Chapter 2

INTRODUCTION TO THE CONTINUUM FLUID

2.1 Properties of the Continuum Fluid

A flow can be of statistical (i.e., molecular) or of continuum nature, depending on the involved length and time scales. Fluid mechanics is normally concerned with the macroscopic behavior of fluids on length scales significantly larger than the mean distance between molecules and on time scales significantly larger than those associated with molecular vibrations. In such a case, a fluid can be approximated as a *continuum*, i.e., as a hypothetical infinitely divisible substance, and can be treated strictly by macroscopic methods. As a consequence of the *continuum hypothesis*, a fluid property is assumed to have a definite value at every point in space. This unique value is defined as the average over a very large number of molecules surrounding a given point within a small distance, which is still large compared with the mean intermolecular distance. Such a collection of molecules occupying a very small volume is called *fluid particle*. Hence, the velocity of a particle is considered equal to the mean velocity of the molecules it contains. The velocity so defined can also be considered to be the velocity of the fluid at the center of mass of the fluid particle. The continuum assumption implies that the values of the various fluid properties are continuous functions of position and of time. This assumption breaks down in rarefied gas flow, where the mean free path of the molecules may be of the same order of magnitude as the physical dimensions of the flow. In this case, a microscopic or statistical approach must be used.

Properties are macroscopic, observable quantities that characterize a state. They are called *extensive*, if they depend on the amount of fluid; otherwise, they are called *intensive*. Therefore, mass, weight, volume and internal energy are extensive properties, whereas temperature, pressure, and density are intensive properties. The *temperature*, T, is a measure of thermal energy, and may vary with position and time. The *pressure*, p, is also a function of position and time, defined as the limit of the ratio of the normal force, ΔF_n, acting on a surface, to the area ΔA of the surface, as $\Delta A \to 0$,

$$p \equiv \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} . \tag{2.1}$$

Hence, the pressure is a kind of *normal stress*. Similarly, the *shear stress* is defined as the limit of the tangential component of the force, ΔF_t, divided by ΔA, as $\Delta A \to 0$. Shear and normal stresses are considered in detail in Chapter 5.

Under equilibrium conditions, i.e., in a static situation, pressure results from random molecular collisions with the surface and is called *equilibrium* or *thermodynamic pressure*. Under flow conditions, i.e., in a dynamic situation, the pressure resulting from the directed molecular collisions with the surface is different from the thermodynamic pressure and is called *mechanical pressure*. The thermodynamic pressure can be determined from *equations of state*, such as the *ideal gas law* for gases and the *van der Waals equation* for liquids. The mechanical pressure can be determined only by means of energy-like *conservation equations* than take into account not just the potential and the thermal energy associated with equilibrium, but also the kinetic energy associated with flow and deformation. The general relationship between thermodynamic and mechanical pressures is considered in Chapter 5.

The density

A fundamental property of continuum is the *mass density*. The density of a fluid at a point is defined as

$$\rho \equiv \lim_{\Delta V \to L^3} \left(\frac{\Delta m}{\Delta V} \right) , \tag{2.2}$$

where Δm is the mass of a very small volume ΔV surrounding the point, and L is a very small characteristic length which, however, is significantly larger than the mean distance between molecules. Density can be inverted to give the *specific volume*

$$\hat{V} \equiv \frac{1}{\rho} , \tag{2.3}$$

or the *molecular volume*

$$V_M \equiv \frac{\hat{V}}{M} , \tag{2.4}$$

where M is the *molecular weight.*

The density of a homogeneous fluid is a function of temperature T, pressure p, and molecular weight:

$$\rho = \rho(T, p, M) . \tag{2.5}$$

Equation (2.5) is an *equation of state* at equilibrium. An example of such an equation is the *ideal gas law*,

$$\rho = \frac{pM}{RT}, \tag{2.6}$$

where R is the *ideal gas constant* which is equal to 8314 Nm/(Kg mole K).

The density of an *incompressible fluid* is independent of the pressure. The density of a *compressible fluid* depends on the pressure, and may vary in time and space, even under isothermal conditions. A measure of the changes in volume and, therefore, in density, of a certain mass of fluid subjected to pressure or normal forces, under constant temperature, is provided by the *compressibility* of the fluid, defined by

$$\beta \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = -\left(\frac{\partial \ln V}{\partial p}\right)_T \tag{2.7}$$

The compressibility of steel is around 5×10^{-12} m^2/N, that of water is 5×10^{-10} m^2/N, and that of air is identical to the inverse of its pressure (around 10^{-3} m^2/N at atmospheric pressure). Under isothermal conditions, solids, liquids and gases are virtually incompressible at low pressures. Gases are compressible at moderate pressures, and their density is a strong function of pressure. Under nonisothermal conditions, all materials behave like compressible ones, unless their *coefficient of thermal expansion*,

$$\alpha \equiv \left(\frac{\partial V}{\partial T}\right)_p, \tag{2.8}$$

is negligible.

Example 2.1.1. Air-density variations

The basic pressure-elevation relation of fluid statics is given by

$$\frac{dp}{dz} = -\rho g, \tag{2.9}$$

where g is the gravitational acceleration, and z is the elevation. Assuming that air is an ideal gas, we can calculate the air density distribution as follows. Substituting Eq. (2.6) into Eq. (2.9), we get

$$\frac{dp}{dz} = -\frac{pMg}{RT} \quad \Longrightarrow \quad \frac{dp}{p} = -\frac{Mg}{RT}\,dz\,.$$

If p_0 and ρ_0 denote the pressure and the density, respectively, at z=0, then

$$\int_{p_0}^{p(z)} \frac{dp}{p} = -\frac{Mg}{RT}\int_0^z dz \quad \Longrightarrow \quad p = p_0 \exp\left(-\frac{Mgz}{RT}\right),$$

and

$$\rho \;=\; \rho_0 \exp\left(-\frac{Mgz}{RT}\right) \;.$$

In reality, the temperature changes with elevation according to

$$T(z) \;=\; T_0 - az$$

where a is called the *atmospheric lapse rate* [1]. If the temperature variation is taken into account,

$$\int_{p_0}^{p(z)} \frac{dp}{p} \;=\; -\frac{Mg}{R}\int_0^z \frac{dz}{T_0 - az}$$

which yields

$$\frac{p(z)}{p_0} \;=\; \left(\frac{T_0 - az}{T_0}\right)^{\frac{Mg}{aR}}$$

and, therefore,

$$\frac{\rho(z)}{\rho_0} \;=\; \frac{p(z)\,T_0}{p_0\,T(z)} \;=\; \left(\frac{T_0 - az}{T_0}\right)^{\frac{Mg}{aR}-1} \;.$$

Thus, the density changes with elevation according to

$$\frac{1}{\rho_0}\frac{d\rho}{dz} \;=\; \left(-\frac{a}{T_0}\right)\left(\frac{Mg}{aR}-1\right)\left(\frac{T_0 - az}{T_0}\right)^{\frac{Mg}{aR}-1} \;.$$

□

The viscosity

A fluid in *static equilibrium* is under normal stress, which is the hydrostatic or thermodynamic pressure given by Eq. (2.1). As explained in Chapter 1, the total stress tensor, $\mathbf{T}$, consists of an isotropic pressure stress component, $-p\mathbf{I}$, and of an anisotropic viscous stress component, $\boldsymbol{\tau}$,

$$\mathbf{T} \;=\; -p\,\mathbf{I} \;+\; \boldsymbol{\tau} \;. \tag{2.10}$$

The stress tensor $\boldsymbol{\tau}$ comes from the relative motion of fluid particles and is zero in static equilibrium. When there is relative motion of fluid particles, the *velocity-gradient tensor*, $\nabla\mathbf{u}$, and the *rate-of-strain tensor*,

$$\mathbf{D} \;\equiv\; \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\,, \tag{2.11}$$

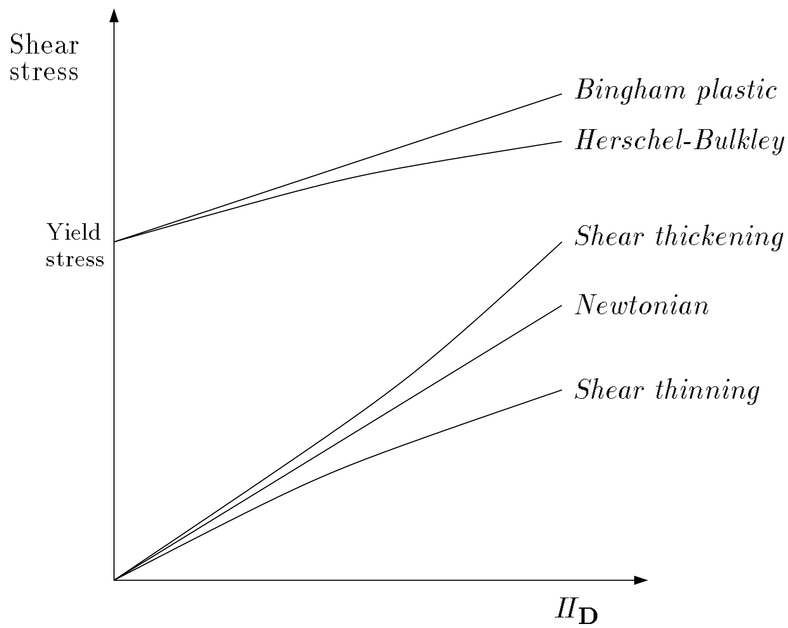

Figure 2.1. *Behavior of various non-Newtonian fluids.*

are not zero. Incompressible *Newtonian* fluids follow *Newton's law of viscosity* (discussed in detail in Chapter 5) which states that the viscous stress tensor $\boldsymbol{\tau}$ is proportional to the rate-of-strain tensor,

$$\boldsymbol{\tau} \;=\; 2\eta\,\mathbf{D} \;=\; \eta\,[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \tag{2.12}$$

or, equivalently,

$$[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \;=\; \frac{\boldsymbol{\tau}}{\eta}\,. \tag{2.13}$$

The proportionality constant, η, which is a coefficient of *momentum transfer* in Eq. (2.12) and *resistance* in Eq. (2.13), is called *dynamic viscosity* or, simply, *viscosity*. The dynamic viscosity divided by density is called *kinematic viscosity* and is usually denoted by ν:

$$\nu \;\equiv\; \frac{\eta}{\rho} \tag{2.14}$$

A fluid is called *ideal* or *inviscid* if its viscosity is zero; fluids of nonzero viscosity are called *viscous*. Viscous fluids not obeying Newton's law are generally called *non-Newtonian fluids*. These are classified into *generalized Newtonian* and *viscoelastic* *fluids*. Note that the same qualifiers are used to describe the corresponding flow, e.g., *ideal flow*, *Newtonian flow*, *viscoelastic flow* etc.

Generalized Newtonian fluids are viscous *inelastic* fluids that still follow Eq. (2.12), but the viscosity itself is a function of the rate of strain tensor **D**; more precisely, the viscosity is a function of the second invariant of **D**, $\eta{=}\eta(II_{\mathbf{D}})$. A fluid is said to be *shear thinning*, if its viscosity is a decreasing function of $II_{\mathbf{D}}$; when the opposite is true, the fluid is said to be *shear thickening*. *Bingham plastic fluids* are generalized Newtonian fluids that exhibit *yield stress*. The material flows only when the applied shear stress exceeds the finite yield stress. A *Herschel-Bulkley fluid* is a generalization of the Bingham fluid, where, upon deformation, the viscosity is either shear thinning or shear thickening. The dependence of the shear stress on $II_{\mathbf{D}}$ is illustrated in Fig. 2.1, for various non-Newtonian fluids.

Fluids that have both viscous and *elastic properties* are called *viscoelastic fluids*. Many fluids of industrial importance, such as polymeric liquids, solutions, melts or suspensions fall into this category. Fluids exhibiting elastic properties are often referred to as *memory fluids*.

