=\; \frac{1}{\eta}\, \frac{\partial p}{\partial x} \;-\; \frac{1}{\nu}\, g_x\,. \tag{6.190}$$

Equation (6.190) is an *elliptic* PDE. Since $\partial p/\partial x$ is a function of x alone and u_x is a function of y and z, Eq. (6.190) can be satisfied only when $\partial p/\partial x$ is constant. Therefore, the right hand side term of Eq. (6.190) is a constant. This inhomogeneous term can be eliminated by introducing a new dependent variable which satisfies the *Laplace equation.*

Two classes of flows governed by Eq. (6.190) are:

(a) Poiseuille flows in tubes of arbitrary but constant cross section; and

(b) gravity-driven rectilinear film flows.

One-dimensional Poiseuille flows have been encountered in Sections 6.1 and 6.2. The most important of them, i.e., plane, round and annular Poiseuille flows, are summarized in Fig. 6.28. In the following, we will discuss two-dimensional Poiseuille flows in tubes of elliptical, rectangular and triangular cross sections, illustrated in Fig. 6.29. In these rather simple geometries, Eq. (6.190) can be solved analytically. Analytical solutions for other cross sectional shapes are given in Refs. [10] and [11].

Example 6.7.1. Poiseuille flow in a tube of elliptical cross section

Consider fully-developed flow of an incompressible Newtonian liquid in an infinitely long tube of elliptical cross section, under constant pressure gradient $\partial p/\partial x$. Gravity is neglected, and thus Eq. (6.190) becomes

$$\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} = \frac{1}{\eta}\frac{\partial p}{\partial x} \quad \text{in} \quad \frac{y^2}{a^2} + \frac{z^2}{b^2} \leq 1 \,, \tag{6.191}$$

where a and b are the semi-axes of the elliptical cross section, as shown in Fig. 6.29a. The velocity is zero at the wall, and thus the boundary condition is:

$$u_x = 0 \quad \text{on} \quad \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \,. \tag{6.192}$$

Let us now introduce a new dependent variable u'_x, such that

$$u_x(y,z) = u'_x(y,z) + c_1 y^2 + c_2 z^2 \,, \tag{6.193}$$

where c_1 and c_2 are non zero constants to be determined so that (a) u'_x satisfies the Laplace equation, and (b) u'_x is constant on the wall. Substituting Eq. (6.193) into Eq. (6.191), we get

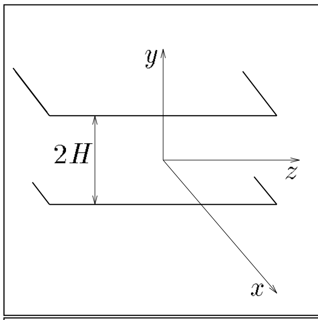
$$\frac{\partial^2 u'_x}{\partial y^2} + \frac{\partial^2 u'_x}{\partial z^2} + 2c_1 + 2c_2 = \frac{1}{\eta}\frac{\partial p}{\partial x} \,. \tag{6.194}$$

Evidently, u'_x satisfies the Laplace equation,

$$\frac{\partial^2 u'_x}{\partial y^2} + \frac{\partial^2 u'_x}{\partial z^2} = 0 \,, \tag{6.195}$$

if

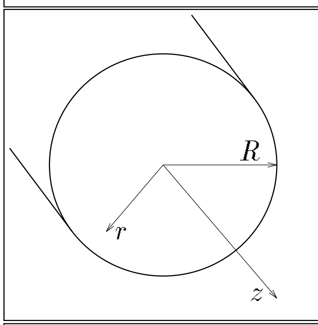
$$2c_1 + 2c_2 = \frac{1}{\eta}\frac{\partial p}{\partial x} \,. \tag{6.196}$$



Plane Poiseuille flow

$$u_x = -\frac{1}{2\eta}\frac{\partial p}{\partial x}(H^2 - y^2)$$

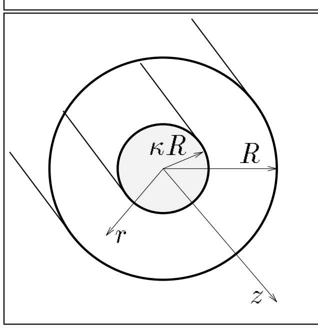
$$Q = -\frac{2}{3\eta}\frac{\partial p}{\partial x}H^3 W$$



Round Poiseuille flow

$$u_z = -\frac{1}{4\eta}\frac{\partial p}{\partial z}\left(R^2 - r^2\right)$$

$$Q = -\frac{\pi}{8\eta}\frac{\partial p}{\partial z}R^4$$



Annular Poiseuille flow

$$u_z = -\frac{1}{4\eta}\frac{\partial p}{\partial z}R^2\left[1 - \left(\frac{r}{R}\right)^2 + \frac{1-\kappa^2}{\ln(1/\kappa)}\ln\frac{r}{R}\right]$$

$$Q = -\frac{\pi}{8\eta}\frac{\partial p}{\partial z}R^4\left[\left(1-\kappa^4\right) - \frac{\left(1-\kappa^2\right)^2}{\ln(1/\kappa)}\right]$$

