Chapter 1

VECTOR AND TENSOR CALCULUS

The physical quantities encountered in fluid mechanics can be classified into three classes: (a) *scalars*, such as pressure, density, viscosity, temperature, length, mass, volume and time; (b) *vectors*, such as velocity, acceleration, displacement, linear momentum and force, and (c) *tensors*, such as stress, rate of strain and vorticity tensors.

Scalars are completely described by their *magnitude* or absolute value, and they do not require direction in space for their specification. In most cases, we shall denote scalars by lower case lightface italic type, such as p for pressure and ρ for density. Operations with scalars, i.e., addition and multiplication, follow the rules of elementary algebra. A *scalar field* is a real-valued function that associates a scalar (i.e., a real number) with each point of a given region in space. Let us consider, for example, the *right-handed Cartesian coordinate system* of Fig. 1.1 and a closed three-dimensional region V occupied by a certain amount of a moving fluid at a given time instance t. The density ρ of the fluid at any point (x, y, z) of V defines a scalar field denoted by $\rho(x, y, z)$. If the density is, in addition, time-dependent, one may write $\rho=\rho(x, y, z, t)$.

Vectors are specified by their *magnitude* and their *direction* with respect to a given frame of reference. They are often denoted by lower case boldface type, such as $\mathbf{u}$ for the velocity vector. A *vector field* is a vector-valued function that associates a vector with each point of a given region in space. For example, the velocity of the fluid in the region V of Fig. 1.1 defines a vector field denoted by $\mathbf{u}(x, y, z, t)$. A vector field which is independent of time is called a *steady-state* or *stationary* vector field. The magnitude of a vector $\mathbf{u}$ is designated by $|\mathbf{u}|$ or simply by u.

Vectors can be represented geometrically as arrows; the direction of the arrow specifies the direction of the vector and the length of the arrow, compared to some chosen scale, describes its magnitude. Vectors having the same length and the same

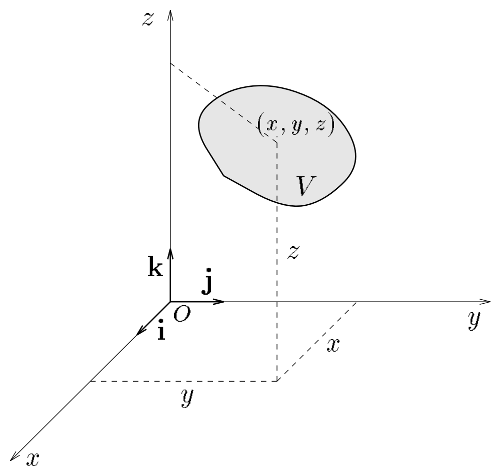

Figure 1.1. *Cartesian system of coordinates.*

direction, regardless of the position of their initial points, are said to be *equal*. A vector having the same length but the opposite direction to that of the vector $\mathbf{u}$ is denoted by $-\mathbf{u}$ and is called the *negative* of $\mathbf{u}$.

The *sum* (or the *resultant*) $\mathbf{u}+\mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ can be found using the *parallelogram law* for vector addition, as shown in Fig. 1.2a. Extensions to sums of more than two vectors are immediate. The difference $\mathbf{u}$-$\mathbf{v}$ is defined as the sum $\mathbf{u}+(-\mathbf{v})$; its geometrical construction is shown in Fig. 1.2b.

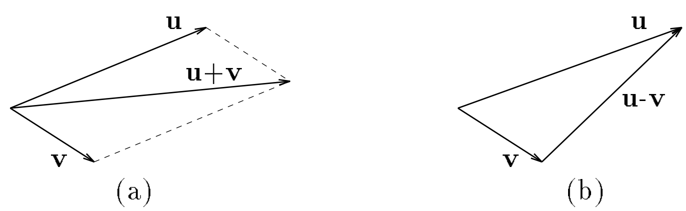

Figure 1.2. *Addition and subtraction of vectors.*

The vector of length zero is called the *zero vector* and is denoted by $\mathbf{0}$. Obviously, there is no natural direction for the zero vector. However, depending on the problem, a direction can be assigned for convenience. For any vector $\mathbf{u}$,

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

and

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}\,.$$

Vector addition obeys the *commutative* and *associative* laws. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors, then

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative law
$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative law

If $\mathbf{u}$ is a nonzero vector and m is a nonzero scalar, then the *product* $m\mathbf{u}$ is defined as the vector whose length is $|m|$ times the length of $\mathbf{u}$ and whose direction is the same as that of $\mathbf{u}$ if $m > 0$, and opposite to that of $\mathbf{u}$ if $m < 0$. If m=0 or $\mathbf{u}$=$\mathbf{0}$, then $m\mathbf{u}$=$\mathbf{0}$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors and m and n are scalars, then

$m\mathbf{u} = \mathbf{u}m$	Commutative law
$m(n\mathbf{u}) = (mn)\mathbf{u}$	Associative law
$(m + n)\mathbf{u} = m\mathbf{u} + n\mathbf{u}$	Distributive law
$m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$	Distributive law

Note also that $(-1)\mathbf{u}$ is just the negative of $\mathbf{u}$,

$$(-1)\mathbf{u} = -\mathbf{u} .$$

A *unit vector* is a vector having unit magnitude. The three vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ which have the directions of the positive x, y and z axes, respectively, in the Cartesian coordinate system of Fig. 1.1 are unit vectors.

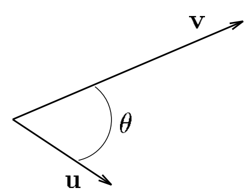

Figure 1.3. *Angle between vectors* $\mathbf{u}$ *and* $\mathbf{v}$*.*

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in a two- or three-dimensional space positioned so that their initial points coincide (Fig. 1.3). The *angle* θ between $\mathbf{u}$ and $\mathbf{v}$ is the angle determined by $\mathbf{u}$ and $\mathbf{v}$ that satisfies $0 \leq \theta \leq \pi$. The *dot product* (or *scalar product*) of $\mathbf{u}$ and $\mathbf{v}$ is a scalar quantity defined by

$$\mathbf{u} \cdot \mathbf{v} \equiv uv \cos\theta . \tag{1.1}$$

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and m is a scalar, then

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	Commutative law
$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$	Distributive law
$m(\mathbf{u} \cdot \mathbf{v}) = (m\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (m\mathbf{v})$	

Moreover, the dot product of a vector with itself is a positive number that is equal to the square of the length of the vector:

$$\mathbf{u}\cdot\mathbf{u} \;=\; u^2 \qquad \Longleftrightarrow \qquad u \;=\; \sqrt{\mathbf{u}\cdot\mathbf{u}}\,. \tag{1.2}$$

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors and

$$\mathbf{u}\cdot\mathbf{v} \;=\; 0\,,$$

then $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* or *perpendicular* to each other.

A vector set $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is said to be an *orthogonal set* or *orthogonal system* if every distinct pair of the set is orthogonal, i.e.,

$$\mathbf{u}_i\cdot\mathbf{u}_j \;=\; 0\,, \qquad i \neq j\,.$$

If, in addition, all its members are unit vectors, then the set $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is said to be *orthonormal*. In such a case,

$$\mathbf{u}_i\cdot\mathbf{u}_j \;=\; \delta_{ij}\,, \tag{1.3}$$

where δ_{ij} is the *Kronecker delta*, defined as

$$\delta_{ij} \;\equiv\; \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases} \tag{1.4}$$

The three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ defining the Cartesian coordinate system of Fig. 1.1 form an orthonormal set:

$$\boxed{\begin{array}{l} \mathbf{i}\cdot\mathbf{i} \;=\; \mathbf{j}\cdot\mathbf{j} \;=\; \mathbf{k}\cdot\mathbf{k} \;=\; 1 \\[1ex] \mathbf{i}\cdot\mathbf{j} \;=\; \mathbf{j}\cdot\mathbf{k} \;=\; \mathbf{k}\cdot\mathbf{i} \;=\; 0 \end{array}} \tag{1.5}$$

The *cross product* (or *vector product* or *outer product*) of two vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector defined as

$$\mathbf{u}\times\mathbf{v} \;\equiv\; uv\sin\theta\,\mathbf{n}\,, \tag{1.6}$$

where $\mathbf{n}$ is the unit vector normal to the plane of $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{n}$ form a *right-handed* orthogonal system, as illustrated in Fig. 1.4. The magnitude of $\mathbf{u}\times\mathbf{v}$ is the same as that of the area of a parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\sin\theta{=}0$ and $\mathbf{u}\times\mathbf{v}{=}\mathbf{0}$. For instance, $\mathbf{u}\times\mathbf{u}{=}\mathbf{0}$.

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and m is a scalar, then

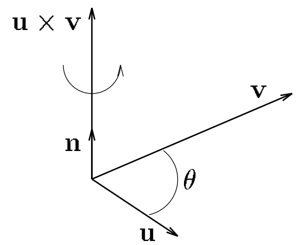

Figure 1.4. *The cross product* $\mathbf{u}\times\mathbf{v}$.

$\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$	Not commutative
$\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$	Distributive law
$m(\mathbf{u}\times\mathbf{v}) = (m\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(m\mathbf{v}) = (\mathbf{u}\times\mathbf{v})m$	

For the three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ one gets:

$$\boxed{\begin{gathered}\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}\,,\\ \mathbf{i}\times\mathbf{j} = \mathbf{k}\,,\quad \mathbf{j}\times\mathbf{k} = \mathbf{i}\,,\quad \mathbf{k}\times\mathbf{i} = \mathbf{j}\,,\\ \mathbf{j}\times\mathbf{i} = -\mathbf{k}\,,\quad \mathbf{k}\times\mathbf{j} = -\mathbf{i}\,,\quad \mathbf{i}\times\mathbf{k} = -\mathbf{j}\,.\end{gathered}} \tag{1.7}$$

Note that the cyclic order $(\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{i},\mathbf{j},\cdots)$, in which the cross product of any neighboring pair in order is the next vector, is consistent with the right-handed orientation of the axes as shown in Fig. 1.1.

The product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is called the *scalar triple product* of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, and is a scalar representing the volume of a parallelepiped with $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ as the edges. The product $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ is a vector called the *vector triple product.* The following laws are valid:

$(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w} \neq \mathbf{u}\,(\mathbf{v}\cdot\mathbf{w})$	Not associative
$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \neq (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$	Not associative
$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$	
$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}$	
$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$	

Thus far, we have presented vectors and vector operations from a geometrical viewpoint. These are treated analytically in Section 1.2.

Tensors may be viewed as *generalized vectors* being characterized by their magnitude and more than one *ordered* directions with respect to a given frame of reference.

Tensors encountered in fluid mechanics are of second order, i.e., they are characterized by an *ordered pair* of coordinate directions. Tensors are often denoted by uppercase boldface type or lower case boldface Greek letters, such as $\boldsymbol{\tau}$ for the stress tensor. A *tensor field* is a tensor-valued function that associates a tensor with each point of a given region in space. Tensor addition and multiplication of a tensor by a scalar are commutative and associative. If $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$ are tensors of the same type, and m and n are scalars, then

$\mathbf{R} + \mathbf{S} = \mathbf{S} + \mathbf{R}$	Commutative law
$(\mathbf{R} + \mathbf{S}) + \mathbf{T} = \mathbf{S} + (\mathbf{R} + \mathbf{T})$	Associative law
$m\mathbf{R} = \mathbf{R}m$	Commutative law
$m(n\mathbf{R}) = (mn)\mathbf{R}$	Associative law
$(m+n)\mathbf{R} = m\mathbf{R} + n\mathbf{R}$	Distributive law
$m(\mathbf{R}+\mathbf{S}) = m\mathbf{R} + m\mathbf{S}$	Distributive law

Tensors and tensor operations are discussed in more detail in Section 1.3.

1.1 Systems of Coordinates

A coordinate system in the three-dimensional space is defined by choosing a set of three *linearly independent* vectors, $B=\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, representing the three fundamental directions of the space. The set B is a *basis* of the three-dimensional space, i.e., each vector $\mathbf{v}$ of this space is uniquely written as a *linear combination* of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$:

$$\mathbf{v} = v_1\,\mathbf{e}_1 + v_2\,\mathbf{e}_2 + v_3\,\mathbf{e}_3 \,. \tag{1.8}$$

The scalars v_1, v_2 and v_3 are the *components* of $\mathbf{v}$ and represent the magnitudes of the *projections* of $\mathbf{v}$ onto each of the fundamental directions. The vector $\mathbf{v}$ is often denoted by $\mathbf{v}(v_1, v_2, v_3)$ or simply by (v_1, v_2, v_3).

In most cases, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are *unit* vectors. In the three coordinate systems that are of interest in this book, i.e., *Cartesian*, *cylindrical* and *spherical* coordinates, the three vectors are, in addition, orthogonal. Hence, in all these systems, the basis $B=\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is orthonormal:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \,. \tag{1.9}$$

(In some cases, nonorthogonal systems are used for convenience; see, for example, [1].) For the cross products of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, one gets:

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk}\,\mathbf{e}_k \,, \tag{1.10}$$

where ϵ_{ijk} is the *permutation symbol*, defined as

$$\epsilon_{ijk} \equiv \begin{cases} 1\,, & \text{if } ijk\text{=123, 231, or 312 (i.e, an even permutation of 123)} \\ -1\,, & \text{if } ijk\text{=321, 132, or 213 (i.e, an odd permutation of 123)} \\ 0\,, & \text{if any two indices are equal} \end{cases} \tag{1.11}$$

A useful relation involving the permutation symbol is the following:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\, a_i b_j c_k \,. \tag{1.12}$$

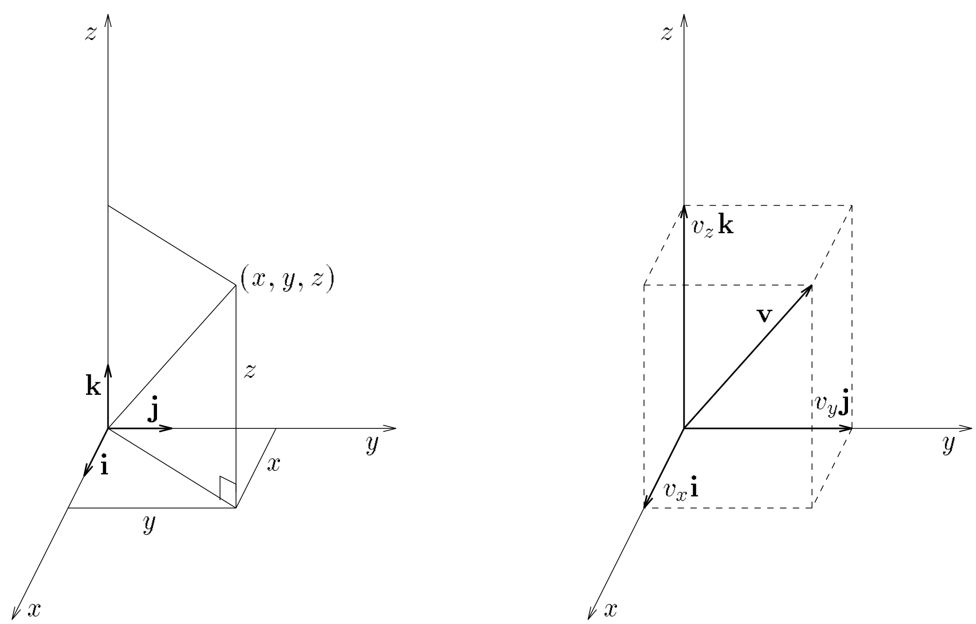

Figure 1.5. *Cartesian coordinates* (x, y, z) *with* $-\infty < x < \infty$, $-\infty < y < \infty$ *and* $-\infty < z < \infty$.