The field of Fluid Mechanics that studies the relation between stress and deformation, called the *constitutive equation*, is called *Rheology* from the Greek words "rheo" (to flow) and "logos" (science or logic), and is the subject of many textbooks [2,3].

The surface tension

Surface tension, σ, is a thermodynamic property which measures the anisotropy of the interactions between molecules on the interface of two immiscible fluids A and B. At equilibrium, the *capillary pressure* (i.e., the effective pressure due to surface tension) on a curved interface is balanced by the difference between the pressures in the fluids across the interface. The jump in the fluid pressure is given by the celebrated *Young-Laplace equation* of capillarity [4],

$$\Delta p \;=\; p_B - p_A \;=\; \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\,, \tag{2.15}$$

where R_1 and R_2 are the principal radii of *curvature*, i.e., the radii of the two mutually perpendicular maximum circles which are tangent to the (two-dimensional) surface at the point of contact. In Chapter 4, these important principles are expanded to include liquids in relative motion.

Example 2.1.2. Capillary pressure

A spherical liquid droplet is in static equilibrium in stationary air at low pressure p_G. How does the pressure p inside the droplet change for droplets of different radii R, for infinite, finite and zero surface tension?

Solution:
In the case of spherical droplets, $R_1{=}R_2{=}R$, and the Young-Laplace equation is reduced to

$$p - p_G \ = \ \frac{2\sigma}{R} \, .$$

The above formula says that the pressure within the droplet is higher than the pressure of the air. The liquid pressure increases with the surface tension and decreases with the size of the droplet. As the surface tension increases, the pressure difference can be supported by bigger liquid droplets. As the pressure difference increases, smaller droplets are formed under constant surface tension. □

Measurement of fluid properties

The density, the viscosity and the surface tension of pure, incompressible, Newtonian liquids are functions of temperature and, to a much lesser extent, functions of pressure. These properties, blended with processing conditions, define a set of *dimensionless numbers* which fully characterize the behavior of the fluid under flow and processing. Three of the most important dimensionless numbers of fluid mechanics are briefly discussed below.

The *Reynolds number* expresses the relative magnitude of inertia forces to viscous forces, and is defined by

$$Re \ \equiv \ \frac{L\bar{u}\rho}{\eta} \, , \tag{2.16}$$

where L is a characteristic length of the flow geometry (i.e., the diameter of a tube), and $\bar{u}$ is a characteristic velocity of the flow (e.g., the mean velocity of the fluid).

The *Stokes number* represents the relative magnitude of gravity forces to viscous forces, and is defined by

$$St \ \equiv \ \frac{\rho g L^2}{\eta \bar{u}} \, . \tag{2.17}$$

The *capillary number* expresses the relative magnitude of viscous forces to surface tension forces, and is defined by

$$Ca \ \equiv \ \frac{\eta \bar{u}}{\sigma} \, . \tag{2.18}$$

The first two dimensionless numbers, Re and St, arise naturally in the dimensionless conservation of momentum equation; the third, Ca, appears in the dimensionless stress condition on a free surface. The procedure of nondimensionalizing these equations is described in Chapter 7, along with the *asymptotic analysis* which is used to construct approximate solutions for limiting values of the dimensionless numbers [5]. The governing equations of motion under these limiting conditions are simplified

		p=0.1 atm		p=1 atm			p=10 atm	
Property	Fluid	4°C	20°C	4°C	20°C	40°C	20°C	40°C
Density (Kg/m^3)	Air	0.129	0.120	1.29	1.20	1.13	12	11.3
	Water	1000	998	1000	998	992	998	992
Viscosity (cP)	Air	0.0158	0.0175	0.0165	0.0181	0.0195	0.0184	0.0198
	Water	1.792	1.001	1.792	1.002	0.656	1.002	0.657
Surface tension	Air	-	-	-	-	-	-	-
with air (dyn/cm)	Water	75.6	73	75.6	73	69.6	73	69.6

Table 2.1. *Density, viscosity and surface tension of air and water at several process conditions.*

by eliminating terms that are multiplied or divided by the limiting dimensionless numbers, accordingly.

Flows of highly viscous liquids are characterized by a vanishingly small Reynolds number and are called *Stokes* or *creeping flows*. Most flows of polymers are creeping flows [6]. The Reynolds number also serves to distinguish between *laminar* and *turbulent* flow. Laminar flows are characterized by the parallel sliding motion of adjacent fluid layers without intermixing, and persist for Reynolds numbers below a critical value that depends on the flow. For example, for flow in a pipe, this critical value is 2,100. Beyond that value, eddies start to develop within the fluid layers that cause intermixing and chaotic, oscillatory fluid motion, which characterizes turbulent flow. Laminar flows at Reynolds numbers sufficiently high that viscous effects are negligible are called *potential* or *Euler* flows. The Stokes number is zero in strictly horizontal flows and high in vertical flows of heavy liquids. The capillary number appears in flows with free surfaces and interfaces [7]. The surface tension, and thus the capillary number, can be altered by the addition of surfactants to the flowing liquids.

The knowledge of the dimensionless numbers and the prediction of the flow behavior demand an a priori measurement of density, viscosity and surface tension of the liquid under consideration. Density is measured by means of *pycnometers*, the function of which is primarily based on the Archimedes principle of buoyancy. Viscosity is measured by means of *viscometers* or *rheometers* in small-scale flows; the torque necessary to drive the flow and the resulting deformation are related according to Newton's law of viscosity. Surface tension is measured by *tensiometers*. These are sensitive devices that record the force which is necessary to overcome the surface tension force, in order to form droplets and bubbles or to break thin films. More sophisticated methods, usually based on optical techniques, are employed when accuracy is vital [8]. The principles of operation of pycnometers, viscometers and tensiometers are highlighted in several chapters starting with Chapter 4. Densities, viscosities and surface tension of air and water at several process conditions are tabulated in Table 2.1.

2.2 Macroscopic and Microscopic Balances

The *control volume* is an arbitrary *synthetic cut* in space which can be either fixed or moving. It is appropriately chosen within or around the system under consideration, in order to apply the laws that describe its behavior. In flow systems, these laws are the equations of conservation (or change) of mass, momentum, and energy. To obtain information on *average* or *boundary quantities* (e.g., of the velocity and the temperature fields *inside* the flow system), without a detailed analysis of the flow, the control volume is usually taken to contain or to coincide with the real flow system. The application of the principles of conservation to this finite system produces the *macroscopic conservation equations.*

However, in order to derive the equations that yield detailed distributions of fields of interest, the control volume must be of infinitesimal dimensions that can shrink to zero, yielding a point-volume. This approach reduces the quantities to point-variables. The application of the conservation principles to this infinitesimal system produces the *microscopic* or *differential conservation equations.* In this case, there is generally no contact between the imaginary boundaries of the control volume and the real boundaries of the system. It is always convenient to choose the shape of the infinitesimal control volume to be similar to that of the geometry of the actual system; a cube for a rectangular geometry, an annulus for a cylindrical geometry and a spherical shell for a spherical geometry.

Conservation of mass

Consider an arbitrary, fixed control volume V, bounded by a surface S, as shown in Fig. 2.2. According to the law of conservation of mass, the rate of increase of the mass of the fluid within the control volume V is equal to the net influx of fluid across the surface S:

$$\left[\begin{array}{c} \textit{Rate of change} \\ \textit{of mass within V} \end{array}\right] = \left[\begin{array}{c} \textit{Rate of addition} \\ \textit{of mass across S} \end{array}\right] . \tag{2.19}$$

The mass m of the fluid contained in V is given by

$$m = \int_V \rho\, dV , \tag{2.20}$$

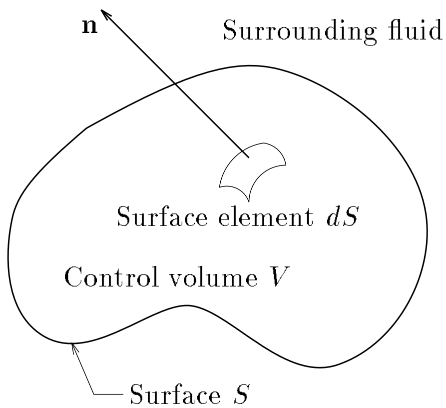

Figure 2.2. *Control volume in a flow field.*

and, hence, the rate of change in mass is

$$\frac{dm}{dt} = \frac{d}{dt} \int_V \rho \, dV \, . \tag{2.21}$$

Since the control volume V is fixed, the time derivative can be brought inside the integral:

$$\frac{dm}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV \, . \tag{2.22}$$

As for the mass rate across S, this is given by

$$-\int_S \mathbf{n} \cdot (\rho \mathbf{u}) \, dS \, ,$$

where $\mathbf{n}$ is the outwardly directed unit vector normal to the surface S, and $\rho\mathbf{u}$ is the *mass flux* (i.e., mass per unit area per unit time). The minus sign accounts for the fact that the mass of the fluid contained in the control volume decreases, when the flow is outward, i.e., when $\mathbf{n} \cdot (\rho\mathbf{u})$ is positive. By substituting the last expression and Eq. (2.22) in Eq. (2.19), we obtain the following form of the equation of mass conservation for a fixed control volume:

$$\frac{dm}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_S \mathbf{n} \cdot (\rho \mathbf{u}) \, dS \, . \tag{2.23}$$

Example 2.2.1. Macroscopic balances

A reactant in water flows down the wall of a cylindrical tank in the form of thin

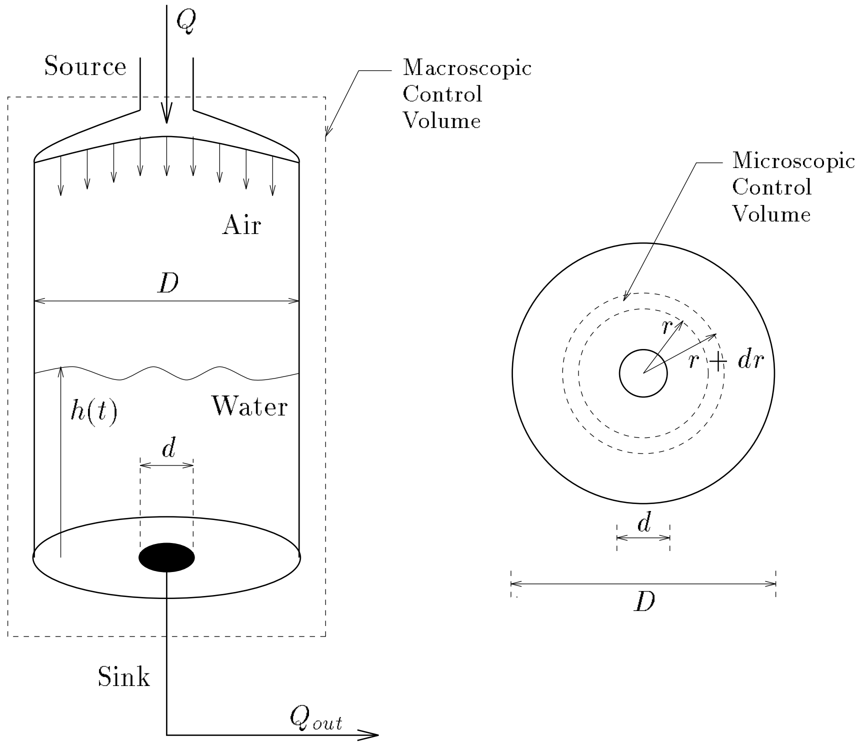

Figure 2.3. *Macroscopic and microscopic balances on a source-sink system.*

film at flow rate Q. The sink at the center of the bottom, of diameter d, discharges water at average velocity $\bar{u} = 2kh$, where k is a constant. Initially, the sink and the source are closed and the level of the water is h_0. What will be the level $h(t)$ after time t?