Figure 6.28. *One-dimensional Poiseuille flows.*

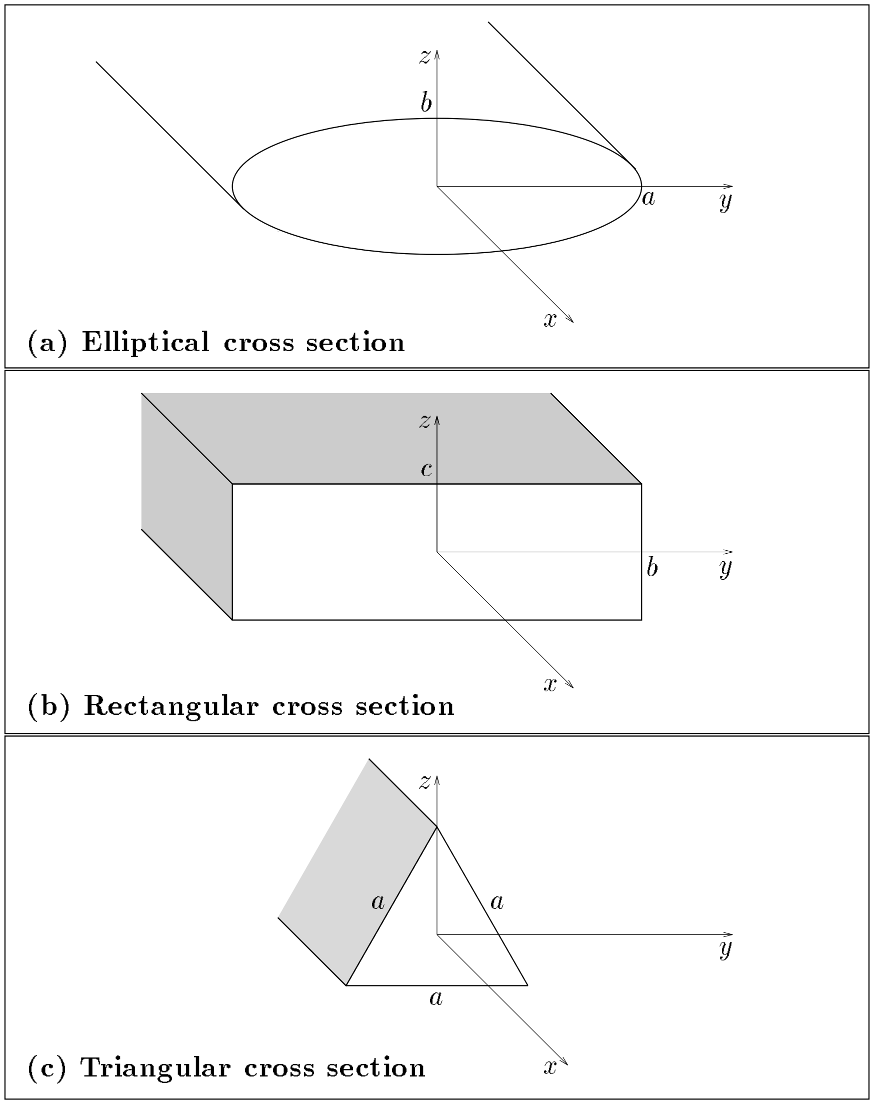


Figure 6.29. *Two-dimensional Poiseuille flow in tubes of various cross sections.*

From boundary condition (6.192), we have

$$u'_x(y,z) \;=\; -c_1 y^2 \;-\; c_2 z^2 \;=\; -c_1\left[y^2 + \frac{c_2}{c_1} z^2\right] \qquad \text{on} \qquad \frac{y^2}{a^2} + \frac{z^2}{b^2} \;=\; 1\,.$$

Setting

$$\frac{c_2}{c_1} \;=\; \frac{a^2}{b^2}\,, \tag{6.197}$$

u'_x becomes constant on the boundary,

$$u'_x(y,z) \;=\; -c_1 a^2 \qquad \text{on} \qquad \frac{y^2}{a^2} + \frac{z^2}{b^2} \;=\; 1\,. \tag{6.198}$$