The Cartesian (or *rectangular*) system of coordinates (x, y, z), with

$$-\infty < x < \infty\,, \quad -\infty < y < \infty \quad \text{and} \quad -\infty < z < \infty\,,$$

has already been introduced, in previous examples. Its basis is often denoted by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. The decomposition of a vector $\mathbf{v}$ into its three components

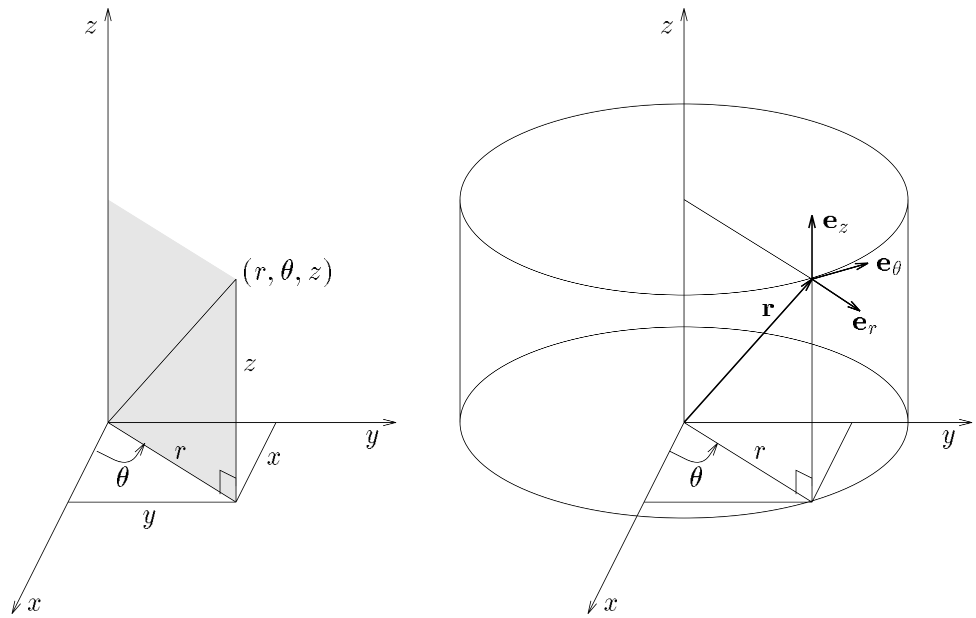

Figure 1.6. *Cylindrical polar coordinates* (r, θ, z) *with* $r \geq 0$, $0 \leq \theta < 2\pi$ *and* $-\infty < z < \infty$, *and the position vector* $\mathbf{r}$.

$(r,\theta,z) \longrightarrow (x,y,z)$	$(x,y,z) \longrightarrow (r,\theta,z)$
Coordinates	
$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$y = r\sin\theta$	$\theta = \begin{cases} \arctan\frac{y}{x}, & x>0,\ y\geq 0 \\ \pi + \arctan\frac{y}{x}, & x<0 \\ 2\pi + \arctan\frac{y}{x}, & x>0,\ y<0 \end{cases}$
$z = z$	$z = z$
Unit vectors	
$\mathbf{i} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta$	$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$
$\mathbf{j} = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta$	$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$
$\mathbf{k} = \mathbf{e}_z$	$\mathbf{e}_z = \mathbf{k}$

Table 1.1. *Relations between Cartesian and cylindrical polar coordinates.*

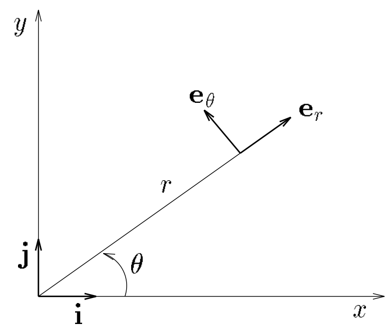

Figure 1.7. *Plane polar coordinates* (r, θ).

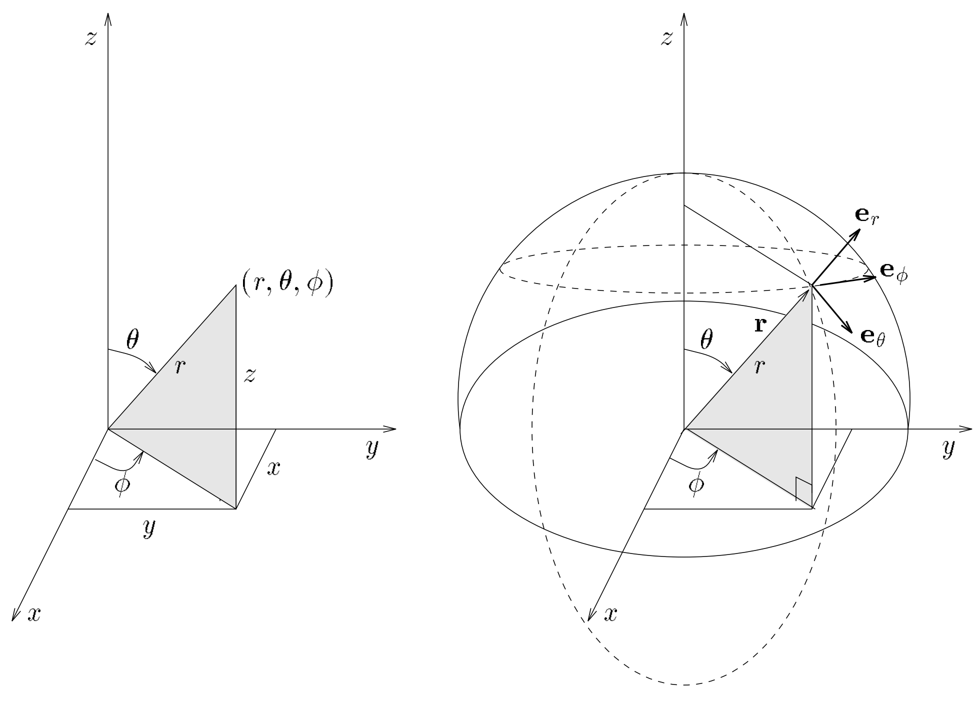

Figure 1.8. *Spherical polar coordinates* (r, θ, ϕ) *with* $r \geq 0$, $0 \leq \theta \leq \pi$ *and* $0 \leq \phi \leq 2\pi$, *and the position vector* $\mathbf{r}$.

$(r,\theta,\phi) \longrightarrow (x,y,z)$	$(x,y,z) \longrightarrow (r,\theta,\phi)$
Coordinates	
$x = r\sin\theta\cos\phi$	$r = \sqrt{x^2+y^2+z^2}$
$y = r\sin\theta\sin\phi$	$\theta = \begin{cases} \arctan\frac{\sqrt{x^2+y^2}}{z}, & z>0 \\ \frac{\pi}{2}, & z=0 \\ \pi + \arctan\frac{\sqrt{x^2+y^2}}{z}, & z<0 \end{cases}$
$z = r\cos\theta$	$\phi = \begin{cases} \arctan\frac{y}{x}, & x>0,\ y\geq 0 \\ \pi + \arctan\frac{y}{x}, & x<0 \\ 2\pi + \arctan\frac{y}{x}, & x>0,\ y<0 \end{cases}$
Unit vectors	
$\mathbf{i} = \sin\theta\cos\phi\,\mathbf{e}_r + \cos\theta\cos\phi\,\mathbf{e}_\theta - \sin\phi\,\mathbf{e}_\phi$	$\mathbf{e}_r = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$
$\mathbf{j} = \sin\theta\sin\phi\,\mathbf{e}_r + \cos\theta\sin\phi\,\mathbf{e}_\theta + \cos\phi\,\mathbf{e}_\phi$	$\mathbf{e}_\theta = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$
$\mathbf{k} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta$	$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$

Table 1.2. *Relations between Cartesian and spherical polar coordinates.*

(v_x, v_y, v_z) is depicted in Fig. 1.5. It should be noted that, throughout this book, we use *right-handed* coordinate systems.

The cylindrical and spherical polar coordinates are the two most important orthogonal *curvilinear* coordinate systems. The cylindrical polar coordinates (r, θ, z), with

$$r \geq 0\,, \quad 0 \leq \theta < 2\pi \quad \text{and} \quad -\infty < z < \infty\,,$$

are shown in Fig. 1.6 together with the Cartesian coordinates sharing the same origin. The basis of the cylindrical coordinate system consists of three orthonormal vectors: the radial vector $\mathbf{e}_r$, the azimuthal vector $\mathbf{e}_\theta$, and the axial vector $\mathbf{e}_z$. Note that the azimuthal angle θ revolves around the z axis. Any vector $\mathbf{v}$ is decomposed into, and is fully defined by its components $\mathbf{v}(v_r, v_\theta, v_z)$ with respect to the cylindrical system. By invoking simple trigonometric relations, any vector, including those of the bases, can be transformed from one system to another. Table 1.1 lists the formulas for making coordinate conversions from cylindrical to Cartesian coordinates and vice versa.

On the xy plane, i.e., if the z coordinate is ignored, the cylindrical polar coordinates are reduced to the familiar *plane polar coordinates* (r, θ) shown in Fig. 1.7.

The spherical polar coordinates (r, θ, ϕ), with

$$r \geq 0\,, \quad 0 \leq \theta \leq \pi \quad \text{and} \quad 0 \leq \phi < 2\pi\,,$$

together with the Cartesian coordinates with the same origin, are shown in Fig. 1.8. It should be emphasized that r and θ in cylindrical and spherical coordinates are not the same. The basis of the spherical coordinate system consists of three orthonormal vectors: the radial vector $\mathbf{e}_r$, the meridional vector $\mathbf{e}_\theta$, and the azimuthal vector $\mathbf{e}_\phi$. Any vector $\mathbf{v}$ can be decomposed into the three components, $\mathbf{v}(v_r, v_\theta, v_\phi)$, which are the scalar projections of $\mathbf{v}$ onto the three fundamental directions. The transformation of a vector from spherical to Cartesian coordinates (sharing the same origin) and vice-versa obeys the relations of Table 1.2.

The choice of the appropriate coordinate system, when studying a fluid mechanics problem, depends on the geometry and symmetry of the flow. Flow between parallel plates is conveniently described by Cartesian coordinates. *Axisymmetric* (i.e., *axially symmetric*) flows, such as flow in an annulus, are naturally described using cylindrical coordinates, and flow around a sphere is expressed in spherical coordinates. In some cases, nonorthogonal systems might be employed too. More details on other coordinate systems and transformations can be found elsewhere [1].

Example 1.1.1. Basis of the cylindrical system

Show that the basis $B=\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ of the cylindrical system is orthonormal.

Solution:

Since $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k}=1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i}=0$, we obtain:

$$\begin{aligned}
\mathbf{e}_r\cdot\mathbf{e}_r &= (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\cdot(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \;=\; \cos^2\theta + \sin^2\theta = 1\\
\mathbf{e}_\theta\cdot\mathbf{e}_\theta &= (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\cdot(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) \;=\; \sin^2\theta + \cos^2\theta = 1\\
\mathbf{e}_z\cdot\mathbf{e}_z &= \mathbf{k}\cdot\mathbf{k} = 1\\
\mathbf{e}_r\cdot\mathbf{e}_\theta &= (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\cdot(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) \;=\; 0\\
\mathbf{e}_r\cdot\mathbf{e}_z &= (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\cdot\mathbf{k} \;=\; 0\\
\mathbf{e}_\theta\cdot\mathbf{e}_z &= (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\cdot\mathbf{k} \;=\; 0
\end{aligned}$$

□

Example 1.1.2. The position vector

The *position vector* $\mathbf{r}$ defines the position of a point in space, with respect to a coordinate system. In Cartesian coordinates,

$$\mathbf{r} \;=\; x\,\mathbf{i} \;+\; y\,\mathbf{j} \;+\; z\,\mathbf{k}\,, \tag{1.13}$$

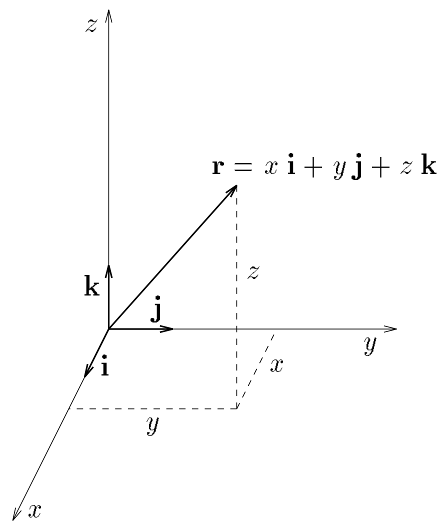

Figure 1.9. *The position vector,* $\mathbf{r}$*, in Cartesian coordinates.*

and thus

$$|\mathbf{r}| \;=\; (\mathbf{r}\cdot\mathbf{r})^{\frac{1}{2}} \;=\; \sqrt{x^2+y^2+z^2}\,. \tag{1.14}$$

The decomposition of $\mathbf{r}$ into its three components (x, y, z) is illustrated in Fig. 1.9.

