Chapter 3

CONSERVATION LAWS

Initiation of relative fluid motion and thus development of velocity gradients occurs under the action of external force gradients, such as those due to pressure, elevation, shear stresses, density, electromagnetic forces, etc. For example, rain falls to earth due to elevation differences (i.e., gravity differential), and butter spreads thin on toast due to the shearing action of a knife. Additionally, industrial liquids are transferred by means of piping systems, after being pushed by pumps or pulled by vacuum, both of which generate pressure differentials. Meteorological phenomena are primarily due to air circulation, as a result of density differences induced by nonisothermal conditions. Finally, conducting liquids flow in non-uniform magnetic fields.

3.1 Control Volume and Surroundings

Mass, momentum and energy within a flowing medium may be transferred by *convection* and/or *diffusion*. Convection is due to bulk fluid motion, and diffusion is due to molecular motions which can take place independently of the presence of bulk motion. These transfer mechanisms, are illustrated in Fig. 3.1, where, without loss of generality, we consider a stationary control volume interacting with its surroundings through the bounding surface, S. Due to the velocity $\mathbf{u}$, fluid entering or leaving the stationary control volume carries by means of convection:

(a) Net mass per unit time,

$$\dot{m}_C = \int_S \rho\,(\mathbf{n}\cdot\mathbf{u})\,dS\ , \tag{3.1}$$

where $\mathbf{n}$ is the local outward-pointing unit normal vector, and ρ is the fluid density (subscript C denotes flux by convection).

(b) Net momentum per unit time,

$$\dot{\mathbf{J}}_C = \int_S \rho\mathbf{u}\,(\mathbf{n}\cdot\mathbf{u})\,dS\ , \tag{3.2}$$

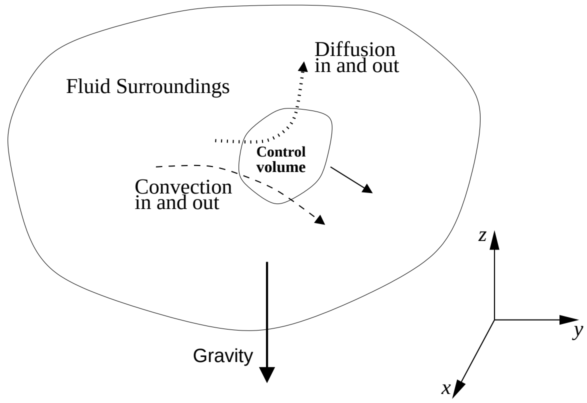

Figure 3.1. *Convection and diffusion between a control volume and its surroundings.*

where $\mathbf{J} = \rho\mathbf{u}$ is the momentum per unit volume.

(c) Net mechanical energy per unit time,

$$\dot{E}_C = \int_S \rho \left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} + gz \right) \, \mathbf{n} \cdot \mathbf{u} \, dS \,, \tag{3.3}$$

where the three scalar quantities in parentheses correspond to the kinetic energy, the flow work and the potential energy per unit mass flow rate; p is the pressure, g is the gravitational acceleration, and z is the vertical distance.

(d) Net thermal energy per unit time,

$$\dot{H}_C = \int_S \rho U (\mathbf{n} \cdot \mathbf{u}) \, dS \,, \tag{3.4}$$

where U is the *internal energy* per unit mass. This is defined as $dU \equiv C_v \, dT$, where C_v is the *specific heat* at constant volume, and T is the temperature.

(e) Total energy per unit time,

$$(\dot{E}_T)_C = \dot{E}_C + \dot{H}_C = \int_S \rho \left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} + gz + U \right) (\mathbf{n} \cdot \mathbf{u}) \, dS \,. \tag{3.5}$$

While *convection* occurs due to bulk motion, *diffusion* is independent of it, and it is entirely due to a gradient that drives to equilibrium. For instance, diffusion, commonly known as *conduction*, of heat occurs whenever there is a temperature gradient (i.e., potential), $\nabla T \neq \mathbf{0}$. Diffusion of mass occurs due to a concentration gradient, $\nabla c \neq \mathbf{0}$, and diffusion of momentum takes place due to velocity, or force gradients. Table 3.1 lists common examples of diffusion.

Quantity	Resistance	Result or Flux
Temperature, T	$1/k$	$-k\nabla T$
Solute, c	$1/D$	$-D\nabla c$
Potential, V	R	$-\frac{1}{R}\nabla V$
Velocity, $\mathbf{u}$	$1/\eta$	$\eta[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$

Table 3.1. *Common examples of diffusion.*

Common forms of diffusion in fluid mechanics are:

(a) Heat conduction, which according to *Fourier's law* is expressed as

$$\dot{H}_D = -\int_S k\,(\mathbf{n}\cdot\nabla T)\,dS\,, \tag{3.6}$$

where k is the *thermal conductivity* (subscript D denotes flux by diffusion).

(b) Momentum diffusion, which according to Newton's law of viscosity is expressed as

$$\mathbf{f} = \int_S \mathbf{n}\cdot\mathbf{T}\;dS\,, \tag{3.7}$$

where $\mathbf{f}$, $\mathbf{T}$, η and $\nabla\mathbf{u}$ are, respectively, the traction force per unit area, the local total stress tensor, the viscosity and the velocity gradient tensor. Momentum diffusion also occurs under the action of body forces, according to Newton's law of gravity,

$$\mathbf{f} = \int_V \rho\,\mathbf{g}\;dV\,, \tag{3.8}$$

where $\mathbf{f}$ is the weight, and $\mathbf{g}$ is the gravitational acceleration vector.