The *maximum principle* for the Laplace equation states that u'_x has both its minimum and maximum values on the boundary of the domain [12]. Therefore, u'_x is constant over the whole domain,

$$u'_x(y,z) \;=\; -c_1 a^2\,. \tag{6.199}$$

Substituting into Eq. (6.193) and using Eq. (6.197), we get

$$u_x(y,z) \;=\; -c_1 a^2 \;+\; c_1 y^2 \;+\; c_2 z^2 \;=\; -c_1 a^2\left[1 \;-\; \frac{y^2}{a^2} \;-\; \frac{c_2}{c_1}\frac{z^2}{a^2}\right] \qquad \Longrightarrow$$

$$u_x(y,z) \;=\; -c_1 a^2\left[1 \;-\; \frac{y^2}{a^2} \;-\; \frac{z^2}{b^2}\right]\,. \tag{6.200}$$

The constant c_1 is determined from Eqs. (6.196) and (6.197),

$$c_1 \;=\; \frac{1}{2\eta}\frac{\partial p}{\partial x}\,\frac{b^2}{a^2+b^2}\,; \tag{6.201}$$

consequently,

$$u_x(y,z) \;=\; -\frac{1}{2\eta}\frac{\partial p}{\partial x}\,\frac{a^2 b^2}{a^2+b^2}\left[1 \;-\; \frac{y^2}{a^2} \;-\; \frac{z^2}{b^2}\right]\,. \tag{6.202}$$

Obviously, the maximum velocity occurs at the origin. Integration of the velocity profile (6.202) over the elliptical cross section yields the volumetric flow rate

$$Q \;=\; -\frac{\pi}{4\eta}\frac{\partial p}{\partial x}\,\frac{a^3 b^3}{a^2+b^2}\,. \tag{6.203}$$

Equation (6.202) degenerates to the circular Poiseuille flow velocity profile when $a{=}b{=}R$,

$$u_x(y,z) \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial x}\,R^2\left[1\;-\;\frac{y^2+z^2}{R^2}\right]\,.$$

Setting $r^2{=}y^2{+}z^2$, and switching to cylindrical coordinates, we get

$$u_z(r) \;=\; -\frac{1}{4\eta}\frac{\partial p}{\partial z}\,(R^2-r^2)\,. \qquad (6.204)$$

If now $a{=}H$ and $b \gg H$, Eq. (6.202) yields the plane Poiseuille flow velocity profile,

$$u_x(y) \;=\; -\frac{1}{2\eta}\frac{\partial p}{\partial x}\,(H^2-y^2)\,. \qquad (6.205)$$

Note that, due to symmetry, the shear stress is zero along symmetry planes. The zero shear stress condition along such a plane applies also in gravity-driven flow of a film of semielliptical cross section. Therefore, the velocity profile for the latter flow can be obtained by replacing $-\partial p/\partial x$ by ρg_x. Similarly, Eqs. (6.204) and (6.205) can be modified to describe the gravity-driven flow of semicircular and planar films, respectively. □

Example 6.7.2. Poiseuille flow in a tube of rectangular cross section

Consider steady pressure-driven flow of an incompressible Newtonian liquid in an infinitely long tube of rectangular cross section of width $2b$ and height $2c$, as shown in Fig. 6.29b. The flow is governed by the Poisson equation

$$\frac{\partial^2 u_x}{\partial y^2} \;+\; \frac{\partial^2 u_x}{\partial z^2} \;=\; \frac{1}{\eta}\,\frac{\partial p}{\partial x}\,. \qquad (6.206)$$

Taking into account the symmetry with respect to the planes $y{=}0$ and $z{=}0$, the flow can be studied only in the first quadrant (Fig. 6.30). The boundary conditions can then be written as follows:

$$\left.\begin{array}{lll}
\dfrac{\partial u_x}{\partial y} = 0 & \text{on} & y = 0 \\[2ex]
u_x = 0 & \text{on} & y = b \\[2ex]
\dfrac{\partial u_x}{\partial z} = 0 & \text{on} & z = 0 \\[2ex]
u_x = 0 & \text{on} & z = c
\end{array}\right\}\,. \qquad (6.207)$$

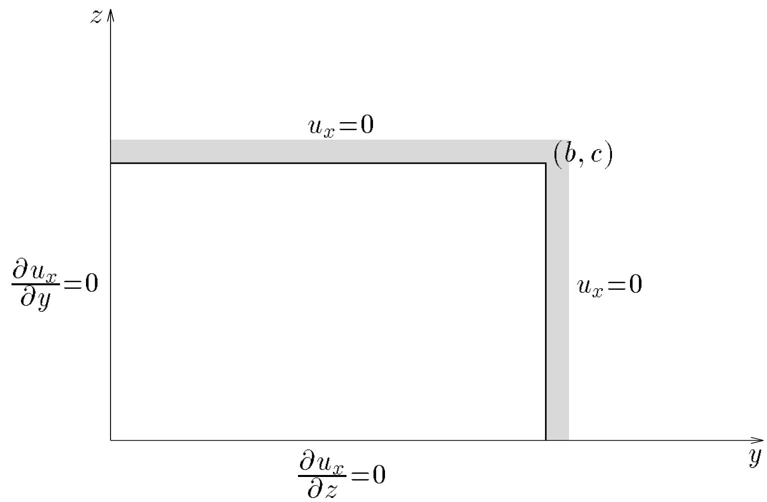


Figure 6.30. *Boundary conditions for the flow in a tube of rectangular cross section.*