In cylindrical coordinates, the position vector is given by

$$\mathbf{r} \;=\; r\,\mathbf{e}_r \;+\; z\,\mathbf{e}_z \quad \text{with} \quad |\mathbf{r}| \;=\; \sqrt{r^2+z^2}\,. \tag{1.15}$$

Note that the magnitude $|\mathbf{r}|$ of the position vector *is not* the same as the radial cylindrical coordinate r. Finally, in spherical coordinates,

$$\mathbf{r} \;=\; r\,\mathbf{e}_r \quad \text{with} \quad |\mathbf{r}| \;=\; r\,, \tag{1.16}$$

that is, $|\mathbf{r}|$ is the radial spherical coordinate r. Even though expressions (1.15) and (1.16) for the position vector are obvious (see Figs. 1.6 and 1.8, respectively), we will derive both of them, starting from Eq. (1.13) and using coordinate transformations.

In cylindrical coordinates,

$$\begin{aligned}
\mathbf{r} &= x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \\
&= r\cos\theta\,(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) \;+\; r\sin\theta\,(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) \;+\; z\,\mathbf{e}_z \\
&= r\,(\cos^2\theta + \sin^2\theta)\,\mathbf{e}_r \;+\; r\,(-\sin\theta\cos\theta + \sin\theta\cos\theta)\,\mathbf{e}_\theta \;+\; z\,\mathbf{e}_z \\
&= r\,\mathbf{e}_r \;+\; z\,\mathbf{e}_z\,.
\end{aligned}$$

In spherical coordinates,

$$\begin{aligned}
\mathbf{r} &= x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \\
&= r\sin\theta\cos\phi\,(\sin\theta\cos\phi\,\mathbf{e}_r + \cos\theta\cos\phi\,\mathbf{e}_\theta - \sin\phi\,\mathbf{e}_\phi) \\
&\quad + r\sin\theta\sin\phi\,(\sin\theta\sin\phi\,\mathbf{e}_r + \cos\theta\sin\phi\,\mathbf{e}_\theta + \cos\phi\,\mathbf{e}_\phi) \\
&\quad + r\cos\theta\,(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) \\
&= r\,[\sin^2\theta\,(\cos^2\phi + \sin^2\phi)\ \cos^2\theta]\,\mathbf{e}_r \\
&\quad + r\sin\theta\cos\theta\,[(\cos^2\phi + \sin^2\phi) - 1]\,\mathbf{e}_\theta \\
&\quad + r\sin\theta\,(-\sin\phi\cos\phi + \sin\phi\cos\phi)\,\mathbf{e}_\phi \\
&= r\,\mathbf{e}_r\,.
\end{aligned}$$

□

Example 1.1.3. Derivatives of the basis vectors

The basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ of the Cartesian coordinates are fixed and do not change with position. This is not true for the basis vectors in curvilinear coordinate systems. From Table 1.1, we observe that, in cylindrical coordinates,

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \quad \text{and} \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\,;$$

therefore, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ change with θ. Taking the derivatives with respect to θ, we obtain:

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} = \mathbf{e}_\theta$$

and

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j} = -\mathbf{e}_r\,.$$

All the other spatial derivatives of $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are zero. Hence,

$$\boxed{\begin{array}{lll}
\frac{\partial \mathbf{e}_r}{\partial r} = \mathbf{0} & \frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0} & \frac{\partial \mathbf{e}_z}{\partial r} = \mathbf{0} \\[2ex]
\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta & \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r & \frac{\partial \mathbf{e}_z}{\partial \theta} = \mathbf{0} \\[2ex]
\frac{\partial \mathbf{e}_r}{\partial z} = \mathbf{0} & \frac{\partial \mathbf{e}_\theta}{\partial z} = \mathbf{0} & \frac{\partial \mathbf{e}_z}{\partial z} = \mathbf{0}
\end{array}} \tag{1.17}$$

Similarly, for the spatial derivatives of the unit vectors in spherical coordinates, we obtain:

$$\boxed{\begin{array}{lll} \frac{\partial \mathbf{e}_r}{\partial r} = \mathbf{0} & \frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0} & \frac{\partial \mathbf{e}_\phi}{\partial r} = \mathbf{0} \\ \frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta & \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r & \frac{\partial \mathbf{e}_\phi}{\partial \theta} = \mathbf{0} \\ \frac{\partial \mathbf{e}_r}{\partial \phi} = \sin\theta\, \mathbf{e}_\phi & \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos\theta\, \mathbf{e}_\phi & \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin\theta\, \mathbf{e}_r - \cos\theta\, \mathbf{e}_\theta \end{array}} \tag{1.18}$$

Equations (1.17) and (1.18) are very useful in converting differential operators from Cartesian to orthogonal curvilinear coordinates.

□

1.2 Vectors

In this section, vector operations are considered from an analytical point of view. Let $B=\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an *orthonormal* basis of the three-dimensional space, which implies that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} , \tag{1.19}$$

and

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk}\, \mathbf{e}_k . \tag{1.20}$$

Any vector $\mathbf{v}$ can be expanded in terms of its components (v_1, v_2, v_3):

$$\mathbf{v} = v_1\, \mathbf{e}_1 + v_2\, \mathbf{e}_2 + v_3\, \mathbf{e}_3 = \sum_{i=1}^{3} v_i\, \mathbf{e_i} . \tag{1.21}$$

Any operation between two or more vectors is easily performed, by first decomposing each vector into its components and then invoking the basis relations (1.19) and (1.20). If $\mathbf{u}$ and $\mathbf{v}$ are vectors, then

$$\mathbf{u} \pm \mathbf{v} = (u_1 \pm v_1)\, \mathbf{e}_1 + (u_2 \pm v_2)\, \mathbf{e}_2 + (u_3 \pm v_3)\, \mathbf{e}_3 = \sum_{i=1}^{3} (u_i \pm v_i)\, \mathbf{e_i} , \tag{1.22}$$

i.e., addition (or subtraction) of two vectors corresponds to adding (or subtracting) their corresponding components. If m is a scalar, then

$$m\mathbf{v} = m \left(\sum_{i=1}^{3} v_i\, \mathbf{e_i} \right) = \sum_{i=1}^{3} m v_i\, \mathbf{e_i} , \tag{1.23}$$

i.e., multiplication of a vector by a scalar corresponds to multiplying each of its components by the scalar.

For the dot product of $\mathbf{u}$ and $\mathbf{v}$, we obtain:

$$\mathbf{u}\cdot\mathbf{v} \;=\; \left(\sum_{i=1}^{3} u_i\,\mathbf{e_i}\right)\cdot\left(\sum_{i=1}^{3} v_i\,\mathbf{e_i}\right) \quad\Longrightarrow$$

$$\mathbf{u}\cdot\mathbf{v} \;=\; u_1v_1 \;+\; u_2v_2 \;+\; u_3v_3 \;=\; \sum_{i=1}^{3} u_iv_i\,. \tag{1.24}$$

The magnitude of $\mathbf{v}$ is thus given by

$$v \;=\; (\mathbf{v}\cdot\mathbf{v})^{\frac{1}{2}} \;=\; \sqrt{v_1^2+v_2^2+v_3^2}\,. \tag{1.25}$$

.

Finally, for the cross product of $\mathbf{u}$ and $\mathbf{v}$, we get

$$\mathbf{u}\times\mathbf{v} \;=\; \left(\sum_{i=1}^{3} u_i\,\mathbf{e_i}\right)\times\left(\sum_{j=1}^{3} v_j\,\mathbf{e_j}\right) \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3} u_iv_j\,\mathbf{e_i}\times\mathbf{e}_j \quad\Longrightarrow$$

$$\mathbf{u}\times\mathbf{v} \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\,u_iv_j\,\mathbf{e_k} \tag{1.26}$$

or

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3-u_3v_2)\mathbf{e}_1-(u_1v_3-u_3v_1)\mathbf{e}_2+(u_1v_2-u_2v_1)\mathbf{e}_3. \tag{1.27}$$

Example 1.2.1. The scalar triple product

For the scalar triple product $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$, we have:

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} \;=\; \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\,u_iv_j\,\mathbf{e_k}\right)\cdot\left(\sum_{k=1}^{3} w_k\,\mathbf{e_k}\right) \quad\Longrightarrow$$

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\,u_iv_jw_k \tag{1.28}$$

or

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \;=\; \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \,. \tag{1.29}$$

Using basic properties of determinants, one can easily show the following identity:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \;=\; (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} \;=\; (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} \,. \tag{1.30}$$

□

In the following subsections, we will make use of the vector differential operator *nabla* (or *del*), ∇. In Cartesian coordinates, ∇ is defined by

$$\nabla \;\equiv\; \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k} \,. \tag{1.31}$$

The *gradient* of a scalar field $f(x, y, z)$ is a vector field defined by

$$\nabla f \;=\; \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} \,. \tag{1.32}$$

The *divergence* of a vector field $\mathbf{v}(x, y, z)$ is a scalar field defined by

$$\nabla \cdot \mathbf{v} \;=\; \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \,. \tag{1.33}$$

More details about ∇ and its forms in curvilinear coordinates are given in Section 1.4.

1.2.1 Vectors in Fluid Mechanics

As already mentioned, the *position* vector, $\mathbf{r}$, defines the position of a point with respect to a coordinate system. The *separation* or *displacement vector* between two points A and B (see Figure 1.10) is commonly denoted by $\Delta\mathbf{r}$, and is defined as

$$\Delta\mathbf{r}_{AB} \;\equiv\; \mathbf{r}_A - \mathbf{r}_B \,. \tag{1.34}$$

The *velocity vector*, $\mathbf{u}$, is defined as the *total time derivative* of the position vector:

$$\mathbf{u} \;\equiv\; \frac{d\mathbf{r}}{dt} \,. \tag{1.35}$$

Geometrically, the velocity vector is tangent to the curve C defined by the motion of the position vector $\mathbf{r}$ (Fig. 1.11). The *relative velocity* of a particle A, with respect to another particle B, is defined accordingly by

$$\mathbf{u}_{AB} \;\equiv\; \frac{d\Delta\mathbf{r}_{AB}}{dt} \;=\; \frac{d\mathbf{r}_A}{dt} - \frac{d\mathbf{r}_B}{dt} \;=\; \mathbf{u}_A - \mathbf{u}_B \,. \tag{1.36}$$

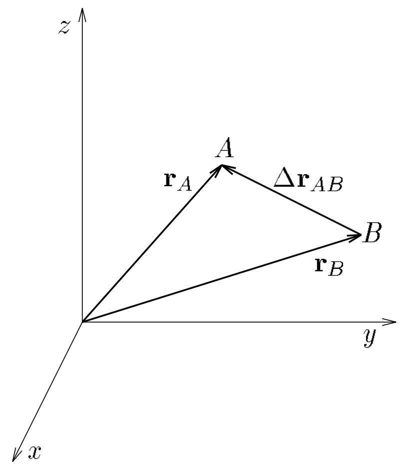

Figure 1.10. *Position and separation vectors.*

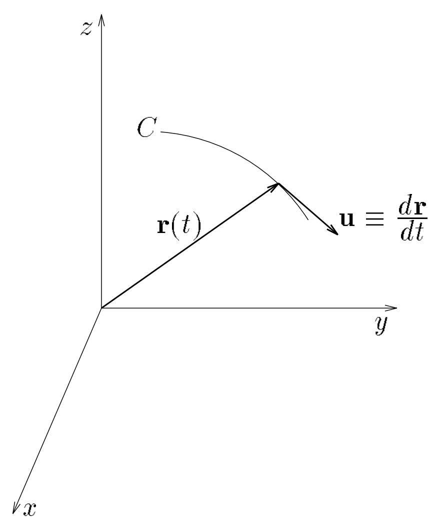

Figure 1.11. *Position and velocity vectors.*

The *acceleration vector*, $\mathbf{a}$, is defined by

$$\mathbf{a} \equiv \frac{d\mathbf{u}}{dt} = \frac{d^2\mathbf{r}}{dt^2} . \tag{1.37}$$

The acceleration of gravity, $\mathbf{g}$, is a vector directed towards the center of earth. In problems where gravity is important, it is convenient to choose one of the axes, usually the z axis, to be collinear with $\mathbf{g}$. In such a case, $\mathbf{g}=-g\mathbf{e}_z$ or $g\mathbf{e}_z$.

Example 1.2.2. Velocity components

In Cartesian coordinates, the basis vectors are fixed and thus time independent. So,

$$\mathbf{u} \equiv \frac{d}{dt}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k} .$$

Hence, the velocity components (u_x, u_y, u_z) are given by:

$$u_x = \frac{dx}{dt} , \quad u_y = \frac{dy}{dt} , \quad u_z = \frac{dz}{dt} . \tag{1.38}$$

In cylindrical coordinates, the position vector is given by $\mathbf{r}=r\,\mathbf{e_r}+z\,\mathbf{e_z}$, where $\mathbf{e_r}$ is time dependent:

$$\mathbf{u} \equiv \frac{d}{dt}(r\,\mathbf{e}_r + z\,\mathbf{e}_z) = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} + \frac{dz}{dt}\mathbf{e}_z = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{d\theta}\,\frac{d\theta}{dt} + \frac{dz}{dt}\mathbf{e}_z \implies$$

$$\mathbf{u} = \frac{dr}{dt}\,\mathbf{e}_r + r\Omega\,\mathbf{e}_\theta + \frac{dz}{dt}\,\mathbf{e}_z ,$$

where $\Omega \equiv d\theta/dt$ is the *angular velocity*. The velocity components (u_r, u_θ, u_z) are given by:

$$u_r = \frac{dr}{dt} , \quad u_\theta = r\frac{d\theta}{dt} = r\Omega , \quad u_z = \frac{dz}{dt} . \tag{1.39}$$

In spherical coordinates, all the basis vectors are time dependent. The velocity components (u_r, u_θ, u_ϕ) are easily found to be:

$$u_r = \frac{dr}{dt} , \quad u_\theta = r\frac{d\theta}{dt} , \quad u_\phi = r\sin\theta\,\frac{d\phi}{dt} . \tag{1.40}$$

□

Example 1.2.3. Circular motion

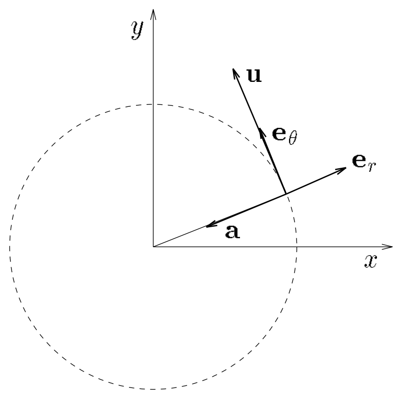

Figure 1.12. *Velocity and acceleration vectors in circular motion.*

Consider plane polar coordinates and suppose that a small solid sphere rotates at a constant distance, R, with constant angular velocity, Ω, around the origin (*uniform rotation*). The position vector of the sphere is $\mathbf{r}{=}R\,\mathbf{e}_r$ and, therefore,

$$\mathbf{u} \;\equiv\; \frac{d\mathbf{r}}{dt} \;=\; \frac{d}{dt}(R\,\mathbf{e}_r) \;=\; R\,\frac{d\mathbf{e}_r}{dt} \;=\; R\,\frac{d\mathbf{e}_r}{d\theta}\,\frac{d\theta}{dt} \qquad \Longrightarrow \qquad \mathbf{u} \;=\; R\Omega\,\mathbf{e}_\theta \;.$$

The acceleration of the sphere is:

$$\mathbf{a} \;\equiv\; \frac{d\mathbf{u}}{dt} \;=\; \frac{d}{dt}(R\Omega\,\mathbf{e}_\theta) \;=\; R\Omega\,\frac{d\mathbf{e}_\theta}{d\theta}\,\frac{d\theta}{dt} \qquad \Longrightarrow \qquad \mathbf{a} \;=\; -\,R\Omega^2\,\mathbf{e}_r \;.$$

This is the familiar *centripetal acceleration* $R\Omega^2$ directed towards the axis of rotation.
□

The *force vector*, $\mathbf{F}$, is combined with other vectors to yield:

$$\text{Work}: \; W = \mathbf{F}\cdot\mathbf{r}_{AB} \;\; ; \tag{1.41}$$

$$\text{Power}: \; P = \frac{dW}{dt} = \mathbf{F}\cdot\frac{d\mathbf{r}_{AB}}{dt} \;\; ; \tag{1.42}$$

$$\text{Moment}: \; \mathbf{M} \equiv \mathbf{r}\times\mathbf{F} \;\; . \tag{1.43}$$

In the first two expressions, the force vector, $\mathbf{F}$, is considered constant.