Production, destruction or *conversion* of fluid quantities may take place within a system or a control volume, such as mechanical energy conversion expressed by

$$\dot{E} = \int_V [\dot{W} - p(\nabla \cdot \mathbf{u}) - (\ \boldsymbol{\tau} : \nabla \mathbf{u})]\, dV \neq 0\,, \tag{3.9}$$

and thermal energy conversion given by

$$\dot{H} = \int_V (\ \boldsymbol{\tau} : \nabla \mathbf{u} + p\nabla \cdot \mathbf{u}) \pm \dot{H}_r\, dV \neq 0\,, \tag{3.10}$$

where $\dot{W}$ is the rate of production of work, and $\dot{H}_r$ is production or consumption of heat by *exothermic* and *endothermic* chemical reactions. While mechanical and thermal energy conversion within a control volume is finite, there is no total mass, or momentum conversion.

According to the sign convention adopted here, mechanical energy is gained by work W done to (+) (e.g., by a pump) or by (-) the control volume (e.g., by a turbine). In addition, mechanical energy is lost to heat due to volume expansion $(\nabla \cdot \mathbf{u})$, and due to *viscous dissipation* $(\ \boldsymbol{\tau} : \nabla \mathbf{u})$, as a result of friction between fluid layers moving at different velocities, and between the fluid and solid boundaries.

Overall change of fluid quantities within the control volume such as mass, momentum and energy is expressed as

$$\frac{d}{dt} \int_V q\, dV\ , \tag{3.11}$$

where q is the considered property per unit volume or, the *density* of the property.

3.2 The General Equations of Conservation

The development of the conservation equations starts with the general *statement of conservation*

$$\left\{ \begin{array}{c} \textit{Rate of} \\ \textit{change} \end{array} \right\} = \left\{ \begin{array}{c} \textit{Net} \\ \textit{convection} \end{array} \right\} \pm \left\{ \begin{array}{c} \textit{Net} \\ \textit{diffusion} \end{array} \right\} \pm \left\{ \begin{array}{c} \textit{Production/} \\ \textit{Destruction} \end{array} \right\}\ , \tag{3.12}$$

which, in mathematical terms, takes the form,

$$\frac{d}{dt} \int_V (\quad)\, dV = -\int_S (\quad) \mathbf{n} \cdot \mathbf{u}\, dS + \int_S k\nabla(\quad) \cdot \mathbf{n}\, dS + \int_V (\quad)\, dV\ . \tag{3.13}$$

Here, V and S are respectively the volume and the bounding surface of the control volume, $\mathbf{n}$ is the outward-pointing unit normal vector along S, $\mathbf{u}$ is the fluid velocity with respect to the control volume, k is a diffusion coefficient, and $\nabla(\quad)$ is the driving gradient responsible for diffusion. By substituting the expressions of Section 3.1 in Eq. (3.13), the *integral forms* of the conservation equations are obtained as follows:

(a) Mass conservation

$$\frac{d}{dt}\int_V \rho \, dV = -\int_S \rho \, (\mathbf{n}\cdot\mathbf{u}) \, dS \, . \tag{3.14}$$

(b) Linear momentum conservation

$$\frac{d}{dt}\int_V \rho\mathbf{u} \, dV = -\int_S \rho\mathbf{u} \, (\mathbf{n}\cdot\mathbf{u}) \, dS + \int_S \mathbf{n}\cdot\mathbf{T} \, dS + \int_V \rho\mathbf{g} \, dV \, . \tag{3.15}$$

(c) Total energy conservation

$$\frac{d}{dt}\int_V \rho E_T \, dV = -\int_S \rho E_T \, (\mathbf{n}\cdot\mathbf{u}) \, dS + \int_S (\mathbf{n}\cdot\mathbf{T})\cdot\mathbf{u} \, dS + \int_V \rho \, (\mathbf{u}\cdot\mathbf{g}) \, dV \, , \tag{3.16}$$

where the total energy is defined as the sum of the mechanical and internal energy, $E_T \equiv E + U$. The last two terms in Eq. (3.16) are the rate of work or power, due to contact and body forces, respectively.

(d) Thermal energy change

$$\begin{aligned}\frac{d}{dt}\int_V \rho U \, dV \quad &= \quad -\int_S \rho U(\mathbf{u}\cdot\mathbf{n}) \, dS + \int_V [(\boldsymbol{\tau} : \nabla\mathbf{u}) + p\,(\nabla\cdot\mathbf{u})] \, dV \\ &\quad \pm \int_V \dot{H}_r \, dV + \int_S k\nabla T\cdot\mathbf{n} \, dS \, ,\end{aligned} \tag{3.17}$$

where $\boldsymbol{\tau}$ is the viscous stress tensor related to the total stress tensor by $\mathbf{T}{=}{-}p\mathbf{I}+\boldsymbol{\tau}$.