Equation (6.206) can be transformed into the Laplace equation by setting

$$u_x(y,z) \;=\; -\frac{1}{2\eta}\,\frac{\partial p}{\partial x}\,(c^2 - z^2) \;+\; u'_x(y,z)\,. \tag{6.208}$$

Note that the first term in the right hand side of Eq. (6.208) is just the Poiseuille flow profile between two infinite plates placed at $z=\pm c$. Substituting Eq. (6.208) into Eqs. (6.206) and (6.207), we get

$$\frac{\partial^2 u'_x}{\partial y^2} \;+\; \frac{\partial^2 u'_x}{\partial z^2} \;=\; 0\,, \tag{6.209}$$

subject to

$$\left.\begin{array}{lll} \dfrac{\partial u'_x}{\partial y} = 0 & \text{on} & y = 0 \\[2ex] u'_x = \dfrac{1}{2\eta}\,\dfrac{\partial p}{\partial x}\,(c^2 - z^2) & \text{on} & y = b \\[2ex] \dfrac{\partial u'_x}{\partial z} = 0 & \text{on} & z = 0 \\[2ex] u'_x = 0 & \text{on} & z = c \end{array}\right\}\,. \tag{6.210}$$

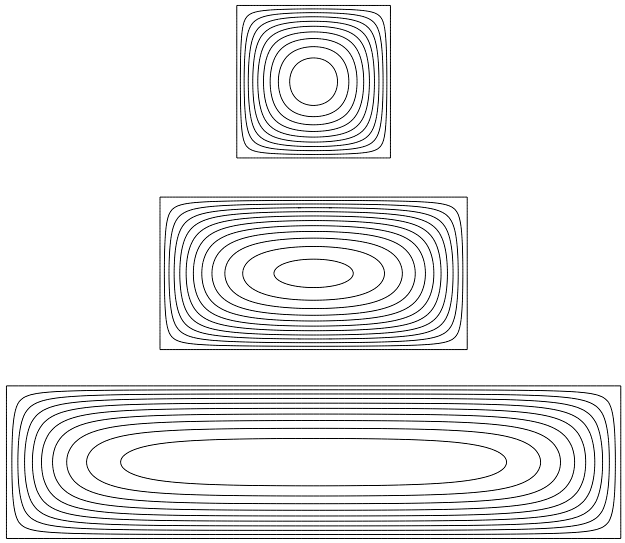


Figure 6.31. *Velocity contours for steady unidirectional flow in tubes of rectangular cross section with width-to-height ratio equal to 1, 2 and 4.*

The above problem can be solved using separation of variables (see Problem 6.13). The solution is

$$u_x(y,z) \;=\; -\frac{1}{2\eta}\,\frac{\partial p}{\partial x}\,c^2\left[1-\left(\frac{z}{c}\right)^2 \;+\; 4\sum_{k=1}^{\infty}\frac{(-1)^k}{\alpha_k^3}\;\frac{\cosh\left(\frac{\alpha_k y}{c}\right)}{\cosh\left(\frac{\alpha_k b}{c}\right)}\,\cos\left(\frac{\alpha_k z}{c}\right)\right] \quad (6.211)$$

where

$$\alpha_k \;=\; (2k-1)\,\frac{\pi}{2}\,,\qquad k=1,2,\cdots \quad (6.212)$$

In Fig. 6.31, we show the velocity contours predicted by Eq. (6.211) for different values of the width-to-height ratio. It is observed that, as this ratio increases, the velocity contours become horizontal away from the two vertical walls. This indicates that the flow away from the two walls is approximately one-dimensional (the dependence of u_x on y is weak).

The volumetric flow rate is given by

$$Q \;=\; -\frac{4}{3\eta}\,\frac{\partial p}{\partial x}\,bc^3\left[1-\frac{6c}{b}\,\sum_{k=1}^{\infty}\frac{\tanh\left(\frac{\alpha_k b}{c}\right)}{\alpha_k^5}\right]. \quad (6.213)$$

□

Example 6.7.3. Poiseuille flow in a tube of triangular cross section
Consider steady pressure-driven flow of a Newtonian liquid in an infinitely long tube whose cross section is an equilateral triangle of side a, as shown in Fig. 6.29c. Once again, the flow is governed by the Poisson equation

$$\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} = \frac{1}{\eta}\frac{\partial p}{\partial x}. \tag{6.214}$$

If the origin is set at the centroid of the cross section, as in Fig. 6.32, the three sides of the triangle lie on the lines

$$2\sqrt{3}z + a = 0, \quad \sqrt{3}z + 3y - a = 0 \quad \text{and} \quad \sqrt{3}z - 3y - a = 0.$$