Solution:

We consider a control volume containing the flow system, as illustrated in Fig. 2.3. The rate of change in mass within the control volume is

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho\, dV = \frac{d}{dt}(\rho V) = \rho\frac{d}{dt}\left(\frac{\pi D^2}{4}h\right) = \rho\frac{\pi D^2}{4}\frac{dh}{dt}\,.$$

We assume that water is incompressible. The net influx of mass across the surface S of the control volume is

$$-\int_S \mathbf{n}\cdot(\rho\mathbf{u})\,dS = \rho\,(Q - Q_{out}) = \rho\left(Q - \frac{\pi d^2}{4}\bar{u}\right) = \rho\left(Q - \frac{\pi d^2}{2}kh\right)\,,$$

where Q and Q_{out} are the *volumetric flow rates* at the inlet and the outlet, respectively, of the flow system (see Fig. 2.3). Therefore, the conservation of mass within the control volume gives:

$$\frac{\pi D^2}{4}\frac{dh}{dt} \;=\; Q - \frac{\pi d^2}{2}\, kh\,. \tag{2.24}$$

The solution to this equation, subjected to the initial condition

$$h(t=0) \;=\; h_0\,,$$

is

$$h(t) \;=\; \frac{2Q}{\pi d^2 k} - \left(\frac{2Q}{\pi d^2 k} - h_0\right) e^{-\left(\frac{2kd^2}{D^2}\right)t}\,. \tag{2.25}$$

The steady-state elevation is

$$h_{ss} \;=\; \lim_{t\to\infty} h(t) \;=\; \frac{2Q}{\pi d^2 k}\,. \tag{2.26}$$

Since, Eq. (2.24) is a macroscopic equation, its solution, given by Eq. (2.25), provides no information on the velocity from the wall to the sink, nor on the pressure distribution within the liquid. These questions are addressed in Example 2.2.2. □

Example 2.2.2. Microscopic balances

Assume now that the system of Example 2.2.1 is a kind of chemical reactor. Find an estimate of the *residence time* of a reactant particle (moving with the liquid) from the wall to the sink.

Solution:

The reactant flows down the vertical wall and enters the radial reacting flow at $r{=}D/2$ directed towards the cylindrical sink at $r{=}d/2$ (r is the distance from the center of the sink). If $u(r)$ is the pointwise radial velocity of the fluid, then

$$u(r) \;=\; \frac{dr}{dt}\,,$$

and, therefore, the residence time of the fluid in the reaction field is given by

$$t \;=\; \int_{D/2}^{d/2} \frac{dr}{u(r)}\,. \tag{2.27}$$

Obviously, we need to calculate $u(r)$ as a function of r. The average velocity $\bar{u}$ found in Example 2.2.1 is of no use here. The velocity $u(r)$ can be found only by performing a microscopic balance. A convenient microscopic control volume is an annulus of radii r and $r + dr$, and of height dz, shown in Fig. 2.3. For this control volume, the conservation of mass states that

$$\frac{d}{dt}(\rho 2\pi r\, dr\, dz) \;=\; [2\pi r\rho u(r)dz]_{r+dr} - [2\pi r\rho u(r)dz]_r \,. \tag{2.28}$$

Assume, for the sake of simplicity, that the reactor operates at steady state, which means that $d/dt{=}0$ and $h{=}h_{ss}$. From Eq. (2.28), we get:

$$[ru(r)]_{r+dr} \;-\; [ru(r)]_r \;=\; 0\,.$$

Dividing the above equation by dr, making the volume to shrink to zero by taking the limit as $dr \to 0$, and invoking the definition of the total derivative, we get a simple, ordinary differential equation:

$$\frac{d}{dr}\,[ru(r)] \;=\; \lim_{dr\to 0}\frac{[ru(r)]_{r+dr} \;-\; [ru(r)]_r}{dr} \;=\; 0\,, \tag{2.29}$$

The solution of the above equation is

$$u(r) \;=\; \frac{c}{r}\,, \tag{2.30}$$

where c is a constant to be determined. The boundary condition at steady state demands that

$$Q \;=\; -2\pi\,\frac{d}{2}h_{ss}u_r\Big|_{r=d/2} \;=\; -\pi d h_{ss}\,\frac{2c}{d} \qquad\Longrightarrow\qquad c \;=\; -\frac{Q}{2\pi h_{ss}}\,.$$

The *velocity profile* is, therefore, given by

$$u(r) \;=\; -\frac{Q}{2\pi h_{ss}}\,\frac{1}{r}\,. \tag{2.31}$$

We can now substitute Eq. (2.31) in Eq. (2.27) and calculate the residence time:

$$t \;=\; -\int_{D/2}^{d/2}\frac{2\pi h_{ss}}{Q}r\,dr \;=\; \frac{\pi h_{ss}}{4Q}\left(D^2 - d^2\right)\,. \tag{2.32}$$

The pressure distribution can be calculated using *Bernoulli's equation*, developed in Chapter 5. Along the radial streamline,

$$\frac{p(r)}{\rho} + \frac{u^2(r)}{2} \;=\; \left[\frac{p(r)}{\rho} + \frac{u^2(r)}{2}\right]_{r=\frac{D}{2}}\,. \tag{2.33}$$

For $d/D \ll 1$, it is reasonable to assume that at $r{=}D/2$, $u \approx 0$ and $p \approx 0$, and, therefore,

$$p(r) \;=\; -\frac{\rho}{2}u^2(r) \;=\; -\frac{\rho Q^2}{8\pi^2 h_{ss}^2}\,\frac{1}{r^2} < 0\,. \tag{2.34}$$

Equation (2.34) predicts an increasingly negative pressure towards the sink. Under these conditions, *cavitation* and even *boiling* may occur, when the pressure $p(r)$ is identical to the vapor pressure of the liquid. These phenomena, which are important in a diversity of engineering applications, cannot be predicted by macroscopic balances. □

Conservation of linear momentum

An *isolated* solid body of mass m moving with velocity $\mathbf{u}$ possesses momentum, $\mathbf{J} \equiv m\mathbf{u}$. According to *Newton's law of motion*, the rate of change of momentum of the solid body is equal to the force $\mathbf{F}$ exerted on the mass m:

$$\frac{d\mathbf{J}}{dt} \;=\; \mathbf{F}\,, \qquad \Longrightarrow \qquad \frac{d}{dt}(m\mathbf{u}) \;=\; \mathbf{F}\,. \tag{2.35}$$

The force $\mathbf{F}$ in Eq. (2.35) is a *body force*, i.e. an external force exerted on the mass m. The most common body force is the gravity force,

$$\mathbf{F}_G \;=\; m\,\mathbf{g}\,, \tag{2.36}$$

which is directed to the center of the Earth ($\mathbf{g}$ is the acceleration of gravity). *Electromagnetic* forces are another kind of body force. Equation (2.35) describes the conservation of *linear* momentum of an isolated body or system:

$$\left[\begin{array}{c} \textit{Rate of change} \\ \textit{of momentum} \\ \textit{of an isolated system} \end{array}\right] = \left[\begin{array}{c} \textit{Body} \\ \textit{force} \end{array}\right]. \tag{2.37}$$

In the case of a non-isolated flow system, i.e., a control volume V, momentum is convected across the bounding surface S due to (a) the flow of the fluid across S, and (b) the molecular motions and interactions at the boundary S. The law of conservation of momentum is then stated as follows:

$$\left[\begin{array}{c} \textit{Rate of} \\ \textit{increase of} \\ \textit{momentum} \\ \textit{within } V \end{array}\right] = \left[\begin{array}{c} \textit{Rate of} \\ \textit{inflow of} \\ \textit{momentum} \\ \textit{across } S \\ \textit{by bulk} \\ \textit{flow} \end{array}\right] + \left[\begin{array}{c} \textit{Rate of} \\ \textit{inflow of} \\ \textit{momentum} \\ \textit{across } S \\ \textit{by molecular} \\ \textit{processes} \end{array}\right] + \left[\begin{array}{c} \textit{Body} \\ \textit{force} \end{array}\right]. \tag{2.38}$$

The momentum $\mathbf{J}$ of the fluid contained within a control volume V is given by

$$\mathbf{J} = \int_V \rho \mathbf{u} \, dV , \tag{2.39}$$

and, therefore,

$$\frac{d\mathbf{J}}{dt} = \frac{d}{dt} \int_V \rho \mathbf{u} \, dV . \tag{2.40}$$

The rate of addition of momentum due to the flow across S is

$$- \int_S \mathbf{n} \cdot (\rho \mathbf{u}) \mathbf{u} \, dS = - \int_S \mathbf{n} \cdot (\rho \mathbf{u}\mathbf{u}) \, dS ,$$

where $\mathbf{n}$ is the unit normal pointing outwards from the surface S. The minus sign in the above expression accounts for the fact that the content of the control volume increases when the velocity vector $\mathbf{u}$ points inwards to the control volume. The dyadic tensor $\rho\mathbf{u}\mathbf{u}$ is the *momentum flux* (i.e., momentum per unit area per unit time). The momentum flux is obviously a symmetric tensor. Its component $\rho u_i u_j \mathbf{ij}$ represents the j component of the momentum convected in the i direction, per unit area per unit time.

The additional momentum flux due to molecular motions and interactions between the fluid and its surroundings is another symmetric tensor, the total stress tensor $\mathbf{T}$, defined in Eq. (2.10). Therefore, the rate of addition of momentum across S, due to molecular processes, is

$$\int_S \mathbf{n} \cdot \mathbf{T} \, dS = \int_S \mathbf{n} \cdot (-p\mathbf{I} + \boldsymbol{\tau}) \, dS . \tag{2.41}$$

As already mentioned, the anisotropic viscous stress tensor $\boldsymbol{\tau}$ accounts for the relative motion of fluid particles. In static equilibrium, the only non-zero stress contribution to the momentum flux comes from the hydrostatic pressure p. The vector $\mathbf{n} \cdot \mathbf{T}$ is the traction produced by $\mathbf{T}$ on a surface element of orientation $\mathbf{n}$. The term (2.41) is often interpreted physically as the resultant of the *surface* (or *contact*) *forces* exerted by the surrounding fluid on the fluid inside the control volume V. It is exactly the hydrodynamic force acting on the boundary S, as required by the principle of action-reaction (Newton's third law).

Assuming that the only body force acting on the fluid within the control volume V is due to gravity, i.e.,

$$\int_V \rho \mathbf{g} \, dV ,$$

and substituting the above expressions into Eq. (2.38), we obtain the following form of the law of conservation of momentum:

$$\int_V \rho \mathbf{u} \, dV = - \int_S \mathbf{n} \cdot (\rho \mathbf{u}\mathbf{u}) \, dS + \int_S \mathbf{n} \cdot (-p\mathbf{I} + \boldsymbol{\tau}) \, dS + \int_V \rho \mathbf{g} \, dV . \tag{2.42}$$

The surface integrals of Eqs. (2.23) and (2.42) can be converted to volume integrals by means of the Gauss divergence theorem. As explained in Chapter 3, this step is necessary for obtaining the differential forms of the corresponding conservation equations.

2.3 Local Fluid Kinematics

Fluids cannot support any shear stress without deforming or flowing, and continue to flow as long as shear stresses persist. The effect of the externally applied shear stress is dissipated away from the boundary due to the viscosity. This gives rise to a relative motion between different fluid particles. The relative motion forces fluid material lines that join two different fluid particles to stretch (or compress) and to rotate as the two fluid particles move with different velocities. In general, the induced deformation gives rise to normal and shear stresses, similar to internal stresses developed in a stretched or twisted rubber cylinder. The difference between the two cases is that, when the externally applied forces are removed, the rubber cylinder returns to its original undeformed and unstressed state, whereas the fluid remains in its deformed state. In the field of *rheology*, it is said that rubber exhibits perfect *memory* of its rest or undeformed state, whereas viscous inelastic liquids, which include the Newtonian liquids, exhibit no memory at all. Viscoelastic materials exhibit *fading memory* and their behavior is between that of ideal elastic rubber and that of viscous inelastic liquids. These distinct behaviors are determined by the *constitutive equation*, which relates deformation to stress.