(e) Mechanical energy change

$$\begin{aligned}\frac{d}{dt}\int_V \rho E \, dV \quad &= \quad \frac{d}{dt}\int_V \rho(E_T - U) \, dV \\ &= \quad -\int_S \rho E(\mathbf{u}\cdot\mathbf{n}) \, dS + \int_S \mathbf{n}\cdot(\mathbf{u}\cdot T) \, dS - \int_V [\boldsymbol{\tau} : \nabla\mathbf{u} + p\nabla\cdot\mathbf{u}] \, dV \\ &\quad + \int_V \rho(\mathbf{u}\cdot\mathbf{g}) \, dV \pm \int_V \dot{H}_r \, dV - \int_S k(\nabla T\cdot\mathbf{n}) \, dS \, .\end{aligned} \tag{3.18}$$

The energy equations are typically expressed in terms of a measurable property, such as temperature, by means of $dU \equiv C_v dT$. For constant C_v, $U = U_0 + C_v(T - T_0)$, where T_0 is a reference temperature of known internal energy U_0.

The minus sign associated with the convection terms is a consequence of the sign convention adopted here: the unit normal vector is positive when pointing outwards. Therefore, a normal velocity *towards* the control volume results in a positive increase of a given quantity, i.e., $d/dt > 0$.

Example 3.2.1

Derive the conservation of mass equation by means of a control volume, moving with the fluid velocity.

Solution:

The total change of mass within the control volume, given by

$$\frac{d}{dt}\int_V \rho \, dV = -\int_S \rho(\mathbf{n} \cdot \mathbf{u}^R) \, dS \; ,$$

is zero because the relative velocity, $\mathbf{u}^R$, between the control volume and its surroundings, is zero. Furthermore, according to Reynolds transport theorem,

$$\frac{d}{dt}\int_V \rho \, dV = \int_V \frac{\partial \rho}{\partial t} \, dV + \int_S \rho(\mathbf{n} \cdot \mathbf{u}) \, dS = 0 \, .$$

By invoking the divergence theorem, we get

$$\int_V \frac{\partial \rho}{\partial t} \, dV = -\int_S (\mathbf{n} \cdot \rho\mathbf{u}) \, dS = -\int_V \nabla \cdot (\rho\mathbf{u}) \, dV \qquad \Longrightarrow$$

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u})\right] dV \; = \; 0 \, . \tag{3.19}$$

Since the control volume is arbitrary,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0 \; , \tag{3.20}$$

which is the familiar form of the continuity equation. □

Example 3.2.2. Flow in an inclined pipe

Apply the integral equations of the conservation of mass, momentum and mechanical energy, to study the steady incompressible flow in an inclined pipe (Fig. 3.2).

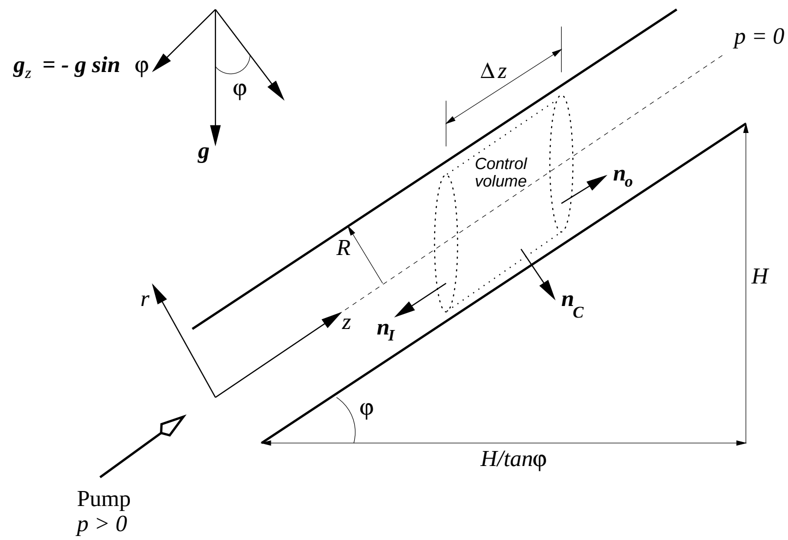

Figure 3.2. *Flow in an inclined pipe and stationary control volume.*

Solution:
For the selected control volume shown in Fig. 3.2, the rate of change of mass for incompressible or steady flow is

$$\frac{d}{dt}\int_V \rho \, dV = \int_V \frac{\partial \rho}{\partial t} \, dV = 0 \, .$$

Therefore, net convection of mass is zero, i.e.,

$$\int_S \rho(\mathbf{n}\cdot\mathbf{u}) \, dS = \int_{S_I} \rho(\mathbf{u}_I\cdot\mathbf{n}_I) \, dS_I + \int_{S_o} \rho(\mathbf{u}_o\cdot\mathbf{n}_o) \, dS_o + \int_{S_C} \rho(\mathbf{u}_C\cdot\mathbf{n}_C) \, dS_C = 0 \, ,$$

where $\mathbf{n}_I, \mathbf{n}_o$ and $\mathbf{n}_C$ are, respectively, the unit normal vectors at the inlet, outlet and cylindrical surfaces of the control volume. The velocities at the corresponding surfaces are denoted by $\mathbf{u}_I, \mathbf{u}_o$ and $\mathbf{u}_C$.