Example 1.2.4. Linear and angular momentum

The *linear momentum*, $\mathbf{J}$, of a body of mass m moving with velocity $\mathbf{u}$ is defined by

$\mathbf{J} \equiv m\mathbf{u}$. The net force $\mathbf{F}$ acting on the body is given by *Newton's law of motion*,

$$\mathbf{F} = \frac{d\mathbf{J}}{dt} = \frac{d}{dt}(m\mathbf{u}) . \tag{1.44}$$

If m is constant, then

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} = m\mathbf{a} , \tag{1.45}$$

where $\mathbf{a}$ is the linear acceleration of the body.

The *angular momentum* (or *moment of momentum*) is defined as

$$\mathbf{J}_\theta \equiv \mathbf{r} \times \mathbf{J} . \tag{1.46}$$

Therefore,

$$\frac{d\mathbf{J}_\theta}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{J}) = \frac{d\mathbf{r}}{dt} \times \mathbf{J} + \mathbf{r} \times \frac{d\mathbf{J}}{dt} = \mathbf{u} \times (m\mathbf{u}) + \mathbf{r} \times \mathbf{F} = \mathbf{0} + \mathbf{r} \times \mathbf{F} \quad \Longrightarrow$$

$$\frac{d\mathbf{J}_\theta}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{M} , \tag{1.47}$$

where the identity $\mathbf{u} \times \mathbf{u}=\mathbf{0}$ has been used. □

1.2.2 Unit Tangent and Normal Vectors

Consider a *smooth* surface S, i.e., a surface at each point of which a *tangent plane* can be defined. At each point of S, one can then define an orthonormal set consisting of two *unit tangent vectors*, $\mathbf{t}_1$ and $\mathbf{t}_2$, lying on the tangent plane, and a *unit normal vector*, $\mathbf{n}$, which is perpendicular to the tangent plane:

$$\mathbf{n} \cdot \mathbf{n} = \mathbf{t}_1 \cdot \mathbf{t}_1 = \mathbf{t}_2 \cdot \mathbf{t}_2 = 1 \quad \text{and} \quad \mathbf{n} \cdot \mathbf{t}_1 = \mathbf{t}_1 \cdot \mathbf{t}_2 = \mathbf{t}_2 \cdot \mathbf{n} = 0 .$$

Obviously, there are two choices for $\mathbf{n}$; the first is the vector

$$\frac{\mathbf{t}_1 \times \mathbf{t}_2}{|\mathbf{t}_1 \times \mathbf{t}_2|} ,$$

and the second one is just its opposite. Once one of these two vectors is chosen as the unit normal vector $\mathbf{n}$, the surface is said to be *oriented*; $\mathbf{n}$ is then called the *orientation* of the surface. In general, if the surface is the boundary of a control volume, we assume that $\mathbf{n}$ is positive when it points away from the system bounded by the surface.

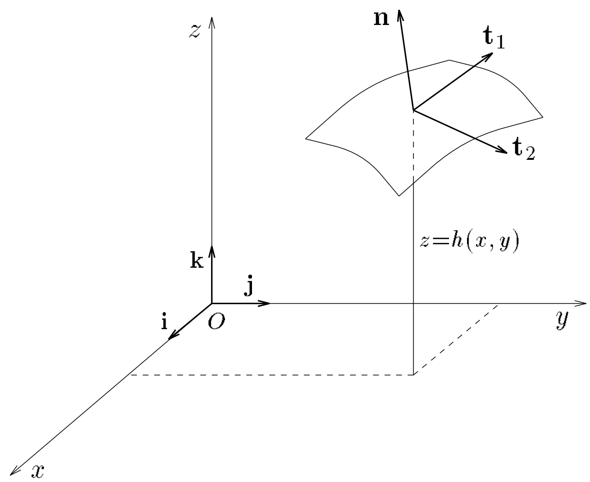

Figure 1.13. *Unit normal and tangent vectors to a surface defined by $z=h(x,y)$.*

The unit normal to a surface represented by

$$f(x,y,z) \;=\; z \;-\; h(x,y) \;=\; 0 \tag{1.48}$$

is given by

$$\mathbf{n} \;=\; \frac{\nabla f}{|\nabla f|} \qquad \Longrightarrow \tag{1.49}$$

$$\mathbf{n} \;=\; \frac{-\dfrac{\partial h}{\partial x}\,\mathbf{i} \;-\; \dfrac{\partial h}{\partial y}\,\mathbf{j} \;+\; \mathbf{k}}{\left[1 \;+\; \left(\dfrac{\partial h}{\partial x}\right)^2 \;+\; \left(\dfrac{\partial h}{\partial y}\right)^2\right]^{1/2}}\,. \tag{1.50}$$

Obviously, $\mathbf{n}$ is defined only if the gradient ∇f is defined and $|\nabla f| \neq 0$. Note that, from Eq. (1.50), it follows that the unit normal vector is considered positive when it is upward, i.e., when its z component is positive, as in Fig. 1.13. One can easily choose two orthogonal unit tangent vectors, $\mathbf{t}_1$ and $\mathbf{t}_2$, so that the set $\{\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2\}$ is orthonormal. Any vector field $\mathbf{u}$ can then be expanded as follows,

$$\mathbf{u} \;=\; u_n\mathbf{n} \;+\; u_{t1}\mathbf{t_1} \;+\; u_{t2}\mathbf{t_2} \tag{1.51}$$

where u_n is the *normal component*, and u_{t1} and u_{t2} are the *tangential components* of $\mathbf{u}$. The dot product $\mathbf{n}\cdot\mathbf{u}$ represents the normal component of $\mathbf{u}$, since

$$\mathbf{n}\cdot\mathbf{u} \;=\; \mathbf{n}\cdot(u_n\mathbf{n} \;+\; u_{t1}\mathbf{t_1} \;+\; u_{t2}\mathbf{t_2}) \;=\; u_n\,.$$

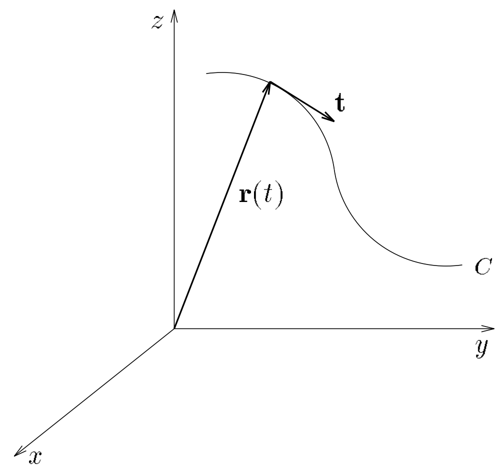

Figure 1.14. *The unit tangent vector to a curve.*

The integral of the normal component of $\mathbf{u}$ over the surface S,

$$Q \equiv \int_S \mathbf{n} \cdot \mathbf{u}\, dS \,, \tag{1.52}$$

is the *flux integral* or *flow rate* of $\mathbf{u}$ *across* S. In fluid mechanics, if $\mathbf{u}$ is the velocity vector, Q represents the *volumetric flow rate* across S. Setting $\mathbf{n}dS{=}d\mathbf{S}$, Eq. (1.52) takes the form

$$Q = \int_S \mathbf{u} \cdot d\mathbf{S} \,. \tag{1.53}$$

A curve C in the three dimensional space can be defined as the graph of the position vector $\mathbf{r}(t)$, as depicted in Fig. 1.14. The motion of $\mathbf{r}(t)$ with parameter t indicates which one of the two possible directions has been chosen as the positive direction to trace C. We already know that the derivative $d\mathbf{r}/dt$ is tangent to the curve C. Therefore, the unit tangent vector to the curve C is given by

$$\mathbf{t} = \frac{\dfrac{d\mathbf{r}}{dt}}{\left|\dfrac{d\mathbf{r}}{dt}\right|} \,, \tag{1.54}$$

and is defined only at those points where the derivative $d\mathbf{r}/dt$ exists and is not zero.

As an example, consider the plane curve of Fig. 1.15, defined by

$$y = h(x) \,, \tag{1.55}$$

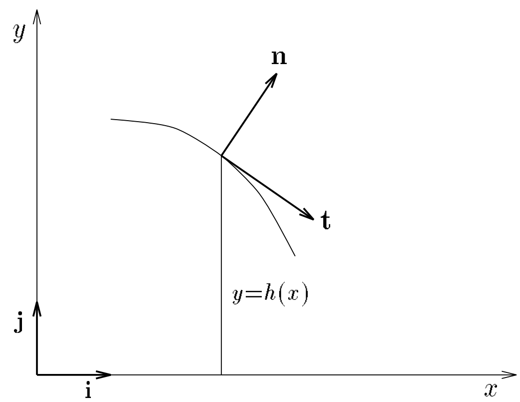

Figure 1.15. *Normal and tangent unit vectors to a plane curve defined by $y=h(x)$.*

or, equivalently, by $\mathbf{r}(t)=x\mathbf{i}+h(x)\mathbf{j}$. The unit tangent vector at a point of C is given by

$$\mathbf{t} \;=\; \frac{\dfrac{d\mathbf{r}}{dt}}{\left|\dfrac{d\mathbf{r}}{dt}\right|} \;=\; \frac{\mathbf{i} \,+\, \dfrac{\partial h}{\partial x}\,\mathbf{j}}{\left[1 \,+\, \left(\dfrac{\partial h}{\partial x}\right)^2\right]^{1/2}} \,. \tag{1.56}$$

By invoking the conditions $\mathbf{n}\cdot\mathbf{t}=0$ and $\mathbf{n}\cdot\mathbf{n}=1$, we find for the unit normal vector $\mathbf{n}$:

$$\mathbf{n} \;=\; \pm\, \frac{-\dfrac{\partial h}{\partial x}\,\mathbf{i} \,+\, \mathbf{j}}{\left[1 \,+\, \left(\dfrac{\partial h}{\partial x}\right)^2\right]^{1/2}} \,.$$

Choosing $\mathbf{n}$ to have positive y-component, as in Fig. 1.15, we get

$$\mathbf{n} \;=\; \frac{-\dfrac{\partial h}{\partial x}\,\mathbf{i} \,+\, \mathbf{j}}{\left[1 \,+\, \left(\dfrac{\partial h}{\partial x}\right)^2\right]^{1/2}} \,. \tag{1.57}$$

Note that the last expression for $\mathbf{n}$ can also be obtained from Eq. (1.50), as a degenerate case.

Let C be an arbitrary *closed* curve in the space, with the unit tangent vector $\mathbf{t}$ oriented in a specified direction, and $\mathbf{u}$ be a vector field. The integral

$$\Gamma \;\equiv\; \oint_C \mathbf{t}\cdot\mathbf{u}\, d\ell \,, \tag{1.58}$$

where ℓ is the *arc length* around C, is called the *circulation* of $\mathbf{u}$ along C. If $\mathbf{r}$ is the position vector, then $\mathbf{t}d\ell=d\mathbf{r}$, and Equation (1.58) is written as follows

$$\Gamma \equiv \oint_C \mathbf{u}\cdot d\mathbf{r}\,. \tag{1.59}$$

1.3 Tensors

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of the three dimensional space. This means that any vector $\mathbf{v}$ of this space can be uniquely expressed as a linear combination of the three coordinate directions $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$,

$$\mathbf{v} = \sum_{i=1}^{3} v_i\,\mathbf{e}_i\,, \tag{1.60}$$

where the scalars v_i are the components of $\mathbf{v}$.

In the previous sections, two kinds of products that can be formed with any two unit basis vectors were defined, i.e. the *dot product*, $\mathbf{e}_i\cdot\mathbf{e}_j$, and the *cross product*, $\mathbf{e}_i\times\mathbf{e}_j$. A third kind of product is the *dyadic product*, $\mathbf{e}_i\mathbf{e}_j$, also referred to as a *unit dyad*. The unit dyad $\mathbf{e}_i\mathbf{e}_j$ represents *an ordered pair of coordinate directions*, and thus $\mathbf{e}_i\mathbf{e}_j \neq \mathbf{e}_j\mathbf{e}_i$ unless $i=j$. The nine possible unit dyads,

$$\{\mathbf{e}_1\mathbf{e}_1,\ \mathbf{e}_1\mathbf{e}_2,\ \mathbf{e}_1\mathbf{e}_3,\ \mathbf{e}_2\mathbf{e}_1,\ \mathbf{e}_2\mathbf{e}_2,\ \mathbf{e}_2\mathbf{e}_3,\ \mathbf{e}_3\mathbf{e}_1,\ \mathbf{e}_3\mathbf{e}_2,\ \mathbf{e}_3\mathbf{e}_3\}\,,$$

form the basis of the space of *second-order tensors*. A second-order tensor, $\boldsymbol{\tau}$, can thus be written as a linear combination of the unit dyads:

$$\boldsymbol{\tau} = \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\,\mathbf{e}_i\mathbf{e}_j\,, \tag{1.61}$$

where the scalars τ_{ij} are referred to as the *components* of the tensor $\boldsymbol{\tau}$. Similarly, a third-order tensor can be defined as the linear combination of all possible *unit triads* $\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$, etc. Scalars can be viewed as zero-order tensors, and vectors as first-order tensors.