Since the conservation equations and the constitutive equation are expressed in terms of relative *kinematics*, i.e., velocities, gradients of velocities, strains and rates of strain, it is important to choose the most convenient way to quantify these variables. The interconnection between these variables requires the investigation and representation of the relative motion of a fluid particle with respect to its neighbors.

Flow kinematics, i.e., the relative motion of fluid particles, can be described by using either a *Lagrangian* or an *Eulerian* description. In the Lagrangian or *material description*, the motion of individual particles is tracked; the position $\mathbf{r}^*$ of a *marked* fluid particle is considered to be a function of time and of its label, such as its initial position $\mathbf{r}_0^*$, $\mathbf{r}^*=\mathbf{r}^*(\mathbf{r}_0^*, t)$. For a fixed $\mathbf{r}_0^*$, we have

$$\mathbf{r}^* = \mathbf{r}^*(t) , \tag{2.43}$$

which is a parametric equation describing the locus of the marked particle, called a *path line*. The independent variables in Lagrangian formulations are the position of a marked fluid particle and time, t. This is analogous to an observer riding a fluid particle and marking his/her position while he/she records the traveling time and other quantities of interest. For example, the pressure p in Lagrangian variables is given by $p{=}p(\mathbf{r}_0^*, t)$.

In the Eulerian description, dependent variables, such as the velocity vector and pressure, are considered to be functions of *fixed* spatial coordinates and of time, e.g., $\mathbf{u}{=}\mathbf{u}(\mathbf{r}, t)$, $p{=}p(\mathbf{r}, t)$, etc. If all dependent variables are independent of time, the flow is said to be *steady*.

Since both Lagrangian and Eulerian variables describe the same flow, there must be a relation between the two. This relation is expressed by the *substantial derivative* which in the Lagrangian description is identical to the common total derivative. The Lagrangian acceleration, $\mathbf{a}^*$, is related to the Eulerian acceleration, $\mathbf{a}{=}\partial\mathbf{u}/\partial t$, as follows:

$$\mathbf{a}^* \;=\; \frac{D\mathbf{u}}{Dt} \;=\; \frac{\partial \mathbf{u}}{\partial t} \;+\; \mathbf{u}\cdot\nabla\mathbf{u}\,. \tag{2.44}$$

Note that the velocity $\mathbf{u}$ in the above equation is the Eulerian one. In steady flows, the Eulerian acceleration, $\mathbf{a}{=}\partial\mathbf{u}/\partial t$, is zero, whereas the Lagrangian one, $\mathbf{a}^*$, may not be so, if finite spatial velocity gradients exist.

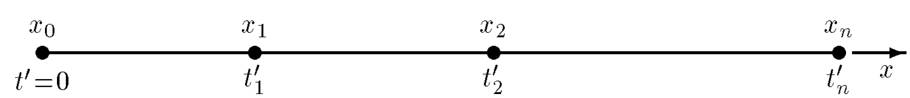

Figure 2.4. *Positions of a fluid particle in one-dimensional motion.*

We will illustrate the two flow descriptions using an idealized one-dimensional example. Consider steady motion of fluid particles along the x-axis, such that

$$x_i^* \;=\; x_{i-1}^* + c\,(t_i - t_{i-1})^2\,, \tag{2.45}$$

where x_i^* is the position of a fluid particle at time t_i (Fig. 2.4), and c is a positive constant. The Lagrangian description of motion gives the position of the particle in terms of its initial position, x_0^*, and the lapsed *traveling time*, t',

$$x^*(x_0^*, t') \;=\; x_0^* + c\,t'^2\,. \tag{2.46}$$

The velocity of the particle is

$$u^*(x_0^*, t') \;=\; \frac{dx^*}{dt'} = 2c\,t'\,, \tag{2.47}$$

which, in this case, is independent of x_0^*. The corresponding acceleration is

$$a^*(x_0^*, t') \;=\; \frac{du^*}{dt'} \;=\; 2c > 0\,. \tag{2.48}$$

The *separation distance* between two particles 1 and 2 (see Fig. 2.4),

$$\Delta x^* \;=\; x_2^* - x_1^* \;=\; c\,(t_2'^2 - t_1'^2)\,, \tag{2.49}$$

changes with time according to

$$\frac{d\Delta x^*}{dt'} \;=\; 2c\,(t_2' - t_1') \;=\; u_2^* - u_1^* > 0\,, \tag{2.50}$$

and is, therefore, continuously stretched, given that $u_2^* > u_1^*$. The velocity gradient is,

$$\frac{du^*}{dx^*} \;=\; \frac{1}{u^*(t')}\frac{du^*}{dt'} \;=\; \frac{2c}{2ct'} \;=\; \frac{1}{t'}\,. \tag{2.51}$$

In the above expressions, the traveling time t' is related to the traveling distance by the simple kinematic argument,

$$dx^* \;=\; u(t')dt'\,, \tag{2.52}$$

and is different from the time t which characterizes an unsteady flow, under the Eulerian description.

In the Eulerian description, the primary variable is

$$u(x) \;=\; 2c^{1/2}(x - x_0)^{1/2}\,. \tag{2.53}$$

Note that time, t, does not appear due to the fact that the motion is steady. Equation (2.44) is easily verified in this steady, one-dimensional flow:

$$\frac{\partial u}{\partial t} \;+\; u\,\frac{\partial u}{\partial x} \;=\; 0 \;+\; 2c^{1/2}\,(x - x_0)^{1/2}\,\frac{1}{2}\,2c^{1/2}\,(x - x_0)^{-1/2} \;=\; 2c \;=\; a^*\,.$$

The Eulerian description may not be convenient to describe path lines but it is more appropriate than the Lagrangian description in calculating *streamlines*. These are lines to which the velocity vector is tangent at any instant. Hence, streamlines can be calculated by

$$\mathbf{u}\times d\mathbf{r} \;=\; \mathbf{0}\,, \tag{2.54}$$

where $\mathbf{r}$ is the position vector describing the streamline. In Cartesian coordinates, Eq. (2.54) is reduced to

$$\frac{dx}{u_x} \;=\; \frac{dy}{u_y} \;=\; \frac{dz}{u_z}\,. \tag{2.55}$$

When the flow is steady, a path line coincides with the streamline that passes through $\mathbf{r}_0^*$. The surface formed instantaneously by all the streamlines that pass through a given closed curve in the fluid is called *streamtube*.

From Eq. (2.55), the equation of a streamline in the xy-plane is given by

$$\frac{dx}{u_x} = \frac{dy}{u_y} \quad \Longrightarrow \quad u_y\, dx - u_x\, dy = 0\,. \tag{2.56}$$

A useful concept related to streamlines, in two-dimensional bidirectional flows, is the *stream function*. In the case of incompressible flow,[1] the stream function, $\psi(x, y)$, is defined by[2]

$$u_x = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y = \frac{\partial \psi}{\partial x}\,. \tag{2.57}$$

An important feature of the stream function is that it automatically satisfies the continuity equation,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0\,, \tag{2.58}$$

as can easily be verified. The stream function is a useful tool in solving creeping, two-dimensional bidirectional flows. Its definitions and use, for various classes of incompressible flow, are examined in detail in Chapter 10.

Substituting Eqs. (2.57) into Eq. (2.56), we get

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0\,. \tag{2.59}$$

Therefore, the stream function, ψ, is constant along a streamline. Moreover, from the definition of a streamline, we realize that there is no flow across a streamline. The volume flow rate, Q, per unit distance in the z direction, across a curve connecting two streamlines (see Fig. 2.5) is the integral of $d\psi$ along the curve. Since the

[1] For steady, compressible flow in the xy-plane, the stream function is defined by

$$\rho\, u_x = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad \rho\, u_y = \frac{\partial \psi}{\partial x}\,.$$

In this case, the difference $\psi_2 - \psi_1$ is the mass flow rate (per unit depth) between the two streamlines.

[2] Note that many authors define the stream function with the opposite sign, i.e.,

$$u_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y = -\frac{\partial \psi}{\partial x}\,.$$

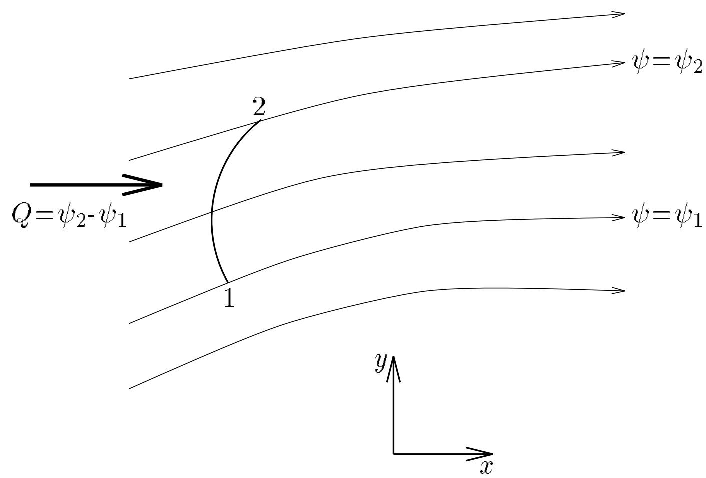

Figure 2.5. *Volume flow rate per unit depth across a curve connecting two streamlines.*

differential of ψ is exact, this integral depends only on the end points of integration, i.e.,

$$Q \;=\; \int_1^2 d\psi \;=\; \psi_2 \;-\; \psi_1 \,. \tag{2.60}$$

Example 2.3.1. Stagnation flow

Consider the steady, two-dimensional *stagnation flow* against a solid wall, shown in Fig. 2.6. Outside a thin boundary layer near the wall, the position of a particle, located initially at $\mathbf{r}_0^*(x_0^*, y_0^*)$, obeys the following relations:

$$x^*(x_0^*, t') \;=\; x_0^*\, e^{\varepsilon t'} \quad \text{and} \quad y^*(y_0^*, t') \;=\; y_0^*\, e^{-\varepsilon t'} \,, \tag{2.61}$$

which is, of course, the Lagrangian description of the flow. The corresponding velocity components are

$$u_x^*(x_0^*, t') \;=\; \frac{dx^*}{dt'} \;=\; \varepsilon x_0^*\, e^{\varepsilon t'} \quad \text{and} \quad u_y^*(y_0^*, t') \;=\; \frac{dy^*}{dt'} \;=\; -\varepsilon y_0^*\, e^{-\varepsilon t'} \,. \tag{2.62}$$

Eliminating the traveling time t' from the above equations results in the equation of the path line,

$$x^* y^* \;=\; x_0^* y_0^* \,, \tag{2.63}$$

which is a hyperbola, in agreement with the physics of the flow.

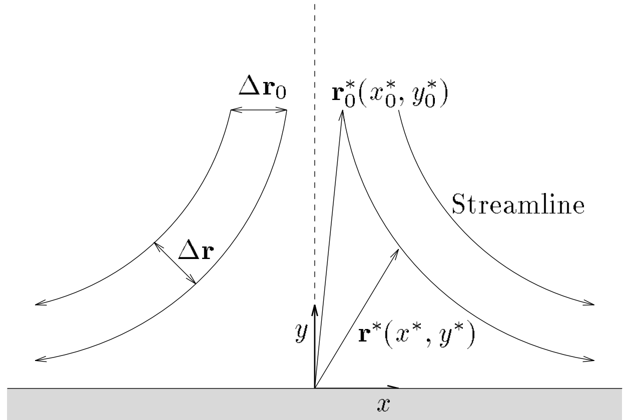

Figure 2.6. *Stagnation flow.*

In the Eulerian description, the velocity components are:

$$u_x = \varepsilon x \quad \text{and} \quad u_y = -\varepsilon y\,. \tag{2.64}$$

The streamlines of the flow are calculated by means of Eq. (2.55):

$$\frac{dx}{u_x} = \frac{dy}{u_y} \implies \frac{dx}{\varepsilon x} = \frac{dy}{-\varepsilon y} \implies xy = x_0 y_0\,. \tag{2.65}$$

Equations (2.65) and (2.63) are identical: since the flow is steady, streamlines and path lines coincide. □

The Lagrangian description is considered a more natural choice to represent the actual kinematics and stresses experienced by fluid particles. However, the use of this description in solving complex flow problems is limited, due to the fact that it requires tracking of fluid particles along a priori unknown streamlines. The approach is particularly convenient in flows of viscoelastic liquids, i.e., of fluids with memory, that require particle tracking and calculation of deformation and stresses along streamlines. The Eulerian formulation is, in general, more convenient to use because it deals only with local or present kinematics. In most cases, all variables of interest, such as strain (deformation), rate of strain, stress, vorticity, streamlines and others, can be calculated from the velocity field. An additional advantage of the Eulerian description is that it involves time, as a variable, only in unsteady flows, whereas the Lagrangian description uses traveling time even in steady-state flows. Finally, quantities following the motion of the liquid can be reproduced easily from the Eulerian variables by means of the substantial derivative.