At the inlet, $\mathbf{n}_I\cdot\mathbf{u}_I = -u_n^I = -u_I(r)$; at the outlet $\mathbf{n}_o\cdot\mathbf{u}_o = u_n^o = u_o(r)$; $\mathbf{n}_C\cdot\mathbf{u}_C$ is the normal velocity to the cylindrical surface which is zero. Moreover,

$$dS_I = d(\pi r_I^2) = (2\pi r dr)_I \, , \qquad dS_o = d(\pi r_o^2) = (2\pi r dr)_o \, , \qquad dS_C = 2\pi R dz \, ,$$

and

$$dV = d(\pi r^2) dz = 2\pi r dr dz \, .$$

The above expressions are substituted in the appropriate terms of the conservation of mass equation, Eq. (3.14), to yield

$$-2\pi \int_0^R [ru(r)]_I \, dr + 2\pi \int_0^R [ru(r)]_o \, dr + 0 = 0 \ ,$$

and

$$\int_0^R \left([ru(r)]_I - [ru(r)]_o \right) \, dr = 0 \ .$$

Since the control volume is arbitrary, we must have

$$[ru(r)]_I = [ru(r)]_o \ ,$$

which yields the well known result for steady pipe flow, $u(r)_I = u(r)_o = u(r)$, i.e., the flow is characterized by a single velocity component which is parallel to the pipe wall and depends only on r.

For the same control volume, the rate of change of linear momentum for steady flow is

$$\frac{d}{dt} \int_V \rho \mathbf{u} \, dV = \int_V \rho \frac{\partial \mathbf{u}}{\partial t} \, dV = \mathbf{0} \ .$$

Convection of momentum in the flow direction (z-direction) is given by

$$\begin{aligned} \mathbf{e}_z \cdot \int_S \rho \mathbf{u}(\mathbf{n} \cdot \mathbf{u}) \, dS &= \rho \mathbf{e}_z \cdot \int_{S_I} \mathbf{u}_I(\mathbf{n}_I \cdot \mathbf{u}_I) \, dS_I + \rho \mathbf{e}_z \cdot \int_{S_o} \mathbf{u}_o(\mathbf{n}_o \cdot \mathbf{u}_o) \, dS_o \\ &\quad + \rho \mathbf{e}_z \cdot \int_{S_C} \mathbf{u}_C(\mathbf{n}_C \cdot \mathbf{u}_C) \, dS_C \\ &= -2\pi\rho \int_0^R r u_I^2(r) \, dr + 2\pi\rho \int_0^R r u_o^2(r) \, dr + 0 = 0 \ . \end{aligned}$$

The contact force (stress) contribution is

$$\begin{aligned} \mathbf{e}_z \cdot \int_S \mathbf{n} \cdot \mathbf{T} dS &= \mathbf{e}_z \cdot \int_{S_I} \mathbf{n}_I \cdot \mathbf{T}_I \, dS_I + \mathbf{e}_z \cdot \int_{S_o} \mathbf{n}_o \cdot \mathbf{T}_o \, dS_o + \mathbf{e}_z \cdot \int_{S_C} \mathbf{n}_C \cdot \mathbf{T}_C \, dS_C \\ &= \mathbf{e}_z \cdot \int_{S_I} \mathbf{n}_I \cdot (-p\mathbf{I} + \boldsymbol{\tau}) \, dS_I + \mathbf{e}_z \cdot \int_{S_o} \mathbf{n}_o \cdot (-p\mathbf{I} + \boldsymbol{\tau}) \, dS_o \\ &\quad + \mathbf{e}_z \cdot \int_{S_C} \mathbf{n}_C \cdot (-p\mathbf{I} + \boldsymbol{\tau}) \, dS_C \\ &= -2\pi \int_0^R (-p + \tau_{zz})_I \, r dr + 2\pi \int_0^R (-p + \tau_{zz})_o \, r dr + 2\pi(\Delta z) R \, \tau_{rz}^w \ , \end{aligned}$$

where τ_{rz}^w is the shear stress at the wall. By means of *macroscopic balances*, the various quantities are approximated by their average values. Therefore,

$$\begin{aligned}\mathbf{e}_z \cdot \int_S \mathbf{n} \cdot \mathbf{T}\, dS &= -2\pi \frac{R^2}{2}[(-p+\tau_{zz})_I - (-p+\tau_{zz})_o] + 2\pi R\, \Delta z\, \tau_{rz}^w \\ &= \pi R^2[-\Delta p + \Delta \tau_{zz}] + 2\pi R \tau_{rz}^w \Delta z\,,\end{aligned}$$

where $\Delta p = p_o - p_I < 0$.

Finally, the body force contribution in the flow direction is

$$\begin{aligned}\mathbf{e}_z \cdot \int_V \rho \mathbf{g}\, dV &= \mathbf{e}_z \cdot \left[\int_0^R \rho(g_r \mathbf{e}_r + g_z \mathbf{e}_z + g_\theta \mathbf{e}_\theta) 2\pi\, r dr\right] \Delta z \\ &= -2\pi \Delta z \int_0^R \rho g \sin\phi\, r dr = -2\pi \frac{R^2}{2} \Delta z \rho g \sin\phi\,.\end{aligned}$$

Therefore, the overall, *macroscopic momentum equation* is

$$-\frac{\Delta p}{\Delta z} + \frac{\Delta \tau_{zz}}{\Delta z} + \frac{2}{R}\tau_{rz}^w - \rho g \sin\phi = 0\,.$$

□

Example 3.2.3. Growing bubble

A spherical gas bubble of radius $R(t)$ grows within a liquid at a rate $\dot{R} = dR/dt$. The gas inside the bubble behaves as incompressible fluid. However, both the mass and volume change due to evaporation of liquid at the interface. By choosing appropriate control volumes show that:
(a) the gas velocity is zero;
(b) the mass flux at $r < R$ is $\rho_G \dot{R}$;
(c) the mass flux at $r > R$ is $-(\rho_L - \rho_G)\dot{R}(R^2/r^2)$.