A tensor, $\boldsymbol{\tau}$, can be represented by means of a square matrix as

$$\boldsymbol{\tau} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\ \tau_{21} & \tau_{22} & \tau_{23}\\ \tau_{31} & \tau_{32} & \tau_{33}\end{bmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} \tag{1.62}$$

and often simply by the matrix of its components,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} . \tag{1.63}$$

Note that the equality sign "=" is loosely used, since $\boldsymbol{\tau}$ is a tensor and *not* a matrix. For a complete description of a tensor by means of Eq. (1.63), the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ should be provided.

The *unit* (or *identity*) *tensor*, $\mathbf{I}$, is defined by

$$\mathbf{I} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\, \mathbf{e}_i \mathbf{e}_j = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3 . \tag{1.64}$$

Each diagonal component of the matrix form of $\mathbf{I}$ is unity and the nondiagonal components are zero:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{1.65}$$

The sum of two tensors, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, is the tensor whose components are the sums of the corresponding components of the two tensors:

$$\boldsymbol{\sigma} + \boldsymbol{\tau} = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}\, \mathbf{e}_i \mathbf{e}_j + \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\, \mathbf{e}_i \mathbf{e}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} (\sigma_{ij} + \tau_{ij})\, \mathbf{e}_i \mathbf{e}_j . \tag{1.66}$$

The product of a tensor, $\boldsymbol{\tau}$, and a scalar, m, is the tensor whose components are equal to the components of $\boldsymbol{\tau}$ multiplied by m:

$$m\, \boldsymbol{\tau} = m \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}\, \mathbf{e}_i \mathbf{e}_j \right) = \sum_{i=1}^{3}\sum_{j=1}^{3} (m\tau_{ij})\, \mathbf{e}_i \mathbf{e}_j . \tag{1.67}$$

The *transpose*, $\boldsymbol{\tau}^T$, of a tensor $\boldsymbol{\tau}$ is defined by

$$\boldsymbol{\tau}^T \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ji}\, \mathbf{e}_i \mathbf{e}_j . \tag{1.68}$$

The matrix form of $\boldsymbol{\tau}^T$ is obtained by interchanging the rows and columns of the matrix form of $\boldsymbol{\tau}$:

$$\boldsymbol{\tau}^T = \begin{bmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} . \tag{1.69}$$

If $\boldsymbol{\tau}^T=\boldsymbol{\tau}$, i.e., if $\boldsymbol{\tau}$ is equal to its transpose, the tensor $\boldsymbol{\tau}$ is said to be *symmetric*. If $\boldsymbol{\tau}^T=-\boldsymbol{\tau}$, the tensor $\boldsymbol{\tau}$ is said to be *antisymmetric* (or *skew symmetric*). Any tensor $\boldsymbol{\tau}$ can be expressed as the sum of a symmetric, $\mathbf{S}$, and an antisymmetric tensor, $\mathbf{U}$,

$$\boldsymbol{\tau} = \mathbf{S} + \mathbf{U} , \tag{1.70}$$

where

$$\mathbf{S} = \frac{1}{2}(\boldsymbol{\tau} + \boldsymbol{\tau}^T) , \tag{1.71}$$

and

$$\mathbf{U} = \frac{1}{2}(\boldsymbol{\tau} - \boldsymbol{\tau}^T) . \tag{1.72}$$

The dyadic product of two vectors $\mathbf{a}$ and $\mathbf{b}$ can easily be constructed as follows:

$$\mathbf{ab} = \left(\sum_{i=1}^{3} a_i\,\mathbf{e}_i\right)\left(\sum_{j=1}^{3} b_j\,\mathbf{e}_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j\,\mathbf{e}_i\mathbf{e}_j . \tag{1.73}$$

Obviously, $\mathbf{ab}$ is a tensor, referred to as *dyad* or *dyadic tensor*. Its matrix form is

$$\mathbf{ab} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} . \tag{1.74}$$

Note that $\mathbf{ab} \neq \mathbf{ba}$ unless $\mathbf{ab}$ is symmetric. Given that $(\mathbf{ab})^T=\mathbf{ba}$, the dyadic product of a vector by itself, $\mathbf{aa}$, is symmetric.

The unit dyads $\mathbf{e}_i\mathbf{e}_j$ are dyadic tensors, the matrix form of which has only one unitary nonzero entry at the (i,j) position. For example,

$$\mathbf{e}_2\mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

The most important operations involving unit dyads are the following:

(i) The *single-dot product* (or *tensor product*) of two unit dyads is a tensor defined by

$$(\mathbf{e}_i\mathbf{e}_j)\cdot(\mathbf{e}_k\mathbf{e}_l) \equiv \mathbf{e}_i\,(\mathbf{e}_j\cdot\mathbf{e}_k)\,\mathbf{e}_l = \delta_{jk}\,\mathbf{e}_i\mathbf{e}_l . \tag{1.75}$$

This operation is *not* commutative.

(ii) The *double-dot product* (or *scalar product* or *inner product*) of two unit dyads is a scalar defined by

$$(\mathbf{e}_i\mathbf{e}_j):(\mathbf{e}_k\mathbf{e}_l) \;\equiv\; (\mathbf{e}_i\cdot\mathbf{e}_l)\,(\mathbf{e}_j\cdot\mathbf{e}_k) \;=\; \delta_{il}\delta_{jk}\,. \tag{1.76}$$

It is easily seen that this operation is commutative.

(iii) The dot product of a unit dyad and a unit vector is a vector defined by

$$(\mathbf{e}_i\mathbf{e}_j)\cdot\mathbf{e}_k \;\equiv\; \mathbf{e}_i\,(\mathbf{e}_j\cdot\mathbf{e}_k) \;=\; \delta_{jk}\,\mathbf{e}_i\,, \tag{1.77}$$

or

$$\mathbf{e}_i\cdot(\mathbf{e}_j\mathbf{e}_k) \;\equiv\; (\mathbf{e}_i\cdot\mathbf{e}_j)\,\mathbf{e}_k \;=\; \delta_{ij}\,\mathbf{e}_k\,. \tag{1.78}$$

Obviously, this operation is *not* commutative.

Operations involving tensors are easily performed by expanding the tensors into components with respect to a given basis and using the elementary unit dyad operations defined in Eqs. (1.75)-(1.78). The most important operations involving tensors are summarized below.

The single-dot product of two tensors

If $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are tensors, then

$$\begin{aligned}
\boldsymbol{\sigma}\cdot\boldsymbol{\tau} &= \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\,\mathbf{e}_i\mathbf{e}_j\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\tau_{kl}\,\mathbf{e}_k\mathbf{e}_l\right)\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sigma_{ij}\tau_{kl}\,(\mathbf{e}_i\mathbf{e}_j)\cdot(\mathbf{e}_k\mathbf{e}_l)\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sigma_{ij}\tau_{kl}\,\delta_{jk}\,\mathbf{e}_i\mathbf{e}_l\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\sigma_{ij}\tau_{jl}\,\mathbf{e}_i\mathbf{e}_l \qquad\Longrightarrow
\end{aligned}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} \;=\; \sum_{i=1}^{3}\sum_{l=1}^{3}\left(\sum_{j=1}^{3}\sigma_{ij}\tau_{jl}\right)\mathbf{e}_i\mathbf{e}_l\,. \tag{1.79}$$

The operation is not commutative. It is easily verified that

$$\boldsymbol{\sigma}\cdot\mathbf{I} \;=\; \mathbf{I}\cdot\boldsymbol{\sigma} \;=\; \boldsymbol{\sigma}\,. \tag{1.80}$$

A tensor $\boldsymbol{\tau}$ is said to be *invertible* if there exists a tensor $\boldsymbol{\tau}^{-1}$ such that

$$\boldsymbol{\tau}\cdot\boldsymbol{\tau}^{-1} \;=\; \boldsymbol{\tau}^{-1}\cdot\boldsymbol{\tau} \;=\; \mathbf{I}\,. \tag{1.81}$$

If $\boldsymbol{\tau}$ is invertible, then $\boldsymbol{\tau}^{-1}$ is unique and is called the *inverse* of $\boldsymbol{\tau}$.

The double-dot product of two tensors

$$\boldsymbol{\sigma}:\boldsymbol{\tau} \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\tau_{ji}\,\mathbf{e}_i\mathbf{e}_j\,. \tag{1.82}$$

The dot product of a tensor and a vector

This is a very useful operation in fluid mechanics. If $\mathbf{a}$ is a vector, we have:

$$\begin{aligned}
\boldsymbol{\sigma}\cdot\mathbf{a} \;&=\; \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\,\mathbf{e}_i\mathbf{e}_j\right)\cdot\left(\sum_{k=1}^{3}a_k\,\mathbf{e}_k\right)\\
&=\; \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sigma_{ij}a_k\,(\mathbf{e}_i\mathbf{e}_j)\cdot\mathbf{e}_k\\
&=\; \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sigma_{ij}a_k\,\delta_{jk}\,\mathbf{e}_i\\
&=\; \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}a_j\,\delta_{jj}\,\mathbf{e}_i \qquad\Longrightarrow
\end{aligned}$$

$$\boldsymbol{\sigma}\cdot\mathbf{a} \;=\; \sum_{i=1}^{3}\left(\sum_{j=1}^{3}\sigma_{ij}a_j\right)\,\mathbf{e}_i\,. \tag{1.83}$$

Similarly, we find that

$$\mathbf{a}\cdot\boldsymbol{\sigma} \;=\; \sum_{i=1}^{3}\left(\sum_{j=1}^{3}\sigma_{ji}a_j\right)\,\mathbf{e}_i\,. \tag{1.84}$$

The vectors $\boldsymbol{\sigma}\cdot\mathbf{a}$ and $\mathbf{a}\cdot\boldsymbol{\sigma}$ are not, in general, equal.

The following identities, in which $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are vectors, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are tensors, and $\mathbf{I}$ is the unit tensor, are easy to prove:

$$(\mathbf{ab})\cdot(\mathbf{cd}) \;=\; (\mathbf{b}\cdot\mathbf{c})\,\mathbf{ad}\,, \tag{1.85}$$

$$(\mathbf{ab}):(\mathbf{cd}) \;=\; (\mathbf{cd}):(\mathbf{ab}) \;=\; (\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c})\,, \tag{1.86}$$

$$(\mathbf{ab})\cdot\mathbf{c} \;=\; (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}\,, \tag{1.87}$$

$$\mathbf{c}\cdot(\mathbf{ab}) \;=\; (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}\,, \tag{1.88}$$
$$\mathbf{a}\cdot\mathbf{I} \;=\; \mathbf{I}\cdot\mathbf{a} \;=\; \mathbf{a}\,, \tag{1.89}$$
$$\boldsymbol{\sigma}:\mathbf{ab} \;=\; (\boldsymbol{\sigma}\cdot\mathbf{a})\cdot\mathbf{b}\,, \tag{1.90}$$
$$\mathbf{ab}:\boldsymbol{\sigma} \;=\; \mathbf{a}\cdot(\mathbf{b}\cdot\boldsymbol{\sigma})\,. \tag{1.91}$$

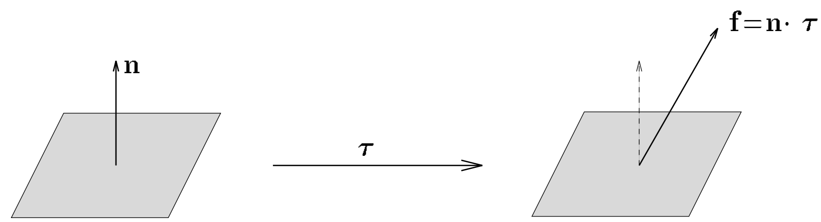

Figure 1.16. *The action of a tensor* $\boldsymbol{\tau}$ *on the normal vector* $\mathbf{n}$*.*

The vectors forming an orthonormal basis of the three-dimensional space are normal to three mutually perpendicular plane surfaces. If $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ is such a basis and $\mathbf{v}$ is a vector, then

$$\mathbf{v} \;=\; \mathbf{n}_1 v_1 \;+\; \mathbf{n}_2 v_2 \;+\; \mathbf{n}_3 v_3\,, \tag{1.92}$$

where v_1, v_2 and v_3 are the components of $\mathbf{v}$ in the coordinate system defined by $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$. Note that *a vector associates a scalar with each coordinate direction.* Since $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ is orthonormal,

$$v_1 \;=\; \mathbf{n}_1\cdot\mathbf{v}\,, \qquad v_2 \;=\; \mathbf{n}_2\cdot\mathbf{v} \quad \text{and} \quad v_3 \;=\; \mathbf{n}_3\cdot\mathbf{v}\,. \tag{1.93}$$

The component $v_i{=}\mathbf{n}_i\cdot\mathbf{v}$ might be viewed as *the result or flux produced by* $\mathbf{v}$ *through the surface* with unit normal $\mathbf{n}_i$, since the contributions of the other two components are tangent to that surface. Hence, the vector $\mathbf{v}$ is fully defined at a point by the *fluxes it produces through three mutually perpendicular infinitesimal surfaces.* We also note that a vector can be defined as an operator which produces a scalar flux when acting on an orientation vector.