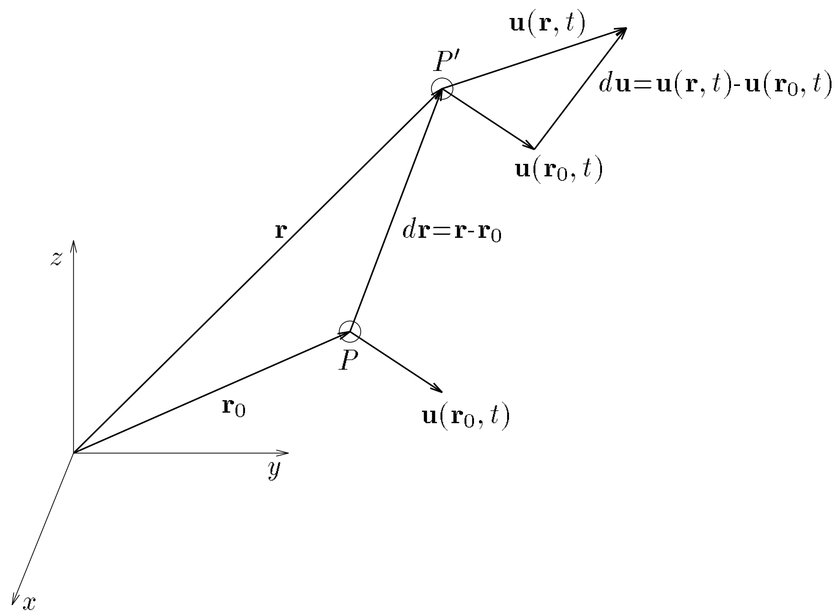

Figure 2.7. *Relative motion of adjacent fluid particles.*

2.4 Elementary Fluid Motions

The relative motion of fluid particles gives rise to velocity gradients that are directly responsible for strain (deformation). Strain, in turn, creates internal shear and extensional stresses that are quantified by the constitutive equation. Therefore, it is important to study how relative motion between fluid particles arises and how this relates to strain and stress.

Consider the adjacent fluid particles P and P' of Fig. 2.7, located at points $\mathbf{r}_0$ and $\mathbf{r}$, respectively, and assume that the distance $d\mathbf{r}$=$\mathbf{r}$-$\mathbf{r}_0$ is vanishingly small. The velocity $\mathbf{u}(\mathbf{r},t)$ of the particle P' can be *locally* decomposed into four elementary motions:

(a) *rigid-body translation*;
(b) *rigid-body rotation*;
(c) *isotropic expansion*; and
(d) *pure straining motion without change of volume.*

Actually, this decomposition is possible for any vector $\mathbf{u}$ in the three-dimensional space.

Expanding $\mathbf{u}(\mathbf{r},t)$ in a Taylor series with respect to $\mathbf{r}$ about $\mathbf{r}_0$, we get

$$\mathbf{u}(\mathbf{r},t) \;=\; \mathbf{u}(\mathbf{r}_0,t) \;+\; d\mathbf{r}\cdot\nabla\mathbf{u} \;+\; O[(d\mathbf{r})^2]\,, \tag{2.66}$$

where $\nabla\mathbf{u}$ is the velocity gradient tensor. Retaining only the linear term, we have

$$\mathbf{u}(\mathbf{r},t) \;=\; \mathbf{u}(\mathbf{r}_0,t) \;+\; d\mathbf{u}\,, \tag{2.67}$$

where the velocity $\mathbf{u}(\mathbf{r}_0,t)$ of P represents, of course, *rigid-body translation*, and

$$d\mathbf{u} \;=\; d\mathbf{r}\cdot\nabla\mathbf{u} \tag{2.68}$$

represents the *relative velocity* of particle P' with respect to P. The rigid-body translation component, $\mathbf{u}(\mathbf{r}_0,t)$, does not give rise to any strain or stress, and can be omitted by placing the frame origin or the observer on a moving particle. All the information for the relative velocity $d\mathbf{u}$ is contained in the velocity gradient tensor. The relative velocity can be further decomposed into two components corresponding to rigid-body rotation and pure straining motion, respectively. Recall that $\nabla\mathbf{u}$ can be written as the sum of a symmetric and an antisymmetric tensor,

$$\nabla\mathbf{u} \;=\; \mathbf{D} \;+\; \mathbf{S}\,, \tag{2.69}$$

where

$$\mathbf{D} \;\equiv\; \frac{1}{2}\,[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \tag{2.70}$$

is the symmetric rate-of-strain tensor, and

$$\mathbf{S} \;\equiv\; \frac{1}{2}\,[\nabla\mathbf{u} - (\nabla\mathbf{u})^T] \tag{2.71}$$

is the antisymmetric vorticity tensor. Substituting Eqs. (2.69) to (2.71) in Eq. (2.68), we get

$$d\mathbf{u} \;=\; d\mathbf{r}\cdot(\mathbf{D}+\mathbf{S}) \;=\; d\mathbf{r}\cdot\frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \;+\; d\mathbf{r}\cdot\frac{1}{2}[\nabla\mathbf{u} - (\nabla\mathbf{u})^T]\,. \tag{2.72}$$

The first term,

$$\mathbf{u}^{(s)} \;=\; d\mathbf{r}\cdot\mathbf{D} \;=\; d\mathbf{r}\cdot\frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \tag{2.73}$$

represents the *pure straining motion* of P' about P. The second term

$$\mathbf{u}^{(r)} \;=\; d\mathbf{r}\cdot\mathbf{S} \;=\; d\mathbf{r}\cdot\frac{1}{2}[\nabla\mathbf{u} - (\nabla\mathbf{u})^T] \tag{2.74}$$

represents the *rigid-body rotation* of P' about P. A flow in which $\mathbf{D}$ is zero everywhere corresponds to rigid-body motion (including translation and rotation). Rigid-body motion does not alter the shape of fluid particles, resulting only in their displacement. On the other hand, straining motion results in deformation of fluid particles.

Note that the matrix forms of $\nabla\mathbf{u}$, $\mathbf{D}$ and $\mathbf{S}$ in Cartesian coordinates are given by

$$\nabla\mathbf{u} \;=\; \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{bmatrix}, \tag{2.75}$$

$$\mathbf{D} \;=\; \frac{1}{2}\begin{bmatrix} 2\,\frac{\partial u_x}{\partial x} & \left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) & \left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right) \\ \left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) & 2\,\frac{\partial u_y}{\partial y} & \left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right) \\ \left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right) & \left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right) & 2\,\frac{\partial u_z}{\partial z} \end{bmatrix}, \tag{2.76}$$

and

$$\mathbf{S} \;=\; \frac{1}{2}\begin{bmatrix} 0 & -\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right) & \left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right) \\ \left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right) & 0 & -\left(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}\right) \\ -\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right) & \left(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}\right) & 0 \end{bmatrix}. \tag{2.77}$$

Any antisymmetric tensor has only three independent components and may, therefore, be associated with a vector, referred to as the *dual vector* of the antisymmetric tensor. The dual vector of the vorticity tensor $\mathbf{S}$ is the vorticity vector,

$$\boldsymbol{\omega} \;\equiv\; \nabla\times\mathbf{u}\,. \tag{2.78}$$

In Cartesian coordinates, it is easy to verify that, if

$$\boldsymbol{\omega} \;=\; \omega_x\mathbf{i} \,+\, \omega_y\mathbf{j} \,+\, \omega_z\mathbf{k}\,, \tag{2.79}$$

then

$$\mathbf{S} \;=\; \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{2.80}$$

and

$$d\mathbf{r}\cdot\mathbf{S} \;=\; \frac{1}{2}\,\boldsymbol{\omega}\times d\mathbf{r}\,. \tag{2.81}$$

The vorticity tensor in Eq. (2.74) can be replaced by its dual vorticity vector, according to

$$\mathbf{u}^{(r)} \;=\; d\mathbf{r}\cdot\frac{1}{2}[\nabla\mathbf{u}-(\nabla\mathbf{u})^T] \;=\; \frac{1}{2}\,(\nabla\times\mathbf{u})\times d\mathbf{r} \;=\; \frac{1}{2}\,\boldsymbol{\omega}\times d\mathbf{r}\,. \tag{2.82}$$

In *irrotational* flows, the vorticity $\boldsymbol{\omega}$ is everywhere zero, and, as a result, the rigid-body rotation component $\mathbf{u}^{(r)}$ is zero. If the vorticity is not everywhere zero, then the flow is called *rotational*. The rigid-body rotation component $\mathbf{u}^{(r)}$ also obeys the relation

$$\mathbf{u}^{(r)} \;\equiv\; \boldsymbol{\Omega}\times d\mathbf{r}\,, \tag{2.83}$$

where $\boldsymbol{\Omega}$ is the *angular velocity*. Therefore, the vorticity vector $\boldsymbol{\omega}$ is twice the angular velocity of the local rigid-body rotation. It should be emphasized that the vorticity acts as a measure of the *local* rotation of fluid particles, and it is not directly connected with the curvature of the streamlines, i.e., it is independent of any *global* rotation of the fluid.

It must be always kept in mind that the pure straining motion component $\mathbf{u}^{(s)}$ represents strain unaffected by rotation, i.e., strain experienced by an observer rotating with the local vorticity. The straining part of the velocity gradient tensor, which is the rate of strain tensor, can be broken into two parts: an extensional one representing isotropic expansion, and one representing pure straining motion without change of volume. In other words, the rate of strain tensor $\mathbf{D}$ can be written as the sum of a properly chosen diagonal tensor and a symmetric tensor of zero trace:

$$\mathbf{D} \;=\; \frac{1}{3}tr(\mathbf{D})\,\mathbf{I} \;+\; [\mathbf{D}-\frac{1}{3}tr(\mathbf{D})\,\mathbf{I}]\,. \tag{2.84}$$

The diagonal elements of the tensor $[\mathbf{D}-\frac{1}{3}tr(\mathbf{D})\mathbf{I}]$ represent normal or extensional strains on three mutually perpendicular surfaces. The off-diagonal elements represent shear strains in two directions on each of the three mutually perpendicular surfaces. Noting that

$$tr(\mathbf{D}) \;=\; \nabla\cdot\mathbf{u}\,, \tag{2.85}$$

Eq. (2.84) takes the form:

$$\mathbf{D} \;=\; \frac{1}{3}\nabla\cdot\mathbf{u}\,\mathbf{I} \;+\; \frac{1}{2}\,[\nabla\mathbf{u}+(\nabla\mathbf{u})^T-\frac{2}{3}(\nabla\cdot\mathbf{u})\,\mathbf{I}]\,. \tag{2.86}$$

Therefore, the strain velocity, $\mathbf{u}^{(s)}=d\mathbf{r}\cdot\mathbf{D}$, can be written as

$$\mathbf{u}^{(s)} \;=\; \mathbf{u}^{(e)} \;+\; \mathbf{u}^{(st)}\,, \tag{2.87}$$

where

$$\mathbf{u}^{(e)} \;=\; d\mathbf{r}\cdot\frac{1}{3}\,(\nabla\cdot\mathbf{u})\,\mathbf{I} \tag{2.88}$$

represents isotropic expansion, and

$$\mathbf{u}^{(st)} \;=\; d\mathbf{r}\cdot\frac{1}{2}\,[\nabla\mathbf{u}+(\nabla\mathbf{u})^T-\frac{2}{3}\,(\nabla\cdot\mathbf{u})\,\mathbf{I}] \tag{2.89}$$

represents pure straining motion without change of volume.