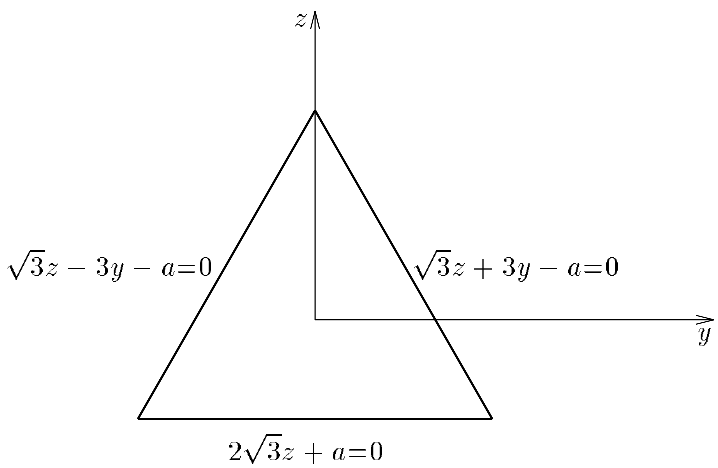


Figure 6.32. *Equations of the sides of an equilateral triangle of side a when the origin is set at the centroid.*

Since the velocity $u_x(y, z)$ is zero on the wall, the following solution form is prompted

$$u_x(y, z) = A\,(2\sqrt{3}z + a)\,(\sqrt{3}z + 3y - a)\,(\sqrt{3}z - 3y - a), \tag{6.215}$$

where A is a constant to be determined so that the governing Eq. (6.214) is satisfied. Differentiation of Eq. (6.215) gives

$$\frac{\partial^2 u_x}{\partial y^2} = -18A\,(2\sqrt{3}z + a) \quad \text{and} \quad \frac{\partial^2 u_x}{\partial z^2} = 18A\,(2\sqrt{3}z - a).$$

It turns out that Eq. (6.214) is satisfied provided that

$$A = -\frac{1}{36\eta}\,\frac{\partial p}{\partial x}\,\frac{1}{a}\,. \tag{6.216}$$

Thus, the velocity profile is given by

$$u_x(y,z) = -\frac{1}{36\eta}\,\frac{\partial p}{\partial x}\,\frac{1}{a}\,(2\sqrt{3}z+a)\,(\sqrt{3}z+3y-a)\,(\sqrt{3}z-3y-a)\,. \tag{6.217}$$

The volumetric flow rate is

$$Q = -\frac{\sqrt{3}}{320\eta}\,\frac{\partial p}{\partial x}\,a^4\,. \tag{6.218}$$

□

The unidirectional flows examined in this chapter are good approximations to many important industrial and processing flows. Channel, pipe and annulus flows are good prototypes of liquid transferring systems. The solutions to these flows provide the means to estimate the power required to overcome friction and force the liquid through, and the residence or traveling time. Analytical solutions are extremely important to the design and operation of viscometers [13]. In fact, the most known viscometers were named after the utilized flow: Couette viscometer, capillary or pressure viscometer and parallel plate viscometer [14].

The majority of the flows studied in this chapter are easily extended to *nearly unidirectional* flows in non-parallel channels or pipes and annuli, and to non-uniform films under the action of surface tension, by means of the *lubrication approximation* [15], examined in detail in Chapter 9. Transient flows that involve vorticity generation and diffusion are dynamically similar to steady flows overtaking submerged bodies giving rise to *boundary layers* [9], which are studied in Chapter 8.

6.8 Problems

6.1. Consider flow of a thin, uniform film of an incompressible Newtonian liquid on an infinite, inclined plate that moves upwards with constant speed V, as shown in Fig. 6.33. The ambient air is assumed to be stationary, and the surface tension is negligible.
(a) Calculate the velocity $u_x(y)$ of the film in terms of V, δ, ρ, η, g and θ.
(b) Calculate the speed V of the plate at which the net volumetric flow rate is zero.

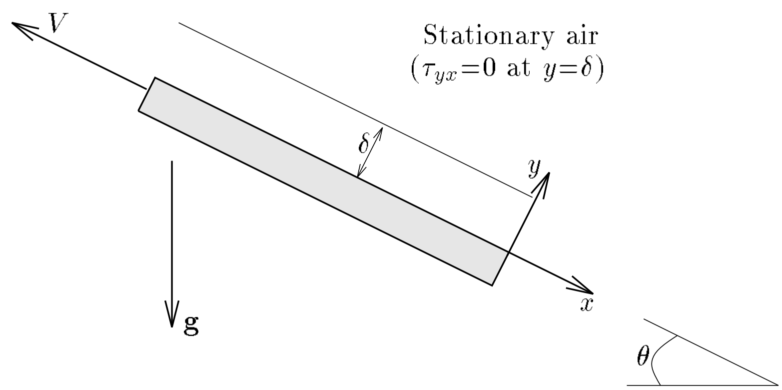


Figure 6.33. *Film flow down a moving inclined plate.*

6.2. A thin Newtonian film of uniform thickness δ is formed on the external surface of a vertical, infinitely long cylinder, which rotates at angular speed Ω, as illustrated in Fig. 6.34. Assume that the flow is steady, the surface tension is zero and the ambient air is stationary.