Along these lines, a tensor can be conveniently defined as an operator of higher order that operates on an orientation vector and produces a *vector flux.* The action of a tensor $\boldsymbol{\tau}$ on the unit normal to a surface, $\mathbf{n}$, is illustrated in Fig. 1.16. The dot product $\mathbf{f}{=}\mathbf{n}\cdot\,\boldsymbol{\tau}$ is a vector that differs from $\mathbf{n}$ in both length and direction. If the vectors

$$\mathbf{f}_1 \;=\; \mathbf{n}_1\cdot\boldsymbol{\tau}\,, \qquad \mathbf{f}_2 \;=\; \mathbf{n}_2\cdot\boldsymbol{\tau} \quad \text{and} \quad \mathbf{f}_3 \;=\; \mathbf{n}_3\cdot\boldsymbol{\tau}\,, \tag{1.94}$$

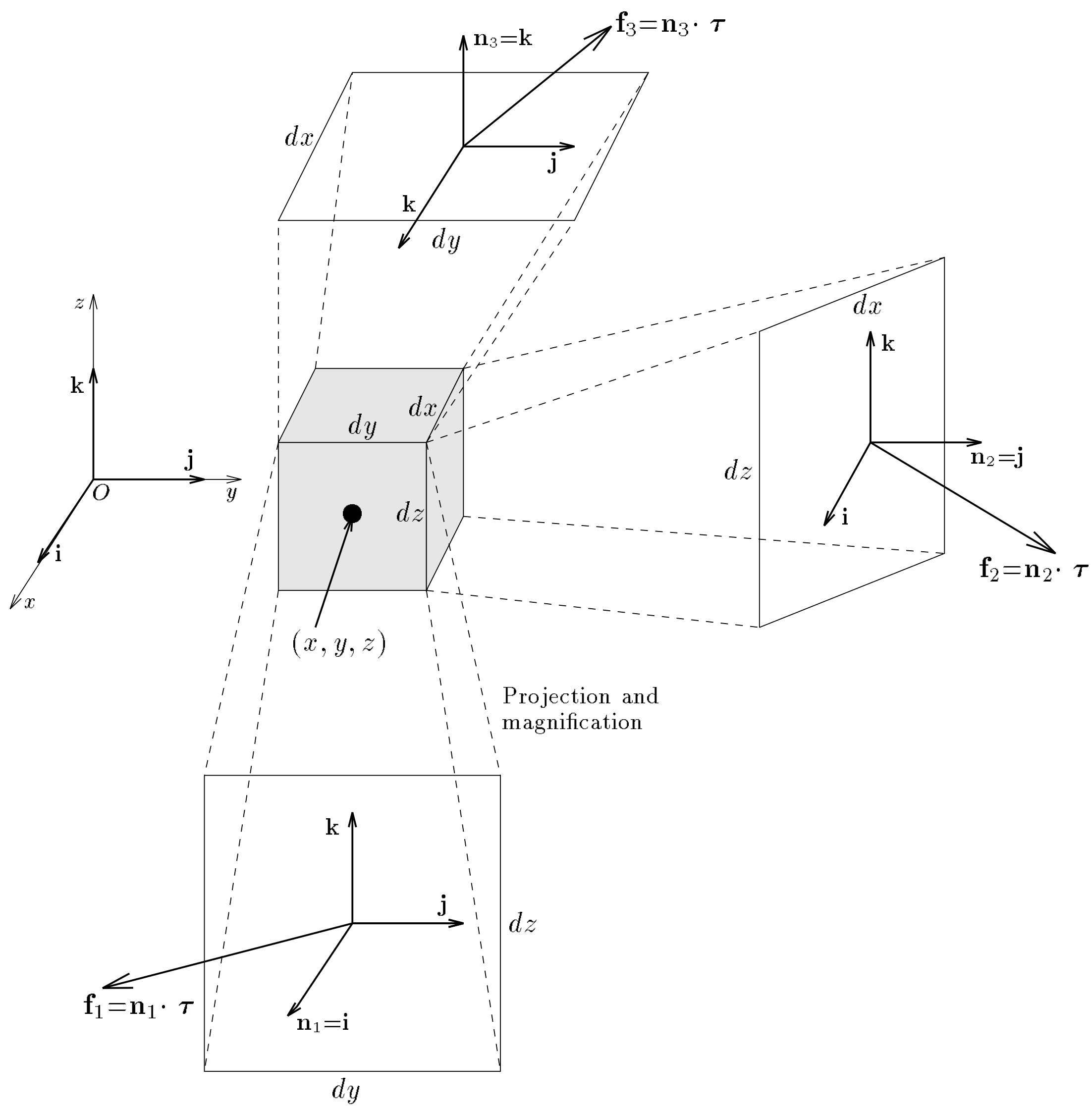

Figure 1.17. *Actions of a tensor* $\boldsymbol{\tau}$ *on three mutually perpendicular infinitesimal plane surfaces.*

are the actions of a tensor $\boldsymbol{\tau}$ on the unit normals $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ of three mutually perpendicular infinitesimal plane surfaces, as illustrated in Fig. 1.17, then $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} \;=\; \mathbf{n}_1\mathbf{f}_1 \;+\; \mathbf{n}_2\mathbf{f}_2 \;+\; \mathbf{n}_3\mathbf{f}_3 \,. \tag{1.95}$$

The tensor $\boldsymbol{\tau}$ is thus represented by the sum of three dyadic products. Note that *a second-order tensor associates a vector with each coordinate direction.* The vectors $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{f}_3$ can be further expanded into measurable scalar components,

$$\begin{aligned} \mathbf{f}_1 &= \tau_{11}\mathbf{n}_1 + \tau_{12}\mathbf{n}_2 + \tau_{13}\mathbf{n}_3 \,, \\ \mathbf{f}_2 &= \tau_{21}\mathbf{n}_1 + \tau_{22}\mathbf{n}_2 + \tau_{23}\mathbf{n}_3 \,, \\ \mathbf{f}_3 &= \tau_{31}\mathbf{n}_1 + \tau_{32}\mathbf{n}_2 + \tau_{33}\mathbf{n}_3 \,. \end{aligned} \tag{1.96}$$

The scalars that appear in Eq. (1.96) are obviously the components of $\boldsymbol{\tau}$ with respect to the system of coordinates defined by $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$:

$$\boldsymbol{\tau} \;=\; \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \,. \tag{1.97}$$

The diagonal elements are the components of the *normal* on each of the three mutually perpendicular surfaces; the nondiagonal elements are the magnitudes of the two *tangential* or *shear* actions or fluxes on each of the three surfaces.

The most common tensor in fluid mechanics is the *stress tensor*, $\mathbf{T}$, which, when acting on a surface of unit normal $\mathbf{n}$, produces *surface stress* or *traction*,

$$\mathbf{f} \;=\; \mathbf{n}\cdot\mathbf{T} \,. \tag{1.98}$$

The traction $\mathbf{f}$ is the force per unit area exerted on an infinitesimal surface element. It can be decomposed into a normal component $\mathbf{f}_N$ that points in the direction of $\mathbf{n}$, and a tangential or shearing component $\mathbf{f}_T$:

$$\mathbf{f} \;=\; \mathbf{f}_N \;+\; \mathbf{f}_T \,. \tag{1.99}$$

The *normal traction*, $\mathbf{f}_N$, is given by

$$\mathbf{f}_N \;=\; (\mathbf{n}\cdot\mathbf{f})\,\mathbf{n} \;=\; \mathbf{n}\cdot(\mathbf{n}\cdot\mathbf{T})\,\mathbf{n} \;=\; (\mathbf{nn}:\mathbf{T})\,\mathbf{n} \,, \tag{1.100}$$

and, therefore, for the *tangetial traction* we obtain:

$$\mathbf{f}_T \;=\; \mathbf{f} - \mathbf{f}_N \;=\; \mathbf{n}\cdot\mathbf{T} \;-\; (\mathbf{nn}:\mathbf{T})\,\mathbf{n} \,. \tag{1.101}$$

It is left to the reader to show that the above equation is equivalent to

$$\mathbf{f}_T \;=\; \mathbf{n}\times(\mathbf{f}\times\mathbf{n}) \;=\; \mathbf{f}\cdot(\mathbf{I}-\mathbf{nn})\,, \tag{1.102}$$

where $\mathbf{I}$ is the unit tensor.

Example 1.3.1. Vector-tensor operations[1]

Consider the Cartesian system of coordinates and the point $\mathbf{r} = \sqrt{3}\mathbf{j}$ m. Measurements of force per unit area on a small test surface give the following time-independent results:

Direction in which the test surface faces	Measured traction on the test surface (force/area)
$\mathbf{i}$	$2\,\mathbf{i}+\mathbf{j}$
$\mathbf{j}$	$\mathbf{i}+4\,\mathbf{j}+\mathbf{k}$
$\mathbf{k}$	$\mathbf{j}+6\,\mathbf{k}$

(a) What is the state of stress at the point $\mathbf{r} = \sqrt{3}\,\mathbf{j}$?

(b) What is the traction on the test surface when it is oriented to face in the direction $\mathbf{n} = (1/\sqrt{3})(\mathbf{i}+\mathbf{j}+\mathbf{k})$?

(c) What is the moment of the traction found in Part (b)?

Solution:
(a) Let

$$\mathbf{n}_1 = \mathbf{i}\,, \quad \mathbf{n}_2 = \mathbf{j}\,, \quad \mathbf{n}_3 = \mathbf{k}\,,$$

$$\mathbf{f}_1 = 2\mathbf{i}+\mathbf{j}\,, \quad \mathbf{f}_2 = \mathbf{i}+4\mathbf{j}+\mathbf{k} \quad \text{and} \quad \mathbf{f}_3 = \mathbf{j}+6\mathbf{k}\,.$$

The stress tensor, $\mathbf{T}$, is given by

$$\begin{aligned}
\mathbf{T} \;&=\; \mathbf{n}_1\mathbf{f}_1 + \mathbf{n}_2\mathbf{f}_2 + \mathbf{n}_3\mathbf{f}_3 \\
&=\; \mathbf{i}(2\mathbf{i}+\mathbf{j}) + \mathbf{j}(\mathbf{i}+4\mathbf{j}+\mathbf{k}) + \mathbf{k}(\mathbf{j}+6\mathbf{k}) \\
&=\; 2\mathbf{ii} + \mathbf{ij} + 0\mathbf{ik} + \mathbf{ji} + 4\mathbf{jj} + \mathbf{jk} + 0\mathbf{ki} + \mathbf{kj} + 6\mathbf{kk}
\end{aligned}$$

The matrix representation of $\mathbf{T}$ with respect to the basis $(\mathbf{i},\mathbf{j},\mathbf{k})$ is

$$\mathbf{T} \;=\; \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 6 \end{bmatrix}\,.$$

[1]Taken from Ref. [2].

Notice that $\mathbf{T}$ is symmetric.
(b) The traction $\mathbf{f}$ on the surface $\mathbf{n}$ is given by

$$\mathbf{f} = \mathbf{n}\cdot\mathbf{T} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(2\mathbf{ii}+\mathbf{ij}+\mathbf{ji}+4\mathbf{jj}+\mathbf{jk}+\mathbf{kj}+6\mathbf{kk}) = \frac{1}{\sqrt{3}}(3\mathbf{i}+6\mathbf{j}+7\mathbf{k}).$$

(c) The moment of the traction at the point $\mathbf{r} = \sqrt{3}\,\mathbf{j}$ is a vector given by

$$\mathbf{M} = \mathbf{r}\times\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \sqrt{3} & 0 \\ \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{3}} & \frac{7}{\sqrt{3}} \end{vmatrix} = 7\mathbf{i} - 3\mathbf{k}\,.$$

□

Example 1.3.2. Normal and tangential tractions

Consider the state of stress given in Example 1.3.1. The normal and tangential components of the traction $\mathbf{f}_1$ are:

$$\mathbf{f}_{1N} = (\mathbf{n}_1\cdot\mathbf{f}_1)\,\mathbf{n}_1 = \mathbf{i}\cdot(2\mathbf{i}+\mathbf{j})\,\mathbf{i} = 2\mathbf{i}$$

and

$$\mathbf{f}_{1T} = \mathbf{f} - \mathbf{f}_{1N} = (2\mathbf{i}+\mathbf{j}) - 2\mathbf{i} = \mathbf{j}\,,$$

respectively. Similarly, for the tractions on the other two surfaces, we get:

$$\mathbf{f}_{2N} = 4\mathbf{j}\,, \qquad \mathbf{f}_{2T} = \mathbf{i}+\mathbf{k}\,;$$

$$\mathbf{f}_{3N} = 6\mathbf{k}\,, \qquad \mathbf{f}_{3T} = \mathbf{j}\,.$$

Note that the normal tractions on the three surfaces are exactly the diagonal elements of the component matrix

$$\mathbf{T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 6 \end{bmatrix}.$$

The nondiagonal elements of each line are the components of the corresponding tangential traction.

□

1.3.1 Principal Directions and Invariants

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of the three dimensional space and $\boldsymbol{\tau}$ be a second-order tensor,

$$\boldsymbol{\tau} \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\,\mathbf{e}_i\mathbf{e}_j\;, \tag{1.103}$$

or, in matrix notation,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} . \tag{1.104}$$

If certain conditions are satisfied, it is possible to identify an orthonormal basis $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ such that

$$\boldsymbol{\tau} \;=\; \lambda_1\,\mathbf{n}_1\mathbf{n}_1 \;+\; \lambda_2\,\mathbf{n}_2\mathbf{n}_2 \;+\; \lambda_3\,\mathbf{n}_3\mathbf{n}_3\;, \tag{1.105}$$

which means that the matrix form of $\boldsymbol{\tau}$ in the coordinate system defined by the new basis is diagonal:

$$\boldsymbol{\tau} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} . \tag{1.106}$$

The orthogonal vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ that *diagonalize* $\boldsymbol{\tau}$ are called the *principal directions*, and λ_1, λ_2 and λ_3 are called the *principal values* of $\boldsymbol{\tau}$. From Eq. (1.105), one observes that the vector fluxes through the surface of unit normal $\mathbf{n}_i$, i=1,2,3, satisfy the relation

$$\mathbf{f}_i \;=\; \mathbf{n}_i\cdot\boldsymbol{\tau} \;=\; \boldsymbol{\tau}\cdot\mathbf{n}_i \;=\; \lambda_i\mathbf{n}_i\;, \qquad i=1,2,3\,. \tag{1.107}$$

What the above equation says is that the vector flux through the surface with unit normal $\mathbf{n}_i$ is collinear with $\mathbf{n}_i$, i.e., $\mathbf{n}_i\cdot\,\boldsymbol{\tau}$ is normal to that surface and its tangential component is zero. From Eq. (1.107) one gets:

$$(\boldsymbol{\tau} - \lambda_i\mathbf{I})\;\cdot\;\mathbf{n}_i \;=\; \mathbf{0}\;, \tag{1.108}$$

where $\mathbf{I}$ is the unit tensor.