Solution:
The problem is solved by applying the mass conservation equation,

$$\frac{d}{dt}\int_{V(t)} \rho\, dV = -\int_{S(t)} \mathbf{n} \cdot \rho(\mathbf{u} - \mathbf{u}_s)\, dS\,,$$

where V is the control volume bounded by the surface S, $\mathbf{u}$ is the velocity of the fluid under consideration, and $\mathbf{u}_s$ is the velocity of the surface bounding the control volume. In the following, the control volume is always a sphere. Therefore, the normal to the surface S is $\mathbf{n} = \mathbf{e}_r$.

(a) The control volume is fixed ($\mathbf{u}_s$=$\mathbf{0}$) of radius r, and contains only gas, i.e., $r < R$. From Reynolds transport theorem, we have

$$\frac{d}{dt}\int_V \rho_G \, dV \;=\; \int_V \frac{\partial \rho_G}{\partial t} \, dV \;=\; 0\,.$$

Therefore, for the mass flux we get

$$-\int_S \mathbf{n}\cdot\rho_G(\mathbf{u}-\mathbf{u}_s)\, dS \;=\; \frac{d}{dt}\int_{V(t)} \rho \, dV \;=\; 0 \qquad \Longrightarrow$$

$$\int_S \mathbf{n}\cdot\rho_G \mathbf{u}\, dS \;=\; 0 \qquad \Longrightarrow \qquad \mathbf{u} \;=\; \mathbf{0} \quad \text{for all} \quad r < R\,.$$

(b) The control volume is moving with the bubble ($\mathbf{u}_s$=$\dot{R}\mathbf{e}_r$) and contains only gas ($r < R$). From Reynolds transport theorem, we get

$$\frac{d}{dt}\int_{V(t)} \rho_G \, dV \;=\; \int_{V(t)} \frac{\partial \rho_G}{\partial t}\, dV + \int_{S(t)} \mathbf{n}\cdot(\rho_G \mathbf{u}_s)\, dS \;=\; 0 + \rho_G \dot{R}\,(4\pi r^2) \;=\; 4\pi\rho_G \dot{R}\, r^2\,.$$

The mass flux is given by

$$\frac{d}{dt}\int_{V(t)} \rho_G \, dV \;=\; -\int_{S(t)} \mathbf{n}\cdot\rho_G(\mathbf{u}-\mathbf{u}_s)\, dS \;=\; q\, 4\pi r^2\,,$$

where q is the relative flux per unit area. Combining the above expressions, we get q=$\rho_G \dot{R}$.

(c) The control volume is fixed ($\mathbf{u}_s$=$\mathbf{0}$) and contains the bubble ($r > R$). From Reynolds transport theorem, we get

$$\begin{aligned}
\frac{d}{dt}\int_V \rho \, dV \;&=\; \frac{d}{dt}\int_{V_G(t)} \rho_G \, dV \;+\; \frac{d}{dt}\int_{V_L(t)} \rho_L \, dV \\
&=\; \int_{V_G(t)} \frac{\partial \rho_G}{\partial t}\, dV + \int_{S(R)} \mathbf{n}\cdot(\rho_G \mathbf{u}_s)\, dS \\
&\quad\; \int_{V_L(t)} \frac{\partial \rho_L}{\partial t}\, dV + \int_{S(r)} \mathbf{n}\cdot(\rho_L \mathbf{u}_s)\, dS \;+\; \int_{S(R)} \mathbf{n}\cdot(\rho_L \mathbf{u}_s)\, dS \\
&=\; 0 \;+\; \int_{S(R)} \mathbf{e}_r\cdot(\rho_G \mathbf{u}_s)\, dS \;+\; 0 \;+\; 0 \;-\; \int_{S(R)} \mathbf{e}_r\cdot(\rho_L \mathbf{u}_s)\, dS \\
&=\; \int_{S(R)} (\rho_G - \rho_L)\, u_s \, dS \;=\; -(\rho_L - \rho_G)\dot{R}\,(4\pi R^2)\,.
\end{aligned}$$

For the mass flux, we have

$$-\int_S \rho \mathbf{n} \cdot (\mathbf{u} - \mathbf{u}_s)\, dS \;=\; q\, 4\pi r^2 \, .$$

Combining the above two equations, we get

$$q \;=\; -(\rho_L - \rho_G)\dot{R}\frac{R^2}{r^2} \, .$$

□

3.3 The Differential Forms of the Conservation Equations

The integral forms of the conservation equations derived in Section 3.2, arise naturally from the conservation statement, Eq. (3.13). However, these equations are not convenient to use in complex flow problems. To address this issue, the conservation equations are expressed in differential form by invoking the integral theorems of Chapter 1.

The general form of the integral equation of change, with respect to a stationary control volume V bounded by a surface S, may be written as

$$\int_V \frac{\partial}{\partial t}(\quad)_1\, dV = -\int_S \mathbf{n} \cdot (\quad)_1 \mathbf{u}\, dS + \int_S \mathbf{n} \cdot (\quad)_2\, dS + \int_V (\quad)_3\, dV \, . \tag{3.21}$$

Here $(\quad)_1$ is a scalar (e.g., energy or density) or a vector (e.g., momentum), $(\quad)_2$ is a vector (e.g., gradient of temperature) or a tensor (e.g., stress tensor), and $(\quad)_3$ is a vector (e.g., gravity) or a scalar (e.g., viscous dissipation or heat release by reaction).