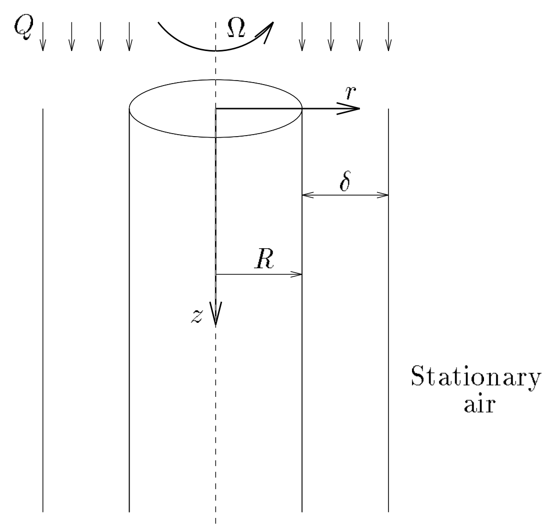


Figure 6.34. *Thin film flow down a vertical rotating cylinder.*

(a) Calculate the two nonzero velocity components.
(b) Sketch the streamlines of the flow.
(c) Calculate the volumetric flow rate Q.

(d) What must be the external pressure distribution, $p(z)$, so that uniform thickness is preserved?

6.3. A spherical bubble of radius R_A and of constant mass m_0 grows radially at a rate

$$\frac{dR_A}{dt} = k ,$$

within a spherical incompressible liquid droplet of density ρ_1, viscosity η_1 and volume V_1. The droplet itself is contained in a bath of another Newtonian liquid of density ρ_2 and viscosity η_2, as shown in Fig. 6.35. The surface tension of the inner liquid is σ_1, and its interfacial tension with the surrounding liquid is σ_2.

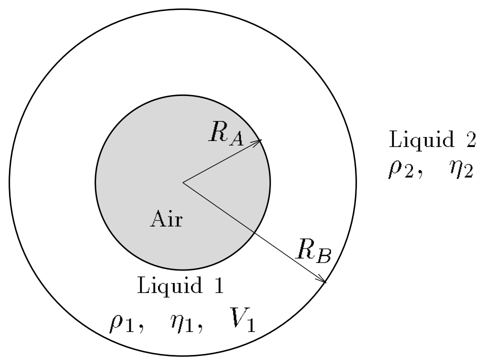


Figure 6.35. *Liquid film growing around a gas bubble.*

(a) What is the growth rate of the droplet?
(b) Calculate the velocity distribution in the two liquids.
(c) What is the pressure distribution within the bubble and the two liquids?
(d) When does the continuity of the thin film of liquid around the bubble break down?

6.4. The equations

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

and

$$\rho \left(u_y \frac{\partial u_x}{\partial y} + u_x \frac{\partial u_x}{\partial x} \right) = \eta \frac{\partial^2 u_x}{\partial y^2}$$

govern the (bidirectional) *boundary layer* flow near a horizontal plate of infinite dimensions coinciding with the xz-plane. The boundary conditions for $u_x(x, y)$ and $u_y(x, y)$ are

$$u_x = u_y = 0 \quad \text{at} \quad y=0$$
$$u_x=V, \quad u_y=0 \quad \text{at} \quad y=\infty$$

Does this problem admit a similarity solution? What is the similarity variable?

6.5. Consider a semi-infinite incompressible Newtonian liquid of viscosity η and density ρ, bounded below by a plate at $y=0$, as illustrated in Fig. 6.36. Both the plate and liquid are initially at rest. Suddenly, at time $t=0^+$, a constant shear stress τ is applied along the plate.

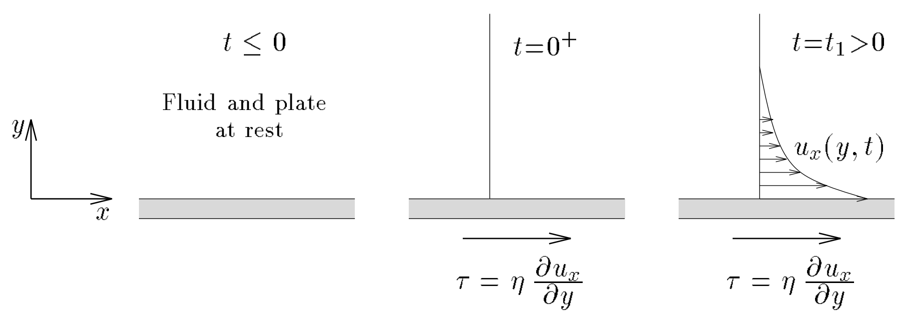


Figure 6.36. *Flow near a plate along which a constant shear stress is suddenly applied.*