In mathematical terminology, Eq. (1.108) defines an *eigenvalue problem.* The principal directions and values of $\boldsymbol{\tau}$ are thus also called the *eigenvectors* and *eigenvalues* of $\boldsymbol{\tau}$, respectively. The eigenvalues are determined by solving the *characteristic equation*,

$$\det(\boldsymbol{\tau} - \lambda\mathbf{I}) \;=\; 0 \tag{1.109}$$

or

$$\begin{vmatrix} \tau_{11}-\lambda & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22}-\lambda & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33}-\lambda \end{vmatrix} = 0\,, \tag{1.110}$$

which guarantees nonzero solutions to the homogeneous system (1.108). The characteristic equation is a cubic equation and, therefore, it has three roots, λ_i, i=1,2,3. After determining an eigenvalue λ_i, one can determine the eigenvectors, $\mathbf{n}_i$, associated with λ_i by solving the *characteristic system* (1.108). When the tensor (or matrix) $\boldsymbol{\tau}$ is symmetric, all eigenvalues and the associated eigenvectors are real. This is the case with most tensors arising in fluid mechanics.

Example 1.3.3. Principal values and directions

(a) Find the principal values of the tensor

$$\boldsymbol{\tau} = \begin{bmatrix} x & 0 & z \\ 0 & 2y & 0 \\ z & 0 & x \end{bmatrix}.$$

(b) Determine the principal directions $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ at the point (0,1,1).

(c) Verify that the vector flux through a surface normal to a principal direction $\mathbf{n}_i$ is collinear with $\mathbf{n}_i$.

(d) What is the matrix form of the tensor $\boldsymbol{\tau}$ in the coordinate system defined by $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$?

Solution:
(a) The characteristic equation of $\boldsymbol{\tau}$ is

$$0 = \det(\boldsymbol{\tau}-\lambda\mathbf{I}) = \begin{vmatrix} x-\lambda & 0 & z \\ 0 & 2y-\lambda & 0 \\ z & 0 & x-\lambda \end{vmatrix} = (2y-\lambda)\begin{vmatrix} x-\lambda & z \\ z & x-\lambda \end{vmatrix} \implies$$

$$(2y-\lambda)\,(x-\lambda-z)\,(x-\lambda+z) = 0.$$

The eigenvalues of $\boldsymbol{\tau}$ are $\lambda_1=2y$, $\lambda_2=x-z$ and $\lambda_3=x+z$.
(b) At the point $(0,1,1)$,

$$\boldsymbol{\tau} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \mathbf{ik} + 2\mathbf{jj} + \mathbf{ki}\,,$$

and λ_1=2, λ_2=−1 and λ_3=1. The associated eigenvectors are determined by solving the corresponding characteristic system:

$$(\boldsymbol{\tau} - \lambda_i \mathbf{I}) \cdot \mathbf{n}_i = \mathbf{0}, \quad i = 1, 2, 3.$$

For λ_1=2, one gets

$$\begin{bmatrix} 0-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 0-2 \end{bmatrix} \begin{bmatrix} n_{x1} \\ n_{y1} \\ n_{z1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \left.\begin{aligned} -2n_{x1} + n_{z1} &= 0 \\ 0 &= 0 \\ n_{x1} - 2n_{z1} &= 0 \end{aligned}\right\} \quad \Longrightarrow$$

$$n_{x1} = n_{z1} = 0.$$

Therefore, the eigenvectors associated with λ_1 are of the form $(0, a, 0)$, where a is an arbitrary nonzero constant. For a=1, the eigenvector is *normalized*, i.e. it is of unit magnitude. We set

$$\mathbf{n}_1 = (0, 1, 0) = \mathbf{j}.$$

Similarly, solving the characteristic systems

$$\begin{bmatrix} 0+1 & 0 & 1 \\ 0 & 2+1 & 0 \\ 1 & 0 & 0+1 \end{bmatrix} \begin{bmatrix} n_{x2} \\ n_{y2} \\ n_{z2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

of λ_2=−1, and

$$\begin{bmatrix} 0-1 & 0 & 1 \\ 0 & 2-1 & 0 \\ 1 & 0 & 0-1 \end{bmatrix} \begin{bmatrix} n_{x3} \\ n_{y3} \\ n_{z3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

of λ_3=1, we find the normalized eigenvectors

$$\mathbf{n}_2 = \frac{1}{\sqrt{2}}(1, 0, -1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$$

and

$$\mathbf{n}_3 = \frac{1}{\sqrt{2}}(1, 0, 1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k}).$$

We observe that the three eigenvectors, $\mathbf{n}_1$ $\mathbf{n}_2$ and $\mathbf{n}_3$ are orthogonal:[2]

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \mathbf{n}_2 \cdot \mathbf{n}_3 = \mathbf{n}_3 \cdot \mathbf{n}_1 = 0.$$

[2] A well known result of linear algebra is that the eigenvectors associated with distinct eigenvalues of a symmetric matrix are orthogonal. If two eigenvalues are the same, then the two linearly independent eigenvectors determined by solving the corresponding characteristic system may not be orthogonal. From these two eigenvectors, however, a pair of orthogonal eigenvectors can be obtained using the *Gram-Schmidt orthogonalization process*; see, for example, [3].

(c) The vector fluxes through the three surfaces normal to $\mathbf{n}_1$ $\mathbf{n}_2$ and $\mathbf{n}_3$ are:

$$\begin{aligned}
\mathbf{n}_1 \cdot \boldsymbol{\tau} &= \mathbf{j} \cdot (\mathbf{ik} + 2\mathbf{jj} + \mathbf{ki}) = 2\mathbf{j} = 2\,\mathbf{n}_1\,, \\
\mathbf{n}_2 \cdot \boldsymbol{\tau} &= \frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{k}) \cdot (\mathbf{ik} + 2\mathbf{jj} + \mathbf{ki}) = \frac{1}{\sqrt{2}}(\mathbf{k}-\mathbf{i}) = -\mathbf{n}_2\,, \\
\mathbf{n}_3 \cdot \boldsymbol{\tau} &= \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k}) \cdot (\mathbf{ik} + 2\mathbf{jj} + \mathbf{ki}) = \frac{1}{\sqrt{2}}(\mathbf{k}+\mathbf{i}) = \mathbf{n}_3\,.
\end{aligned}$$

(d) The matrix form of $\boldsymbol{\tau}$ in the coordinate system defined by $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ is

$$\boldsymbol{\tau} = 2\mathbf{n}_1\mathbf{n}_1 - \mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

The *trace*, $tr\boldsymbol{\tau}$, of a tensor $\boldsymbol{\tau}$ is defined by

$$tr\boldsymbol{\tau} \equiv \sum_{i=1}^{3} \tau_{ii} = \tau_{11} + \tau_{22} + \tau_{33}\,. \tag{1.111}$$

An interesting observation for the tensor $\boldsymbol{\tau}$ of Example 1.3.3 is that its trace is the same (equal to 2) in both coordinate systems defined by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$. Actually, it can be shown that the trace of a tensor is independent of the coordinate system to which its components are referred. Such quantities are called *invariants* of a tensor.[3] There are three independent invariants of a second-order tensor $\boldsymbol{\tau}$:

$$I \equiv tr\boldsymbol{\tau} = \sum_{i=1}^{3} \tau_{ii}\,, \tag{1.112}$$

$$II \equiv tr\boldsymbol{\tau}^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\tau_{ji}\,, \tag{1.113}$$

$$III \equiv tr\boldsymbol{\tau}^3 = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \tau_{ij}\tau_{jk}\tau_{ki}\,, \tag{1.114}$$

where $\boldsymbol{\tau}^2 = \boldsymbol{\tau} \cdot \boldsymbol{\tau}$ and $\boldsymbol{\tau}^3 = \boldsymbol{\tau} \cdot \boldsymbol{\tau}^2$. Other invariants can be formed by simply taking combinations of I, II and III. Another common set of independent invariants is the

[3] From a vector $\mathbf{v}$, only one independent invariant can be constructed. This is the magnitude $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ of $\mathbf{v}$.

following:

$$I_1 \;=\; I \;=\; tr\boldsymbol{\tau}\,, \tag{1.115}$$

$$I_2 \;=\; \frac{1}{2}\,(I^2 - II) \;=\; \frac{1}{2}\,[(tr\boldsymbol{\tau})^2 - tr\boldsymbol{\tau}^2]\,, \tag{1.116}$$

$$I_3 \;=\; \frac{1}{6}\,(I^3 - 3I\;\,II + 2III) \;=\; \det\boldsymbol{\tau}\,. \tag{1.117}$$

I_1, I_2 and I_3 are called *basic invariants* of $\boldsymbol{\tau}$. The characteristic equation of $\boldsymbol{\tau}$ can be written as[4]

$$\lambda^3 \;-\; I_1\lambda^2 \;+\; I_2\lambda \;-\; I_3 \;=\; 0\,. \tag{1.118}$$

If λ_1, λ_2 and λ_3 are the eigenvalues of $\boldsymbol{\tau}$, the following identities hold:

$$I_1 \;=\; \lambda_1 + \lambda_2 + \lambda_3 \;=\; tr\boldsymbol{\tau}\,, \tag{1.119}$$

$$I_2 \;=\; \lambda_1\lambda_2 \;+\; \lambda_2\lambda_3 \;+\; \lambda_3\lambda_1 \;=\; \frac{1}{2}\,[(tr\boldsymbol{\tau})^2 - tr\boldsymbol{\tau}^2]\,, \tag{1.120}$$

$$I_3 \;=\; \lambda_1\lambda_2\lambda_3 \;=\; \det\boldsymbol{\tau}\,. \tag{1.121}$$

The theorem of Cayley-Hamilton states that a square matrix (or a tensor) is a root of its characteristic equation, i.e.,

$$\boldsymbol{\tau}^3 \;-\; I_1\boldsymbol{\tau}^2 \;+\; I_2\boldsymbol{\tau} \;-\; I_3\,\mathbf{I} \;=\; \mathbf{O}\,. \tag{1.122}$$

Note that in the last equation, the boldface quantities $\mathbf{I}$ and $\mathbf{O}$ are the unit and zero tensors, respectively. As implied by its name, the *zero tensor* is the tensor whose all components are zero.

Example 1.3.4. The first invariant

Consider the tensor

$$\boldsymbol{\tau} \;=\; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \;=\; \mathbf{ik} \;+\; 2\mathbf{jj} \;+\; \mathbf{ki}\,,$$

encountered in Example 1.3.3. Its first invariant is

$$I \;\equiv\; tr\boldsymbol{\tau} \;=\; 0 + 2 + 0 \;=\; 2\,.$$

[4] The component matrices of a tensor in two different coordinate systems are *similar*. An important property of similar matrices is that they have the same characteristic polynomial; hence, the coefficients I_1, I_2 and I_3 and the eigenvalues λ_1, λ_2 and λ_3 are invariant under a change of coordinate system.

Verify that the value of I is the same in cylindrical coordinates.

Solution:
Using the relations of Table 1.1, we have

$$\begin{aligned}
\boldsymbol{\tau} &= \mathbf{ik} + 2\mathbf{jj} + \mathbf{ki} \\
&= (\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta)\,\mathbf{e}_z + 2\,(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta)\,(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) \\
&\quad + \mathbf{e}_z\,(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) \\
&= 2\,\sin^2\theta\,\mathbf{e}_r\mathbf{e}_r + 2\,\sin\theta\,\cos\theta\,\mathbf{e}_r\mathbf{e}_\theta + \cos\theta\,\mathbf{e}_r\mathbf{e}_z + \\
&\quad 2\,\sin\theta\,\cos\theta\,\mathbf{e}_\theta\mathbf{e}_r + 2\,\cos^2\theta\,\mathbf{e}_\theta\mathbf{e}_\theta - \sin\theta\,\mathbf{e}_\theta\mathbf{e}_z + \\
&\quad \cos\theta\,\mathbf{e}_z\mathbf{e}_r - \sin\theta\,\mathbf{e}_z\mathbf{e}_\theta + 0\,\mathbf{e}_z\mathbf{e}_z\,.
\end{aligned}$$

Therefore, the component matrix of $\boldsymbol{\tau}$ in cylindrical coordinates $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is

$$\boldsymbol{\tau} = \begin{bmatrix} 2\,\sin^2\theta & 2\,\sin\theta\,\cos\theta & \cos\theta \\ 2\,\sin\theta\,\cos\theta & 2\,\cos^2\theta & -\sin\theta \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}.$$

Notice that $\boldsymbol{\tau}$ remains symmetric. Its first invariant is

$$I = tr\boldsymbol{\tau} = 2\left(\sin^2\theta + \cos^2\theta\right) + 0 = 2\,,$$

as it should be. □

1.3.2 Index Notation and Summation Convention

So far, we have used three different ways for representing tensors and vectors:
(a) the compact *symbolic notation*, e.g., $\mathbf{u}$ for a vector and $\boldsymbol{\tau}$ for a tensor;
(b) the so-called *Gibbs' notation*, e.g.,

$$\sum_{i=1}^{3} u_i\,\mathbf{e}_i \quad \text{and} \quad \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\,\mathbf{e}_i\mathbf{e}_j$$

for $\mathbf{u}$ and $\boldsymbol{\tau}$, respectively; and
(c) the *matrix notation*, e.g.,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

for $\boldsymbol{\tau}$.

Very frequently, in the literature, use is made of the *index notation* and the so-called *Einstein's summation convention*, in order to simplify expressions involving vector and tensor operations by omitting the summation symbols.

In index notation, a vector $\mathbf{v}$ is represented as

$$v_i \;\equiv\; \sum_{i=1}^{3} v_i\,\mathbf{e}_i \;=\; \mathbf{v}\,. \tag{1.123}$$

A tensor $\boldsymbol{\tau}$ is represented as

$$\tau_{ij} \;\equiv\; \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\,\mathbf{e}_i\,\mathbf{e}_j \;=\; \boldsymbol{\tau}\,. \tag{1.124}$$

The nabla operator, for example, is represented as

$$\frac{\partial}{\partial x_i} \;\equiv\; \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\mathbf{e}_i \;=\; \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \;=\; \nabla\,, \tag{1.125}$$

where x_i is the general Cartesian coordinate taking on the values of x, y and z. The unit tensor $\mathbf{I}$ is represented by Kronecker's delta:

$$\delta_{ij} \;\equiv\; \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\,\mathbf{e}_i\,\mathbf{e}_j \;=\; \mathbf{I}\,. \tag{1.126}$$

It is evident that an explicit statement must be made when the tensor τ_{ij} is to be distinguished from its (i,j) element.