By invoking the Gauss divergence theorem, the surface integrals of Eq. (3.21) are expressed as volume integrals:

$$\int_S \mathbf{n} \cdot (\quad)_1 \mathbf{u}\, dS \;=\; \int_V \nabla \cdot [(\quad)_1 \mathbf{u}]\; dV \, ,$$

$$\int_S \mathbf{n} \cdot (\quad)_2\, dS \;=\; \int_V \nabla \cdot (\quad)_2\, dV \, .$$

Equation (3.21) then becomes

$$\int_V \left[\frac{\partial}{\partial t}(\quad)_1 + \nabla \cdot [(\quad)_1 \mathbf{u}] - \nabla \cdot (\quad)_2 - (\quad)_3\right] dV \;=\; 0 \, . \tag{3.22}$$

Since the choice of the volume V is arbitrary, we deduce that

$$\frac{\partial}{\partial t}(\quad)_1 + \nabla \cdot [(\quad)_1 \mathbf{u}] - \nabla \cdot (\quad)_2 - (\quad)_3 \;=\; 0\,. \tag{3.23}$$

Equation (3.23) is the *differential analogue* of Eq. (3.21). It states that driving gradients $\nabla(\quad)_2$, or equivalent mechanisms, $(\quad)_{1,3}$, compete to generate change, $\partial(\quad)/\partial t$. The term $\nabla \cdot (\quad)_2$ contains the transfer or resistance coefficients according to Table 3.1. These coefficients are scalar quantities for isotropic media, vectors for media with two-directional anisotropies, and tensors for media with three-directional anisotropies. Typical transfer coefficients are the scalar viscosity of Newtonian liquids, the vector-conductivity (and mass diffusivity) in long-fiber composite materials, and the tensor-permeability of three-dimensional porous media. As shown below, particular conservation equations are obtained by filling the parentheses of Eq. (3.23) with the appropriate variables.

Mass conservation (continuity equation)

For any fluid, conservation of mass is expressed by the scalar equation

$$\frac{\partial}{\partial t}(\rho)_1 + \nabla \cdot [(\,\rho\,)_1 \mathbf{u}] \quad \Longrightarrow$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0\,. \tag{3.24}$$

Hence, a velocity profile represents an *admissible (real) flow*, if and only if it satisfies the continuity equation. For incompressible fluids, Eq. (3.24) reduces to

$$\nabla \cdot \mathbf{u} = 0\,. \tag{3.25}$$

Momentum equation

For any fluid, the momentum equation is

$$\frac{\partial}{\partial t}(\rho \mathbf{u})_1 + \nabla \cdot [(\rho \mathbf{u})_1 \mathbf{u}] - \nabla \cdot (\mathbf{T})_2 - (\rho \mathbf{g})_3 = 0\,. \tag{3.26}$$

Since $\mathbf{T}{=}{-}p\mathbf{I}{+}\,\boldsymbol{\tau}$, the momentum equation takes the form

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \nabla \cdot (-p\mathbf{I} + \boldsymbol{\tau}) + \rho \mathbf{g}\,. \tag{3.27}$$

Equation (3.27) is a vector equation and can be decomposed further into three scalar components by taking the scalar product with the basis vectors of an appropriate orthogonal coordinate system. By setting $\mathbf{g} = -g\nabla z$, where z is the distance from an arbitrary reference elevation in the direction of gravity, Eq. (3.27) can be also expressed as

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \nabla \cdot (-p\mathbf{I} + \boldsymbol{\tau}) + \nabla(-\rho g z)\,, \tag{3.28}$$

where D/Dt is the substantial derivative introduced in Chapter 1. The momentum equation then states that the acceleration of a particle following the motion is the result of a net force, expressed by the gradient of pressure, viscous and gravity forces.

Mechanical energy equation

This takes the form

$$\begin{aligned} \frac{\partial}{\partial t} \left(\rho \frac{\mathbf{u}^2}{2} \right) + \mathbf{u} \cdot \nabla \left(\rho \frac{\mathbf{u}^2}{2} \right) \quad &= \quad p(\nabla \cdot \mathbf{u}) - \nabla \cdot (p\mathbf{u}) - \boldsymbol{\tau} : \nabla \mathbf{u} \\ &\quad + \nabla \cdot (\boldsymbol{\tau} \cdot \mathbf{u}) + \rho(\mathbf{u} \cdot \mathbf{g})\,. \end{aligned} \tag{3.29}$$

To derive the above equation, we used the identities

$$\mathbf{u} \cdot \nabla p = \nabla \cdot (p\mathbf{u}) - p\nabla \cdot \mathbf{u}\,, \qquad \mathbf{u} \cdot \nabla \cdot \boldsymbol{\tau} = \nabla \cdot (\boldsymbol{\tau} \cdot \mathbf{u}) - \boldsymbol{\tau} : \nabla \mathbf{u}$$

and the continuity equation, Eq. (3.24).

Thermal energy equation

Conservation of thermal energy is expressed by

$$\rho \left[\frac{\partial U}{\partial t} + \mathbf{u} \cdot \nabla U \right] = [\boldsymbol{\tau} : \nabla \mathbf{u} + p\nabla \cdot \mathbf{u}] + \nabla(\kappa \nabla T) \pm \dot{H}_r\,, \tag{3.30}$$

where U is the internal energy per unit mass, and $\dot{H}_r$ is the heat of reaction.