(a) Specify the governing equation, the boundary and the initial conditions for this flow problem.
(b) Assuming that the velocity u_x is of the form

$$u_x = \frac{\tau}{\eta} \sqrt{\nu\, t}\, f(\xi)\,, \tag{6.219}$$

where

$$\xi = \frac{y}{\sqrt{\nu t}}\,, \tag{6.220}$$

show that

$$f(\xi) - \xi\, f'(\xi) = 2\, f''(\xi)\,. \tag{6.221}$$

(The primes denote differentiation with respect to ξ.)
(c) What are the boundary conditions for $f(\xi)$?
(d) Show that

$$u_x = \frac{\tau}{\eta} \sqrt{\nu\, t} \left\{ \frac{2}{\sqrt{\pi}}\, e^{-\xi^2/4} - \xi \left[1 - \text{erf}\left(\frac{\xi}{2}\right) \right] \right\}\,. \tag{6.222}$$

6.6. A Newtonian liquid is contained between two horizontal, infinitely long and wide plates, separated by a distance $2H$, as illustrated in Fig. 6.37. The liquid is initially at rest; at time t=0^+, both plates start moving with constant speed V.

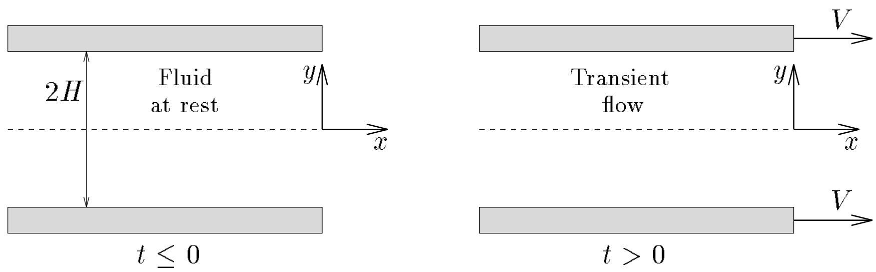


Figure 6.37. *Transient Couette flow (Problem 6.6).*

(a) Identify the governing equation, the boundary and the initial conditions for this transient flow.
(b) What is the solution for $t \leq 0$?
(c) What is the solution for $t \to \infty$?
(d) Find the time-dependent solution $u_x(y,t)$ using separation of variables.
(e) Sketch the velocity profiles at t=0, 0^+, t_1 >0 and ∞.

6.7. A Newtonian liquid is contained between two horizontal, infinitely long and wide plates, separated by a distance H, as illustrated in Fig. 6.38. Initially, the liquid flows steadily, driven by the motion of the upper plate which moves with constant speed V, while the lower plate is held stationary. Suddenly, at time t=0^+, the speed of the upper plate changes to $2V$, resulting in transient flow.
(a) Identify the governing equation, the boundary and the initial conditions for this transient flow.
(b) What is the solution for $t \leq 0$?
(c) What is the solution for $t \to \infty$?
(d) Find the time-dependent solution $u_x(y,t)$.
(e) Sketch the velocity profiles at t=0, 0^+, t_1 >0 and ∞.

6.8. Using separation of variables, show that Eq. (6.154) is indeed the solution of the transient plane Poiseuille flow, described in Example 6.6.4.

6.9. A Newtonian liquid, contained between two concentric, infinitely long, vertical cylinders of radii R_1 and R_2, where $R_2 > R_1$, is initially at rest. At time t=0^+, the

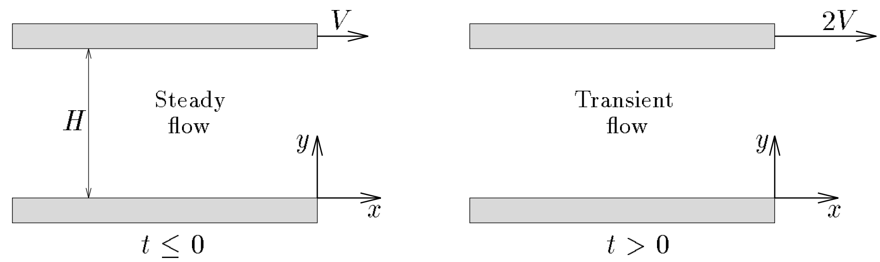


Figure 6.38. *Transient Couette flow (Problem 6.7).*

inner cylinder starts rotating about its axis with constant angular velocity Ω_1.
(a) Specify the governing equation for this transient flow.
(b) Specify the boundary and the initial conditions.
(c) Calculate the velocity $u_\theta(r,t)$.