In summary, the velocity of a fluid particle in the vicinity of the point $\mathbf{r}_0$ is decomposed as

$$\mathbf{u}(\mathbf{r},t) \;=\; \mathbf{u}(\mathbf{r}_0,t) \;+\; \mathbf{u}^{(r)} \;+\; \mathbf{u}^{(e)} \;+\; \mathbf{u}^{(st)}\,, \tag{2.90}$$

or, in terms of the vorticity vector, the rate of strain tensor and the divergence of the velocity vector,

$$\mathbf{u}(\mathbf{r},t) \;=\; \mathbf{u}(\mathbf{r}_0,t)+\frac{1}{2}\boldsymbol{\omega}\times d\mathbf{r}+d\mathbf{r}\cdot\frac{1}{3}\nabla\cdot\mathbf{u}\,\mathbf{I}+d\mathbf{r}\cdot\frac{1}{2}[\nabla\mathbf{u}+(\nabla\mathbf{u})^T-\frac{2}{3}(\nabla\cdot\mathbf{u})\,\mathbf{I}]\,. \tag{2.91}$$

Alternative expressions for all the components of the velocity are given in Table 2.2.

The isotropic expansion component $\mathbf{u}^{(e)}$ accounts for any expansion or contraction due to compressibility. For incompressible fluids, $tr(\mathbf{D})=\nabla\cdot\mathbf{u}=0$, and, therefore, $\mathbf{u}^{(e)}$ is zero. In Example 1.5.3, we have shown that the *local rate of expansion* per unit volume is equal to the divergence of the velocity field,

$$\Delta \;=\; \lim_{V(t)\to 0}\frac{1}{V(t)}\,\frac{dV(t)}{dt} \;=\; \nabla\cdot\mathbf{u}\,. \tag{2.92}$$

Since $\mathbf{D}$ is a symmetric tensor, it has three real eigenvalues, λ_1, λ_2 and λ_3, and three mutually orthogonal eigenvectors. Hence, in the system of the orthonormal basis $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$ of its eigenvectors, $\mathbf{D}$ takes the diagonal form:

$$\mathbf{D}' \;=\; \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} \tag{2.93}$$

If $\mathbf{r}'=(r'_1,r'_2,r'_3)$ is the position vector in the system $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$, then

$$\frac{\Delta d\mathbf{r}'}{\Delta t} \;=\; \mathbf{u}^{(s)} \;=\; d\mathbf{r}'\cdot\mathbf{D}'\,. \tag{2.94}$$

Velocity in the vicinity of $\mathbf{r}_0$

$$\mathbf{u}(\mathbf{r},t) \;=\; \mathbf{u}(\mathbf{r}_0,t) \;+\; d\mathbf{u}$$

or

$$\mathbf{u}(\mathbf{r},t) \;=\; \mathbf{u}(\mathbf{r}_0,t) \;+\; \mathbf{u}^{(r)} \;+\; \mathbf{u}^{(s)}$$

or

$$\mathbf{u}(\mathbf{r},t) \;=\; \mathbf{u}(\mathbf{r}_0,t) \;+\; \mathbf{u}^{(r)} \;+\; \mathbf{u}^{(e)} \;+\; \mathbf{u}^{(st)}$$

Rigid − body translation

$$\mathbf{u}(\mathbf{r}_0,t)$$

Relative velocity

$$d\mathbf{u} \;=\; d\mathbf{r}\cdot\nabla\mathbf{u} \;=\; \mathbf{u}^{(r)} \;+\; \mathbf{u}^{(s)}$$

Rigid − body rotation

$$\mathbf{u}^{(r)} \;=\; d\mathbf{r}\cdot\mathbf{S} \;=\; d\mathbf{r}\cdot\tfrac{1}{2}\,[\nabla\mathbf{u}-(\nabla\mathbf{u})^T] \;=\; \tfrac{1}{2}\,\boldsymbol{\omega}\times d\mathbf{r} \;=\; \boldsymbol{\Omega}\times d\mathbf{r}$$

Pure straining motion

$$\mathbf{u}^{(s)} \;=\; d\mathbf{r}\cdot\mathbf{D} \;=\; d\mathbf{r}\cdot\tfrac{1}{2}\,[\nabla\mathbf{u}+(\nabla\mathbf{u})^T] \;=\; \mathbf{u}^{(e)} \;+\; \mathbf{u}^{(st)}$$

Isotropic expansion

$$\mathbf{u}^{(e)} \;=\; d\mathbf{r}\cdot\tfrac{1}{3}\,tr(\mathbf{D})\,\mathbf{I} \;=\; d\mathbf{r}\cdot\tfrac{1}{3}\,(\nabla\cdot\mathbf{u})\,\mathbf{I}$$

Pure straining motion without change of volume

$$\mathbf{u}^{(st)} \;=\; d\mathbf{r}\cdot[\mathbf{D}-\tfrac{1}{3}\,tr(\mathbf{D})\,\mathbf{I}] \;=\; d\mathbf{r}\cdot\tfrac{1}{2}\,[\nabla\mathbf{u}+(\nabla\mathbf{u})^T-\tfrac{2}{3}\,(\nabla\cdot\mathbf{u})\,\mathbf{I}]$$

Table 2.2. *Decomposition of the velocity* $\mathbf{u}(\mathbf{r},t)$ *of a fluid particle in the vicinity of the point* $\mathbf{r}_0$.

This vector equation is equivalent to three linear differential equations,

$$\frac{\Delta dr_i'}{\Delta t} \;=\; \lambda_i \, dr_i' \,, \qquad i=1,2,3 \,. \tag{2.95}$$

The rate of change of the unit length along the axis of $\mathbf{e}_i'$ at t=0 is, therefore, equal to λ_i. The vector field $d\mathbf{r}' \cdot \mathbf{D}'$ is merely expanding or contracting along each of the axes $\mathbf{e}_i'$. For the rate of change of the volume V of a rectangular parallelepiped whose sides dr_1', dr_2' and dr_3' are parallel to the three eigenvectors of $\mathbf{D}$, we get

$$\frac{\Delta V}{\Delta t} \;=\; \frac{\Delta}{\Delta t}(dr_1', dr_2', dr_3') \;=\; \frac{\Delta dr_1'}{\Delta t} dr_2' dr_3' \;+\; dr_1' \frac{\Delta dr_2'}{\Delta t} dr_3' \;+\; dr_1' dr_2' \frac{\Delta dr_3'}{\Delta t} \qquad \Longrightarrow$$

$$\frac{\Delta V}{\Delta t} \;=\; (\lambda_1 + \lambda_2 + \lambda_3)\, V \,. \tag{2.96}$$

The trace of a tensor is invariant under orthogonal transformations. Hence,

$$\frac{1}{V}\,\frac{\Delta V}{\Delta t} \;=\; \lambda_1 + \lambda_2 + \lambda_3 \;=\; tr\mathbf{D}' \;=\; tr\mathbf{D} \;=\; \nabla \cdot \mathbf{u} \,. \tag{2.97}$$

This result is equivalent to Eq. (2.92).

Another way to see that $\mathbf{u}^{(e)}$ accounts for the local rate of expansion is to show that $\nabla \cdot \mathbf{u}^{(e)}$=$\Delta$. Recall that $\nabla \cdot \mathbf{u}$ is evaluated at $\mathbf{r}_0$, and $d\mathbf{r}$ is the position vector of particle P' with respect to a coordinate system centered at P. Hence,

$$\nabla \cdot \mathbf{u}^{(e)} \;=\; \nabla \cdot \left(d\mathbf{r} \cdot \frac{1}{3} \, (\nabla \cdot \mathbf{u}) \, \mathbf{I} \right) \;=\; \frac{1}{3} \, (\nabla \cdot \mathbf{u}) \, \nabla \cdot (d\mathbf{r} \cdot \mathbf{I}) \;=\; \frac{1}{3} \, (\nabla \cdot \mathbf{u}) \, \nabla \cdot d\mathbf{r} \qquad \Longrightarrow$$

$$\nabla \cdot \mathbf{u}^{(e)} \;=\; \nabla \cdot \mathbf{u} \;=\; \Delta \,. \tag{2.98}$$

Moreover, it is easily shown that the velocity $\mathbf{u}^{(e)}$ is irrotational, i.e., it produces no vorticity:

$$\nabla \times \mathbf{u}^{(e)} \;=\; \nabla \times \left(d\mathbf{r} \cdot \frac{1}{3} \, (\nabla \cdot \mathbf{u}) \, \mathbf{I} \right) \;=\; \frac{1}{3} (\nabla \cdot \mathbf{u}) \, \nabla \times (d\mathbf{r} \cdot \mathbf{I}) \;=\; \frac{1}{3} (\nabla \cdot \mathbf{u}) \, \nabla \times d\mathbf{r} \qquad \Longrightarrow$$

$$\nabla \times \mathbf{u}^{(e)} \;=\; \mathbf{0} \,. \tag{2.99}$$

In deriving Eqs. (2.98) and (2.99), the identities $\nabla \cdot d\mathbf{r}$=3 and $\nabla \times d\mathbf{r}$=$\mathbf{0}$ were used (see Example 1.4.1).

Due to the conditions $\nabla \cdot \mathbf{u}^{(e)}$=$\Delta$ and $\nabla \times \mathbf{u}^{(e)}$=$\mathbf{0}$, the velocity $\mathbf{u}^e$ can be written as the gradient of a scalar field $\phi^{(e)}$,

$$\mathbf{u}_e \;=\; \nabla \phi^{(e)} \,, \tag{2.100}$$

which satisfies the Poisson equation:

$$\nabla^2\phi^{(e)} \;=\; \Delta\,. \tag{2.101}$$

A solution to Eqs. (2.100) and (2.101) is given by

$$\phi^{(e)}(\mathbf{r}) \;=\; -\frac{1}{4\pi}\int_V \Delta(\mathbf{r}')\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,dV(\mathbf{r}') \tag{2.102}$$

and

$$u^{(e)}(\mathbf{r}) \;=\; \frac{1}{4\pi}\int_V \Delta(\mathbf{r}')\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dV(\mathbf{r}')\,, \tag{2.103}$$

where V is the volume occupied by the fluid.