Temperature equation

By invoking the definition of the internal energy, $dU \equiv C_v dT$, Eq. (3.30) becomes,

$$\rho C_v \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = \boldsymbol{\tau} : \nabla \mathbf{u} + p\nabla \cdot \mathbf{u} + \nabla(k\nabla T) \pm \dot{H}_r\,. \tag{3.31}$$

For heat conduction in solids, i.e., when $\mathbf{u} = \mathbf{0}$, $\nabla \mathbf{u} = \mathbf{0}$, and $C_v = C$, the resulting equation is

$$\rho C \frac{\partial T}{\partial t} = \nabla(k\nabla T) \pm \dot{H}_r\,. \tag{3.32}$$

For *phase change*, the latent heat rate per unit volume must be added as a source term to the energy equation.

Total energy and enthalpy equations
By adding Eqs. (3.29) and (3.30) and rearranging terms, we get

$$\rho\left[\frac{\partial}{\partial t}\left(\frac{\mathbf{u}^2}{2}+U\right)+\mathbf{u}\cdot\nabla\left(\frac{\mathbf{u}^2}{2}+gz+U\right)\right]=-\nabla\cdot\mathbf{u}+\nabla\cdot(\boldsymbol{\tau}\cdot\mathbf{u})+\nabla\cdot(k\nabla T). \quad (3.33)$$

By invoking the definition of *enthalpy*, $H\equiv U+p/\rho$, we get

$$\nabla H=\nabla U+\nabla(pV)=\nabla U+p\nabla\left(\frac{1}{\rho}\right)+\frac{1}{\rho}\nabla p\,. \quad (3.34)$$

Equation (3.33) then becomes

$$\rho\left[\frac{\partial}{\partial t}\left(\frac{\mathbf{u}^2}{2}+U\right)+\mathbf{u}\cdot\nabla\left(\frac{\mathbf{u}^2}{2}+gz+H\right)\right]=-p\nabla\cdot\mathbf{u}+\nabla\cdot(\boldsymbol{\tau}\cdot\mathbf{u})+\nabla\cdot(k\nabla T)\,. \quad (3.35)$$

The term $(p\nabla\cdot\mathbf{u})$ represents work done by expansion or compression. This term is important for gases and compressible liquids, but vanishes for incompressible liquids. Notice also that the viscous dissipation term disappears from the total energy and enthalpy equations.

The equations of motion of any incompressible fluid are tabulated in Tables 3.2 to 3.4 for the usual orthogonal coordinate systems. The above equations are specialized for incompressible, laminar flow of Newtonian fluids by means of the Newton's law of viscosity

$$\mathbf{T}=-p\mathbf{I}+\boldsymbol{\tau}=-p\mathbf{I}+\eta\left[\nabla\mathbf{u}+\nabla(\mathbf{u})^T\right]\,. \quad (3.36)$$

In the context of this book, we mostly deal with continuity, and the three components of the momentum equation. The first four equations under consideration are commonly known as *equations of motion.*

Example 3.3.1

Repeat Example 3.2.2 by using now the differential form of the equations of Table 3.3. First derive the appropriate differential equations by simplifying the conservation equations; then state appropriate assumptions based on the geometry, the symmetry of the problem, and your intuition.

Solution:
We employ a cylindrical coordinate system with the z-axis alligned with the axis of

Continuity equation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

Momentum equation

$x-$component :

$$\rho \left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right) =$$

$$= -\frac{\partial p}{\partial x} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] + \rho g_x$$

$y-$component :

$$\rho \left(\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial} \right) =$$

$$= -\frac{\partial p}{\partial y} + \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right] + \rho g_y$$

$z-$component :

$$\rho \left(\frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \right) =$$

$$= -\frac{\partial p}{\partial z} + \left[\frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right] + \rho g_z$$

Table 3.2. *The equations of motion for incompressible fluids in Cartesian coordinates.*

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}+(r u_r)\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0$$

Momentum equation

$r-$component :

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=$$

$$=-\frac{\partial p}{\partial r}+\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right]+\rho g_r$$

$\theta-$component :

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=$$

$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}\right]+\rho g_\theta$$

$z-$component :

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=$$

$$=-\frac{\partial p}{\partial z}+\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right]+\rho g_z$$

Table 3.3. *The equations of motion for incompressible fluids in cylindrical coordinates.*

Continuity equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0$$

Momentum equation

$r-$component :

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + \frac{u_\theta}{r\sin\theta}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta^2 + u_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$

$$+\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{r\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right] + \rho g_r$$

$\theta-$component :

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta}$$

$$+\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot\theta}{r}\tau_{\phi\phi}\right] + \rho g_\theta$$

$\phi-$component :

$$\rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi}{r}\cot\theta\right) =$$

$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} + \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi}) + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{r\phi}}{r} + 2\frac{\cot\theta}{r}\tau_{\theta\phi}\right] + \rho g_\phi$$

Table 3.4. *The equations of motion for incompressible fluids in spherical coordinates.*

symmetry of the pipe. It is obvious then that $u_r = u_\theta = 0$; since the flow is axisymmetric, $\partial u_z / \partial \theta = 0$. The continuity equation from Table 3.3 then yields $\partial u_z / \partial z = 0$. Therefore, the axial velocity is only a function of r, $u_z = u_z(r)$. Using $g_z = -g \sin\phi$, the z-component of the momentum equation becomes

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z \, . \tag{3.37}$$

The above microscopic, differential equation has a form similar to the macroscopic one (final result of Example 3.2.2). As discussed in Chapter 5, Eq. (3.37) can be solved for the unknown velocity profile, $u_z(r)$, given an appropriate *constitutive equation* that relates velocity to viscous stresses. □

3.4 Problems

3.1. Repeat Example 3.2.1 for the conservation of linear momentum. Assume that the control volume travels with the fluid, i.e., it is a *material volume*.