6.10. An infinitely long, vertical rod of radius R is initially held fixed in an infinite pool of Newtonian liquid. At time $t{=}0^+$, the rod starts rotating about its axis with constant angular velocity Ω.
(a) Specify the governing equation for this transient flow.
(b) Specify the boundary and the initial conditions.
(c) Calculate the velocity $u_\theta(r,t)$.

6.11. Consider a Newtonian liquid contained between two concentric, infinitely long, horizontal cylinders of radii κR and R, where $\kappa < 1$. Assume that the liquid is initially at rest. At time $t{=}0^+$, the outer cylinder starts translating parallel to its axis with constant speed V. The geometry of the flow is shown in Fig. 6.13.
(a) Specify the governing equation for this transient flow.
(b) Specify the boundary and the initial conditions.
(c) Calculate the velocity $u_z(r,t)$.

6.12. A Newtonian liquid is initially at rest in a vertical, infinitely long cylinder of radius R. At time $t{=}0^+$, the cylinder starts both translating parallel to itself with constant speed V and rotating about its axis with constant angular velocity Ω.
(a) Calculate the corresponding steady-state solution.
(b) Specify the governing equation for the transient flow.
(c) Specify the boundary and the initial conditions.
(d) Examine whether the superposition principle holds for this transient *bidirectional* flow.
(e) Show that the time-dependent velocity and pressure profiles evolve to the steady-

state solution as $t \to \infty$.

6.13. Using separation of variables, show that Eq. (6.211) is the solution of steady Newtonian Poiseuille flow in a tube of rectangular cross section, described in Example 6.7.2.

6.14. Consider steady Newtonian Poiseuille flow in a horizontal tube of square cross section of side $2b$. Find the velocity distribution in the following cases:
(a) The liquid does not slip on any wall.
(b) The liquid slips on only two opposing walls with constant slip velocity u_w.
(c) The liquid slips on all walls with constant slip velocity u_w.
(d) The liquid slips on only two opposing walls according to the slip law

$$\tau_w = \beta\, u_w\,, \tag{6.223}$$

where τ_w is the shear stress, and β is a material slip parameter. (Note that, in this case, the slip velocity u_w is not constant.)

6.15. Integrate $u_x(y, z)$ over the corresponding cross sections, to calculate the volumetric flow rates of the Poiseuille flows discussed in the three examples of Section 6.7.

6.16. Consider steady, unidirectional, gravity-driven flow of a Newtonian liquid in an inclined, infinitely long tube of rectangular cross section of width $2b$ and height $2c$, illustrated in Fig. 6.39.

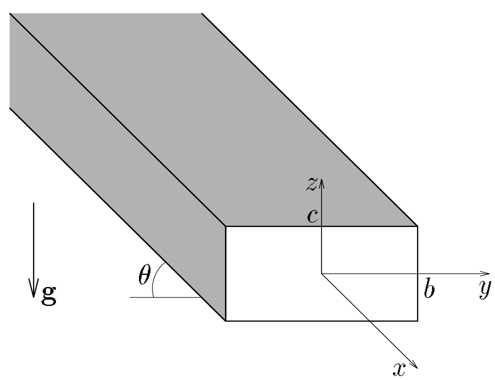


Figure 6.39. *Gravity-driven flow in an inclined tube of rectangular cross section.*

(a) Simplify the three components of the Navier-Stokes equation for this two-dimensional unidirectional flow.
(b) Calculate the pressure distribution $p(z)$.
(c) Specify the boundary conditions on the first quadrant.
(d) Calculate the velocity $u_x(y, z)$. How is this related to Eq. (6.211)?

6.17. Consider steady, gravity-driven flow of a Newtonian rectangular film in an inclined infinitely long channel of width $2b$, illustrated in Fig. 6.40. The film is assumed to be of uniform thickness H, the surface tension is negligible, and the air above the free surface is considered stationary.

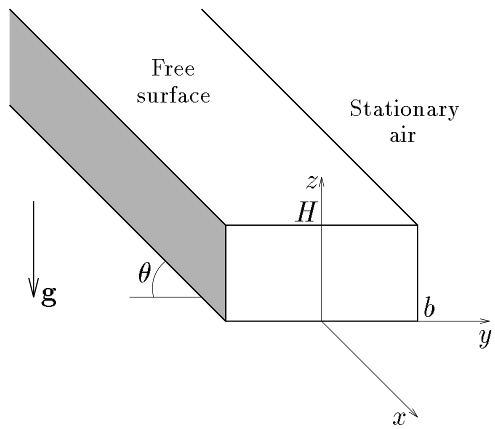


Figure 6.40. *Gravity-driven film flow in an inclined channel.*

(a) Taking into account possible symmetries, specify the governing equation and the boundary conditions for this two-dimensional unidirectional flow.
(b) Is the present flow related to that of the previous problem?
(c) Calculate the velocity $u_x(y,z)$.

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