The curl of the rotational velocity $\mathbf{u}^{(r)}$ is, in fact, equal to the vorticity $\boldsymbol{\omega}$. Invoking the vector identity

$$\nabla\times(\mathbf{a}\times\mathbf{b}) \;=\; \mathbf{a}\,\nabla\cdot\mathbf{b} \;-\; \mathbf{b}\,\nabla\cdot\mathbf{a} \;+\; (\mathbf{b}\cdot\nabla)\,\mathbf{a} \;-\; (\mathbf{a}\cdot\nabla)\,\mathbf{b}\,,$$

we get

$$\begin{aligned}\nabla\times\mathbf{u}^{(r)} \;&=\; \frac{1}{2}\nabla\times(\boldsymbol{\omega}\times d\mathbf{r}) \\ &=\; \frac{1}{2}\left[\boldsymbol{\omega}\,\nabla\cdot d\mathbf{r} \;-\; d\mathbf{r}\,\nabla\cdot\boldsymbol{\omega} \;+\; (d\mathbf{r}\cdot\nabla)\,\boldsymbol{\omega} \;-\; (\boldsymbol{\omega}\cdot\nabla)\,d\mathbf{r}\right].\end{aligned}$$

Since $\nabla\cdot d\mathbf{r}$=3, $\nabla\cdot\boldsymbol{\omega}=\mathbf{0}$ (the vorticity is solenoidal), $(d\mathbf{r}\cdot\nabla)\boldsymbol{\omega}=\mathbf{0}$ (evaluated at $\mathbf{r}_0$), and $(\boldsymbol{\omega}\cdot\nabla)d\mathbf{r}=\boldsymbol{\omega}$, one gets

$$\nabla\times\mathbf{u}^{(r)} \;=\; \boldsymbol{\omega}\,. \tag{2.104}$$

Given that rigid motion is volume preserving, the divergence of the rotational velocity is zero,

$$\nabla\cdot\mathbf{u}^{(r)} \;=\; \frac{1}{2}\nabla\cdot(\boldsymbol{\omega}\times d\mathbf{r}) \;=\; \frac{1}{2}\left[d\mathbf{r}\cdot(\nabla\times\boldsymbol{\omega}) - \boldsymbol{\omega}\cdot\nabla\times d\mathbf{r}\right] \;=\; \frac{1}{2}\left(d\mathbf{r}\cdot\mathbf{0} - \boldsymbol{\omega}\cdot\mathbf{0}\right) \quad\Longrightarrow$$

$$\nabla\cdot\mathbf{u}^{(r)} \;=\; 0\,, \tag{2.105}$$

which can be verified by the fact that the vorticity tensor has zero trace. Equations (2.104) and (2.105) suggest a solution of the form,

$$\mathbf{u}^{(r)} \;=\; \nabla\times\mathbf{B}^{(r)}\,, \tag{2.106}$$

where $\mathbf{B}^{(r)}$ is a vector potential for $\mathbf{u}^{(r)}$ that satisfies Eq. (2.105) identically. From Eq. (2.104), one gets

$$\nabla\times(\nabla\times\mathbf{B}^{(r)}) \;=\; \boldsymbol{\omega} \quad\Longrightarrow\quad \nabla(\nabla\cdot\mathbf{B}^{(r)}) \;-\; \nabla^2\mathbf{B}^{(r)} \;=\; \boldsymbol{\omega}\,. \tag{2.107}$$

If $\nabla \cdot \mathbf{B}^{(r)}=0$,

$$\nabla^2 \mathbf{B}^{(r)} = -\boldsymbol{\omega} \, . \tag{2.108}$$

The solution to Eqs. (2.106) to (2.108) is given by

$$\mathbf{B}^{(r)}(\mathbf{r}) = \frac{1}{4\pi} \int_V \frac{\boldsymbol{\omega}}{|\mathbf{r}-\mathbf{r}'|} \, dV(\mathbf{r}') \tag{2.109}$$

and

$$\mathbf{u}^{(r)}(\mathbf{r}) = -\frac{1}{4\pi} \int_V \frac{(\mathbf{r}-\mathbf{r}') \times \boldsymbol{\omega}}{|\mathbf{r}-\mathbf{r}'|^3} \, dV(\mathbf{r}') \, , \tag{2.110}$$

which suggest that rotational velocity, at a point $\mathbf{r}$, is induced by the vorticity at neighboring points, $\mathbf{r}'$.

Due to the fact that the expansion, Δ, and the vorticity, $\boldsymbol{\omega}$, are accounted for by the expansion and rotational velocities, respectively, the straining velocity, $\mathbf{u}^{(st)}$, is both *solenoidal* and *irrotational*. Therefore,

$$\nabla \cdot \mathbf{u}^{(st)} = \frac{1}{2} \nabla \cdot \left\{ d\mathbf{r} \cdot \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \, \mathbf{I} \right] \right\} = 0 \tag{2.111}$$

and

$$\nabla \times \mathbf{u}^{(st)} = \frac{1}{2} \nabla \times \left\{ d\mathbf{r} \cdot \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \, \mathbf{I} \right] \right\} = \mathbf{0} \, . \tag{2.112}$$

A potential function $\phi^{(st)}$, such that

$$\mathbf{u}^{(st)} = \nabla \phi^{(st)} \, , \tag{2.113}$$

satisfies Eq. (2.112) and reduces Eq. (2.111) to the Laplace equation,

$$\nabla^2 \phi^{(st)} = 0 \, . \tag{2.114}$$

The Laplace equation has been studied extensively, and many solutions are known [9]. The key to the solution of potential flow problems is the selection of proper solutions that satisfy the boundary conditions. By means of the divergence and Stokes theorems, we get from Eqs. (2.111) and (2.112)

$$\int_V \nabla \cdot \mathbf{u}^{(st)} \, dV = \int_S \mathbf{n} \cdot \mathbf{u}^{(st)} \, dS = 0 \tag{2.115}$$

and

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{u}^{(st)}) \, dS = \int_C \mathbf{t} \cdot \mathbf{u}^{(st)} \, d\ell = 0 \, . \tag{2.116}$$

It is clear that the solution $\mathbf{u}^{(st)}$ depends entirely on boundary data.

More details on the mechanisms, concepts and closed form solutions of local and relative kinematics are given in numerous theoretical Fluid Mechanics [10-12], Rheology [13] and Continuum Mechanics [14] publications.

Example 2.4.1. Local kinematics of stagnation flow

Consider the two-dimensional flow of Fig. 2.6, with Eulerian velocities

$$u_x \;=\; \varepsilon x \quad \text{and} \quad u_y \;=\; -\varepsilon y\,.$$

For the velocity gradient tensor we get

$$\nabla\mathbf{u} \;=\; \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix} \;=\; \varepsilon\,\mathbf{ii} - \varepsilon\,\mathbf{jj}\,.$$

Since $\nabla\mathbf{u}$ is symmetric,

$$\mathbf{D} \;=\; \nabla\mathbf{u} \;=\; \varepsilon\,\mathbf{ii} - \varepsilon\,\mathbf{jj}\,,$$

and

$$\mathbf{S} \;=\; \mathbf{O}\,.$$

Therefore, the flow is irrotational. It is also incompressible, since

$$tr(\mathbf{D}) \;=\; \nabla\cdot\mathbf{u} \;=\; \varepsilon - \varepsilon \;=\; 0\,.$$

For the velocities $\mathbf{u}^{(r)}$, $\mathbf{u}^{(e)}$ and $\mathbf{u}^{(st)}$, we find:

$$\begin{aligned}
\mathbf{u}^{(r)} &= d\mathbf{r}\cdot\mathbf{S} \;=\; \mathbf{0}\,,\\
\mathbf{u}^{(e)} &= d\mathbf{r}\cdot\frac{1}{3}\,(\nabla\cdot\mathbf{u})\,\mathbf{I} \;=\; \mathbf{0}\,,\\
\mathbf{u}^{(st)} &= d\mathbf{r}\cdot\frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \frac{2}{3}\,(\nabla\cdot\mathbf{u})\,\mathbf{I}\right] \;=\; d\mathbf{r}\cdot(\varepsilon\,\mathbf{ii} - \varepsilon\,\mathbf{jj})\,.
\end{aligned}$$

Therefore, expansion and rotation are zero, and there is only extension of the material vector $d\mathbf{r}$. If $d\mathbf{r}$ is of the form,

$$d\mathbf{r} \;=\; adx\,\mathbf{i} \,+\, bdy\,\mathbf{j}\,,$$

then

$$\mathbf{u}^{(st)} \;=\; a\varepsilon dx\,\mathbf{i} \;-\; b\varepsilon dy\,\mathbf{j}\,.$$

If, for instance, $d\mathbf{r}{=}adx\,\mathbf{i}$, then $\mathbf{u}^{(st)}{=}a\varepsilon dx\,\mathbf{i}$ and extension is in the x-direction. □

Example 2.4.2. Local kinematics of rotational shear flow

We consider here shear flow in a channel of width $2H$. If the x-axis lies on the plane of symmetry and points in the direction of the flow, the Eulerian velocity profiles are

$$u_x = c\,(H^2 - y^2) \quad \text{and} \quad u_y = u_z = 0\,,$$

where c is a positive constant. The resulting velocity gradient tensor is

$$\nabla \mathbf{u} = \begin{bmatrix} 0 & 0 \\ -2c\,y & 0 \end{bmatrix} = -2c\,y\,\mathbf{ji}\,,$$

and thus

$$\mathbf{D} = \frac{1}{2}\,[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] = \begin{bmatrix} 0 & -c\,y \\ -c\,y & 0 \end{bmatrix} = -c\,y\,(\mathbf{ij} + \mathbf{ji})\,,$$

and

$$\mathbf{S} = \frac{1}{2}\,[\nabla \mathbf{u} - (\nabla \mathbf{u})^T] = \begin{bmatrix} 0 & c\,y \\ -c\,y & 0 \end{bmatrix} = c\,y\,(\mathbf{ij} - \mathbf{ji})\,.$$

Since

$$tr(\mathbf{D}) = \nabla \cdot \mathbf{u} = 0\,,$$

the flow is incompressible,

If $d\mathbf{r}$ is of the form,

$$d\mathbf{r} = adx\,\mathbf{i} + bdy\,\mathbf{j}\,,$$

then

$$\begin{aligned}
\mathbf{u}^{(r)} &= d\mathbf{r} \cdot \mathbf{S} = (adx\,\mathbf{i} + bdy\,\mathbf{j}) \cdot c\,y\,(\mathbf{ij} - \mathbf{ji}) = c\,y\,(-b\,dy\,\mathbf{i} + a\,dx\,\mathbf{j})\,, \\
\mathbf{u}^{(e)} &= d\mathbf{r} \cdot \frac{1}{3}\,\nabla \cdot \mathbf{u}\,\mathbf{I} = \mathbf{0}\,, \\
\mathbf{u}^{(st)} &= d\mathbf{r} \cdot [\mathbf{D} - \frac{1}{3}\,\nabla \cdot \mathbf{u}\,\mathbf{I}] = (adx\,\mathbf{i} + bdy\,\mathbf{j}) \cdot c\,y\,(-\mathbf{ij} - \mathbf{ji}) \\
&= -c\,y\,(b\,dy\,\mathbf{i} + a\,dx\,\mathbf{j})\,.
\end{aligned}$$

Despite the fact that the fluid is not rotating globally (the streamlines are straight lines), the flow is rotational,

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = -\frac{du_x}{dy}\,\mathbf{k} = 2c\,y\,\mathbf{k} \neq \mathbf{0}\,.$$

The vorticity is maximum along the wall ($y{=}H$), and zero along the centerline ($y{=}0$). The existence of vorticity gives rise to extensional strain. This is known as *vorticity induced extension*, to avoid confusion with the *strain induced extension*, represented by $d\mathbf{u}^{(e)}$. Unlike the latter, the vorticity induced extensional strain does not generate any normal stresses, but it does contribute to shear stresses. □

The rate of strain tensor $\mathbf{D}$ results in extensional and shear strain. Consider again the relative velocity between the particles P and P' of Fig. 2.7,

$$d\mathbf{u} \;=\; d\mathbf{r}\cdot\nabla\mathbf{u} \;=\; (\nabla\mathbf{u})^T\cdot d\mathbf{r}\,. \tag{2.117}$$

By definition,

$$d\mathbf{u} \;\equiv\; \frac{Dd\mathbf{r}}{Dt} \quad\Longrightarrow\quad \frac{Dd\mathbf{r}}{Dt} \;=\; d\mathbf{r}\cdot\nabla\mathbf{u} \;=\; (\nabla\mathbf{u})^T\cdot d\mathbf{r}\,. \tag{2.118}$$