3.2. Derive the equation of change of mechanical energy under the conditions of Example 3.2.2.

3.3. Prove that the velocity in the surrounding liquid at distance $r > R(t)$ of the growing bubble of Example 3.2.3 is

$$u_r = \left(\frac{\rho_L - \rho_G}{\rho_L}\right) \frac{R^2(t)}{r^2} \frac{dR(t)}{dt} \, ,$$

using as a control volume either
(a) a fixed sphere of radius $r > R(t)$, or
(b) a sphere of constant mass with radius $r > R(t)$
that contains the growing bubble and the adjacent part of the liquid.

3.4. Starting from the macroscopic mechanical energy equation, Eq. (3.18), show how the corresponding differential one, Eq. (3.29), is obtained. Explain the physical significance of each of the terms in Eq. (3.29). Repeat for Eqs. (3.17) and (3.30), and Eqs. (3.16) and (3.35).

3.5. For a three-dimensional source at the origin, the radial velocity $\mathbf{u}$ is given by

$$\mathbf{u} = \frac{k}{r^2}\, \mathbf{e}_r \, ,$$

where k is a constant. This expression represents the Eulerian description of the flow. Determine the Lagrangian description of this velocity field. Show that the flow is dynamically admissible.

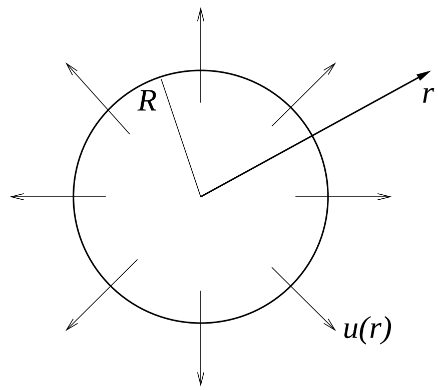

Figure 3.3. *Radial flow from a porous sphere.*

3.6. Analyze the *purely radial flow* of water through a porous sphere of radius R_0 by first identifying, and then simplifying the appropriate equations of motion.

3.7. What are the appropriate conservation equations for steady, isothermal, compressible flow in a pipe?

3.8. The *momentum equation* for Newtonian liquid is

$$\rho\left(\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) \;=\; -\nabla p + \eta\nabla^2\mathbf{u} + \rho\mathbf{g}\,.$$

Assuming that the liquid is incompressible, and by using vector-vector, vector-tensor, and differential operations, show how to derive the following equations:

(a) *Conservation of vorticity*, $\boldsymbol{\omega} = \nabla\times\mathbf{u}$

(b) *Kinetic energy change*, $E_k = 1/2(\mathbf{u}\cdot\mathbf{u})\rho$

(c) *Conservation of angular momentum*, $\mathbf{J}_\theta = \mathbf{r}\times\mathbf{J} = \mathbf{r}\times\rho\mathbf{u}$

Explain the physical significance of the terms in each equation.

3.9. *Incompressibility paradox* [7]. Here is a proof that the only velocity field that satisfies incompressibility is a zero velocity! Starting with

$$\nabla\cdot\mathbf{u} = 0\,, \tag{3.38}$$

where $\mathbf{u}$ is the velocity field, and using the divergence theorem, we find that

$$\int_S \mathbf{n}\cdot\mathbf{u}\,dS \;=\; \int_V \nabla\cdot\mathbf{u}\,dV \;=\; 0\,. \tag{3.39}$$

As a result of Eq. (3.38), there is a stream function, $\mathbf{A}$, such that

$$\mathbf{u} \;=\; \nabla \times \mathbf{A}\,,$$

and, therefore, Eq. (3.39) implies that

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{A})\; dS \;=\; 0\,.$$

Using Stokes' theorem we get,

$$\oint_C (\mathbf{A} \cdot \mathbf{t})\; d\ell \;=\; \int_S \mathbf{n} \cdot (\nabla \times \mathbf{A})\; dS \;=\; 0\,.$$

The circulation of $\mathbf{A}$ is path-independent and, therefore, there exists a scalar function, ψ, such that

$$\mathbf{A} \;=\; \nabla\psi\,,$$

and

$$\mathbf{u} \;=\; \nabla \times \mathbf{A} \;=\; \nabla \times \nabla\psi \;=\; \mathbf{0}\,.$$

What went wrong in this derivation?

3.10. *Conservative force and work* [8]. A conservative force, $\mathbf{F}$, is such that

$$\mathbf{F} \;=\; -\nabla\phi\,,$$

where ϕ is a scalar field, called potential.
(a) Show that any work done by a conservative force is path-independent.
(b) Show that the sum of the potential and the kinetic energy of a system under only conservative force action is constant.
(c) Consider a sphere moving along an inclined surface in a uniform gravity field. Identify the developed forces, characterize them as conservative or not, and evaluate the work done by them during a translation $d\mathbf{r}$. Show that the system is not conservative. Under what conditions does the system approach a conservative one?

3.5 References

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