Let $\mathbf{a}$ be the unit vector in the direction of $d\mathbf{r}$ and $ds{=}|d\mathbf{r}|$, i.e., $d\mathbf{r}{=}\mathbf{a}ds$. Then, from Eq. (2.118) we get:

$$\frac{D\mathbf{a}ds}{Dt} = \mathbf{a}ds\cdot\nabla\mathbf{u} \;=\; (\nabla\mathbf{u})^T\cdot\mathbf{a}ds \quad\Longrightarrow\quad \mathbf{a}\,\frac{1}{ds}\,\frac{Dds}{Dt} \;=\; \mathbf{a}\cdot\nabla\mathbf{u} \;=\; (\nabla\mathbf{u})^T\cdot\mathbf{a} \quad\Longrightarrow$$

$$\frac{1}{ds}\,\frac{Dds}{Dt} = (\mathbf{a}\cdot\nabla\mathbf{u})\cdot\mathbf{a} = \mathbf{a}\cdot[(\nabla\mathbf{u})^T\cdot\mathbf{a}] \quad\Longrightarrow\quad \frac{1}{ds}\,\frac{Dds}{Dt} = \mathbf{a}\cdot\frac{1}{2}[\nabla\mathbf{u}+(\nabla\mathbf{u})^T]\cdot\mathbf{a} \quad\Longrightarrow$$

$$\frac{1}{ds}\,\frac{Dds}{Dt} \;=\; \mathbf{a}\cdot\mathbf{D}\cdot\mathbf{a}\,. \tag{2.119}$$

Equation (2.119) describes the extension of the material length ds with time. The term $\mathbf{a}\cdot\mathbf{D}\cdot\mathbf{a}$ is called *extensional strain rate*. The extensional strain rate of a material vector aligned with one Cartesian axis, $d\mathbf{r}{=}\mathbf{e}_i ds$, is equal to the corresponding diagonal element of $\mathbf{D}$:

$$\left.\frac{1}{ds}\,\frac{Dds}{Dt}\right|_{\mathbf{e}_i ds} \;=\; \mathbf{e}_i\cdot\mathbf{D}\cdot\mathbf{e}_i \;=\; D_{ii} \;=\; \frac{\partial u_i}{\partial x_i}\,. \tag{2.120}$$

Similar expressions can be obtained for the shear (or angular) strain. The shearing of fluid particles depends on how the angle between material vectors evolves with time. If $\mathbf{a}$ and $\mathbf{b}$ are unit material vectors originally at right angle, i.e., $\mathbf{a}\cdot\mathbf{b}{=}0$, then the angle θ, between the two material vectors, evolves according to

$$\left.\frac{D\theta}{Dt}\right|_{\theta=\frac{\pi}{2}} \;=\; -2\,\mathbf{a}\cdot\mathbf{D}\cdot\mathbf{b}\,. \tag{2.121}$$

The right-hand side of the above equation is the *shear strain rate*. Since $\mathbf{D}$ is symmetric, the order of $\mathbf{a}$ and $\mathbf{b}$ in Eq. (2.121) is immaterial. The shear strain rate between material vectors along two axes x_i and x_j of the Cartesian coordinate system is opposite to the ij-component of the rate-of-strain tensor:

$$\left.\frac{D\theta}{Dt}\right|_{\mathbf{e}_i,\mathbf{e}_j} = -2\,\mathbf{e}_i\cdot\mathbf{D}\cdot\mathbf{e_j} = -2D_{ij} = -\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) . \tag{2.122}$$

Example 2.4.3. Deformation of material lines

We revisit here the two flows studied in Examples 2.4.1 and 2.4.2.

Irrotational extensional flow

For the material vector $d\mathbf{r}$=$\mathbf{a}ds$ with

$$\mathbf{a} = \frac{a_1\,\mathbf{i}+a_2\,\mathbf{j}}{\sqrt{a_1^2+a_2^2}} ,$$

the extensional strain rate is

$$\frac{1}{ds}\,\frac{Dds}{Dt} = \mathbf{a}\cdot\mathbf{D}\cdot\mathbf{a} = \frac{a_1\,\mathbf{i}+a_2\,\mathbf{j}}{\sqrt{a_1^2+a_2^2}}\cdot(\varepsilon\,\mathbf{ii}-\varepsilon\,\mathbf{ij})\cdot\frac{a_1\,\mathbf{i}+a_2\,\mathbf{j}}{\sqrt{a_1^2+a_2^2}} \implies$$

$$\frac{1}{ds}\,\frac{Dds}{Dt} = \frac{a_1^2-a_2^2}{a_1^2+a_2^2}\,\varepsilon .$$

We observe that if a_1=$\pm a_2$, the material length ds does not change with time. A material vector along the x-direction ($d\mathbf{r}$=$\mathbf{i}ds$) changes its length according to

$$\frac{D(\ln ds)}{Dt} = \frac{1}{ds}\,\frac{Dds}{Dt} = \varepsilon \quad\implies\quad ds = (ds)_0\,e^{\varepsilon t} .$$

Similarly, for $d\mathbf{r}$=$\mathbf{j}ds$, we find that ds=$(ds)_0\,e^{-\varepsilon t}$.

The shear strain rate for $\mathbf{a}$=$\mathbf{i}$ and $\mathbf{b}$=$\mathbf{j}$ is

$$\left.\frac{D\theta}{Dt}\right|_{\mathbf{i},\mathbf{j}} = -2\,\mathbf{a}\cdot\mathbf{D}\cdot\mathbf{b} = -2\,\mathbf{i}\cdot(\varepsilon\,\mathbf{ii}-\varepsilon\,\mathbf{jj})\cdot\mathbf{j} = 0 ,$$

in agreement with the fact that shearing is not present in irrotational extensional flows.

Rotational shear flow

We consider a material vector of arbitrary orientation,

$$d\mathbf{r} = \mathbf{a}ds = \frac{a_1\,\mathbf{i}+a_2\,\mathbf{j}}{\sqrt{a_1^2+a_2^2}}\,ds ,$$

for which

$$\frac{D(\ln ds)}{Dt} = \frac{1}{ds}\frac{Dds}{Dt} = \mathbf{a}\cdot\mathbf{D}\cdot\mathbf{a}$$
$$= \frac{a_1\,\mathbf{i}+a_2\,\mathbf{j}}{\sqrt{a_1^2+a_2^2}}\cdot\left[-c\,y\,(\mathbf{ij}+\mathbf{ji})\right]\cdot\frac{a_1\,\mathbf{i}+a_2\,\mathbf{j}}{\sqrt{a_1^2+a_2^2}} = -\frac{2a_1a_2}{a_1^2+a_2^2}\,c\,y\,,$$

or

$$\frac{D(\ln ds)}{Dt} = \frac{a_1a_2}{a_1^2+a_2^2}\,\frac{\partial u_x}{\partial y}\,.$$

We easily deduce that a material vector parallel to the x-axis does not change length.

The shear strain rate for $\mathbf{a}$=$\mathbf{i}$ and $\mathbf{b}$=$\mathbf{j}$ is

$$\left.\frac{D\theta}{Dt}\right|_{\mathbf{i},\mathbf{j}} = -2\,\mathbf{a}\cdot\mathbf{D}\cdot\mathbf{b} = -2\,\mathbf{i}\cdot\left[-c\,y\,(\mathbf{ij}+\mathbf{ji})\right]\cdot\mathbf{j} = 2c\,y\,,$$

or

$$\left.\frac{D\theta}{Dt}\right|_{\mathbf{i},\mathbf{j}} = -\frac{\partial u_x}{\partial y}\,.$$

□

2.5 Problems

2.1. Repeat Example 2.1.2 for cylindrical droplets of radius R and length $L \gg R$. How does the inside pressure change with R, L and σ?

2.2. The Eulerian description of a two-dimensional flow is given by

$$u_x = ay \quad \text{and} \quad u_y = 0\,,$$

where a is a positive constant.
(a) Calculate the Lagrangian kinematics and compare with the Eulerian ones.
(b) Calculate the velocity-gradient, the rate-of-strain and the vorticity tensors.
(c) Find the deformation of material vectors parallel to the x- and y-axes.
(d) Find the deformation of material vectors diagonal to the two axes. Explain the physics behind your findings.

2.3. Write down the Young-Laplace equation for interfaces of the following configurations: spherical, cylindrical, planar, elliptical, parabolic, and hyperbolic.

2.4. The motion of a solid body on the xy-plane is described by

$$\mathbf{r}(t) = \mathbf{i}\,a\cos\omega t + \mathbf{j}\,b\sin\omega t\,,$$

where a, b and ω are constants. How far is the body from the origin at any time t? Find the velocity and the acceleration vectors. Show that the body moves on an elliptical path.

2.5. Derive the equation that governs the pressure distribution in the atmosphere by means of momentum balance on an appropriate control volume. You must utilize the integral theorems of Chapter 1.

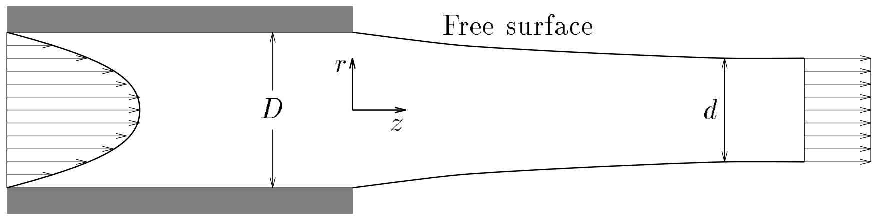

Figure 2.8. *Contraction of a round Newtonian jet at a high Reynolds number.*

2.6. Consider the *high Reynolds number* flow of a Newtonian jet issuing from a capillary of diameter D, as illustrated in Fig. 2.8. Upstream the exit of the capillary, the flow is assumed to be *fully-developed*, i.e., the axial velocity is parabolic,

$$u_z = \frac{32\,Q}{\eta\,D^4}\left(\frac{D^2}{4} - r^2\right) ,$$

where η is the viscosity of the liquid, ρ is its density, and Q denotes the volumetric flow rate. The liquid leaves the capillary as a free round jet and, after some rearrangement, the flow downstream becomes *plug*, i.e.,

$$u_z = V .$$

Using appropriate conservation statements, calculate the velocity V and the final diameter d of the jet. Repeat the procedure for a plane jet issuing from a slit of thickness H and width W.

2.7. Use the substantial derivative,

$$\frac{D(ds)}{Dt} = \frac{\partial(ds)}{\partial t} + \mathbf{u}\cdot\nabla(ds) \tag{2.123}$$

to find how material lengths, ds, change along streamlines. Consider vectors tangent and perpendicular to streamlines. Apply your findings to the following flows:

(a) $u_x=\varepsilon x$ and $u_y=-\varepsilon y$;
(b) $u_x=ay$ and $u_y=0$.

2.8. A material vector **a** enters perpendicularly a shear field given by $u_x=ay$ and $u_y=0$. Describe its motion and deformation as it travels in the field. Repeat for the extensional field given by $u_x=\varepsilon x$ and $u_y=-\varepsilon y$.

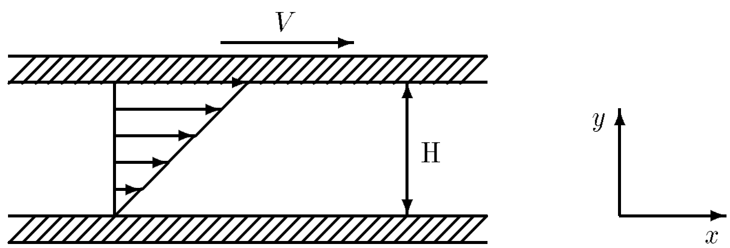

Figure 2.9. *Plane Couette flow.*

2.9. Calculate the configuration of a material square in the *plane Couette flow*, the geometry of which is depicted in Fig. 2.9. The lower wall is fixed, the upper wall is moving with speed V, and the x-component of the velocity is given by

$$u_x = \frac{y}{H} V . \tag{2.124}$$

Consider three entering locations: adjacent to each of the walls and at $y=H/2$. How would you use this flow to measure velocity, vorticity and stress?

2.10. The velocity vector

$$\mathbf{u}(t) = \Omega(t)\, r\, \mathbf{e}_\theta + u_r(t)\, \mathbf{e}_r + u_z\, \mathbf{e}_z$$

describes a *spiral flow* in cylindrical coordinates.
(a) Calculate the acceleration vector $\mathbf{a}(t)$ and the position vector $\mathbf{r}(t)$.
(b) How things change when $u_z=0$, $\Omega(t)=\Omega_0$ and $u_r(t)=u_0$? Sketch a representative streamline.

2.6 References

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