With Einstein's summation convention, if an index appears twice in an expression, then summation is implied with respect to the *repeated* index, over the range of that index. The number of the *free indices*, i.e., the indices that appear only once, is the number of directions associated with an expression; it thus determines whether an expression is a scalar, a vector or a tensor. In the following expressions, there are no free indices, and thus these are scalars:

$$u_i v_i \;\equiv\; \sum_{i=1}^{3} u_i v_i \;=\; \mathbf{u}\cdot\mathbf{v}\,, \tag{1.127}$$

$$\tau_{ii} \;\equiv\; \sum_{i=1}^{3} \tau_{ii} \;=\; tr\boldsymbol{\tau}\,, \tag{1.128}$$

$$\frac{\partial u_i}{\partial x_i} \equiv \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \nabla \cdot \mathbf{u}\,, \tag{1.129}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_i} \text{ or } \frac{\partial^2 f}{\partial x_i^2} \equiv \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f\,, \tag{1.130}$$

where ∇^2 is the *Laplacian operator* to be discussed in more detail in Section 1.4. In the following expression, there are two sets of double indices, and summation must be performed over both sets:

$$\sigma_{ij}\tau_{ji} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}\tau_{ji} = \boldsymbol{\sigma} : \boldsymbol{\tau}\,. \tag{1.131}$$

The following expressions, with one free index, are vectors:

$$\epsilon_{ijk} u_i v_j \equiv \sum_{k=1}^{3} \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \epsilon_{ijk} u_i v_j \right) \mathbf{e}_k = \mathbf{u} \times \mathbf{v}\,, \tag{1.132}$$

$$\frac{\partial f}{\partial x_i} \equiv \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \mathbf{e}_i = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = \nabla f\,, \tag{1.133}$$

$$\tau_{ij} v_j \equiv \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \tau_{ij} v_j \right) \mathbf{e}_i = \boldsymbol{\tau} \cdot \mathbf{v}\,. \tag{1.134}$$

Finally, the following quantities, having two free indices, are tensors:

$$u_i v_j \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j \,\mathbf{e}_i \mathbf{e}_j = \mathbf{u}\mathbf{v}\,, \tag{1.135}$$

$$\sigma_{ik}\tau_{kj} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} \left(\sum_{k=1}^{3} \sigma_{ik}\tau_{kj} \right) \mathbf{e}_i \mathbf{e}_j = \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\,, \tag{1.136}$$

$$\frac{\partial u_j}{\partial x_i} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i} \mathbf{e}_i \mathbf{e}_j = \nabla \mathbf{u}\,. \tag{1.137}$$

Note that $\nabla \mathbf{u}$ in the last equation is a dyadic tensor.[5]

[5] Some authors use even simpler expressions for the nabla operator. For example, $\nabla \cdot \mathbf{u}$ is also represented as $\partial_i u_i$ or $u_{i,i}$, with a comma to indicate the derivative, and the dyadic ∇u is represented as $\partial_i u_j$ or $u_{i,j}$.

1.3.3 Tensors in Fluid Mechanics

Flows in the physical world are three dimensional, and so are the tensors involved in the governing equations. Many flow problems, however, are often approximated as two- or even one-dimensional, in which cases, the involved tensors and vectors degenerate to two- or one-dimensional forms. In this subsection, we give only a brief description of the most important tensors in fluid mechanics. More details are given in following chapters.

The *stress tensor*, $\mathbf{T}$, represents the state of the stress in a fluid. When operating on a surface, $\mathbf{T}$ produces a traction $\mathbf{f}{=}\mathbf{n}{\cdot}\mathbf{T}$, where $\mathbf{n}$ is the unit normal to the surface. In static equilibrium, the stress tensor is identical to the *hydrostatic pressure tensor*,

$$\mathbf{T}^{SE} = -p_H\,\mathbf{I}\,, \tag{1.138}$$

where p_H is the scalar hydrostatic pressure. The traction on any submerged surface is given by

$$\mathbf{f}^{SE} \;=\; \mathbf{n}\cdot\mathbf{T}^{SE} \;=\; \mathbf{n}\cdot(-p_H\,\mathbf{I}) \;=\; -p_H\,\mathbf{n}\,, \tag{1.139}$$

and is normal to the surface; its magnitude is identical to the hydrostatic pressure:

$$|\mathbf{f}^{SE}| \;=\; |-p_H\,\mathbf{n}| \;=\; p_H\,.$$

Since the resulting traction is independent of the orientation of the surface, the pressure tensor is *isotropic*, i.e., its components are unchanged by rotation of the frame of reference.

In flowing incompressible media, the stress tensor consists of an isotropic or pressure part, which is, in general, different from the hydrostatic pressure tensor, and an *anisotropic* or *viscous part*, which resists relative motion:[6]

$$\begin{array}{ccccc} \mathbf{T} & = & -p\,\mathbf{I} & + & \boldsymbol{\tau} \\ \\ \left[\begin{array}{c} \textit{Total} \\ \textit{Stress} \end{array}\right] & = & \left[\begin{array}{c} \textit{Isotropic} \\ \textit{Pressure} \\ \textit{Stress} \end{array}\right] & + & \left[\begin{array}{c} \textit{Anisotropic} \\ \textit{Viscous} \\ \textit{Stress} \end{array}\right] \end{array} \tag{1.140}$$

The viscous stress tensor $\boldsymbol{\tau}$ is, of course, zero in static equilibrium. It is, in general, anisotropic, i.e., the viscous traction on a surface depends on its orientation: it

[6] In some books (e.g., in [4] and [9]), a different sign convention is adopted for the total stress tensor $\mathbf{T}$, so that

$$\mathbf{T} \,=\, p\,\mathbf{I} \,-\, \tau\,.$$

An interesting discussion about the two sign conventions can be found in [9].

can be normal, shear (i.e., tangential) or mixture of the two. In matrix notation, Eq. (1.140) becomes

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}, \tag{1.141}$$

and, in index notation,

$$T_{ij} = -p\,\delta_{ij} + \tau_{ij}\,. \tag{1.142}$$

The diagonal components, T_{ii}, of $\mathbf{T}$ are *normal stresses*, and the nondiagonal ones are *shear stresses*.

Equation (1.140) is the standard decomposition of the stress tensor, inasmuch as the measurable quantities are, in general, the total stress components T_{ij} and not p or τ_{ij}. For educational purposes, the following decomposition appears to be more illustrative:

$$\begin{array}{ccccccccc} \mathbf{T} & = & -p_H\,\mathbf{I} & & -\,p_E\,\mathbf{I} & & +\,\boldsymbol{\tau}^N & & +\,\boldsymbol{\tau}^{SH} \\ \begin{bmatrix} \textit{Total} \\ \textit{Stress} \end{bmatrix} & = & \begin{bmatrix} \textit{Hydrostatic} \\ \textit{Pressure} \\ \textit{Stress} \end{bmatrix} & + & \begin{bmatrix} \textit{Extra} \\ \textit{Pressure} \\ \textit{Stress} \end{bmatrix} & + & \begin{bmatrix} \textit{Viscous} \\ \textit{Normal} \\ \textit{Stress} \end{bmatrix} & + & \begin{bmatrix} \textit{Viscous} \\ \textit{Shear} \\ \textit{Stress} \end{bmatrix} \end{array} \tag{1.143}$$

or, in matrix form,

$$\begin{aligned} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} &= \begin{bmatrix} -p_H & 0 & 0 \\ 0 & -p_H & 0 \\ 0 & 0 & -p_H \end{bmatrix} + \begin{bmatrix} -p_E & 0 & 0 \\ 0 & -p_E & 0 \\ 0 & 0 & -p_E \end{bmatrix} \\ &+ \begin{bmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{bmatrix} + \begin{bmatrix} 0 & \tau_{12} & \tau_{13} \\ \tau_{21} & 0 & \tau_{23} \\ \tau_{31} & \tau_{32} & 0 \end{bmatrix} \end{aligned} \tag{1.144}$$

The *hydrostatic pressure stress*, $-p_H\mathbf{I}$, is the only nonzero stress component in static equilibrium; it is due to the weight of the fluid and is a function of the position or elevation z, i.e.,

$$p_H(z) \;=\; p_0 - \rho g\,(z - z_0)\,, \tag{1.145}$$

where p_0 is the reference pressure at $z = z_0$, ρ is the density of the fluid, and g is the gravitational acceleration.

The *extra pressure stress*, $-p_E\mathbf{I}$, arises in flowing media due to the perpendicular motion of the particles towards a material surface, and is proportional to the convective momentum carried by the moving molecules. In *inviscid motions*, where either the viscosity of the medium is vanishingly small or the velocity gradients are negligible, the hydrostatic and extra pressure stresses are the only nonzero stress components.

The *viscous normal stress*, $\boldsymbol{\tau}^N$, is due to accelerating or decelerating perpendicular motions towards material surfaces and is proportional to the viscosity of the medium and the velocity gradient along the streamlines.

Finally, the *viscous shear stress*, $\boldsymbol{\tau}^{SH}$, is due to shearing motions of adjacent material layers next to material surfaces. It is proportional to the viscosity of the medium and to the velocity gradient in directions perpendicular to the streamlines. In *stretching* or *extensional flows*, where there are no velocity gradients in the directions perpendicular to the streamlines, the viscous shear stress is zero and thus $\boldsymbol{\tau}^N$ is the only nonzero viscous stress component. In *shear flows*, such as flows in rectilinear channels and pipes, $\boldsymbol{\tau}^N$ vanishes.

In summary, the stress (or force per unit area) is the result of the momentum carried by N molecules across the surface according to Newton's law of motion:

$$\mathbf{n}\cdot\mathbf{T} \;=\; \mathbf{f} \;=\; \frac{\mathbf{F}}{\Delta S} \;=\; \frac{1}{\Delta S}\sum_{i=1}^{N}\frac{d}{dt}(m_i\mathbf{u}_i) \tag{1.146}$$

Any flow is a superposition of the above mentioned motions, and, therefore, the appropriate stress expression is that of Eqs. (1.140) and (1.143). Each of the stress components is expressed in terms of physical characteristics of the medium (i.e., viscosity, density, and elasticity which are functions of temperature in nonisothermal situations) and the velocity field by means of the *constitutive equation* which is highlighted in Chapter 5.

The *strain tensor*, $\mathbf{C}$, represents the state of strain in a medium and is commonly called the *Cauchy strain tensor*. Its inverse, $\mathbf{B}=\mathbf{C}^{-1}$, is known as the *Finger strain tensor*. Both tensors are of primary use in non-Newtonian fluid mechanics. Dotted with the unit normal to a surface, the Cauchy strain tensor (or the Finger strain tensor) yields the strain of the surface due to shearing and stretching. The components of the two tensors are the spatial derivatives of the coordinates with respect to the coordinates at an earlier (Cauchy) or later (Finger) time of the moving fluid particle [9].

The *velocity gradient tensor*, $\nabla\mathbf{u}$, measures the rate of change of the separation vector, $\mathbf{r}_{AB}$, between neighboring fluid particles at A and B, according to

$$\nabla\mathbf{u} = \nabla\frac{d\mathbf{r}_{AB}}{dt}\,, \tag{1.147}$$

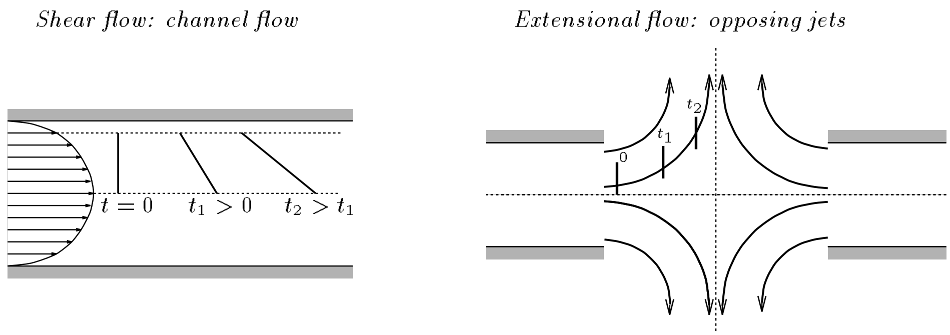

Figure 1.18. *Rotational (weak) and irrotational (strong) deformation of material lines in shear and extensional flows, respectively.*

and represents the rate of change of the magnitude (stretching or compression) and the orientation (rotation) of the material vector $\mathbf{r}_{AB}$. $\nabla\mathbf{u}$ is the dyadic tensor of the generalized derivative vector ∇ and the velocity vector $\mathbf{u}$, as explained in Section 1.4. Like any tensor, $\nabla\mathbf{u}$ can be decomposed into a symmetric, $\mathbf{D}$, and an antisymmetric part, $\mathbf{S}$:[7]

$$\nabla\mathbf{u} \;=\; \mathbf{D} \,+\, \mathbf{\Omega}\,. \tag{1.148}$$

The symmetric tensor

$$\mathbf{D} \;=\; \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right] \tag{1.149}$$

is the *rate of strain* (or *rate of deformation*) *tensor*, and represents the state of the intensity or rate of strain. The antisymmetric tensor

$$\mathbf{S} \;=\; \frac{1}{2}\left[\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right] \tag{1.150}$$

is the *vorticity tensor*.[8] If $\mathbf{n}$ is the unit normal to a surface, then the dot product $\mathbf{n}\cdot\mathbf{D}$ yields the rate of change of the distances in three mutually perpendicular

[7] Some authors define the rate-of-strain and vorticity tensors as

$$\mathbf{D} \;=\; \nabla\mathbf{u} + (\nabla\mathbf{u})^T \qquad \text{and} \qquad \mathbf{S} \;=\; \nabla\mathbf{u} - (\nabla\mathbf{u})^T\,,$$

so that

$$2\,\nabla\mathbf{u} \;=\; \mathbf{D} \,+\, \mathbf{\Omega}\,.$$

[8] Other symbols used for the rate-of-strain and the vorticity tensors are $\mathbf{d}$, $\dot{\gamma}$ and $\mathbf{E}$ for $\mathbf{D}$, and $\mathbf{\Omega}$, ω and $\mathbf{\Xi}$ for $\mathbf{S}$.

Tensor	Orientation	Operation	Result or Vector – Flux
Stress, $\mathbf{T}$	unit normal, $\mathbf{n}$	$\mathbf{n}\cdot\mathbf{T}$	Traction
Rate of strain, $\mathbf{D}$	unit normal, $\mathbf{n}$	$\mathbf{n}\cdot\mathbf{D}$	Rate of stretching
	unit tangent, $\mathbf{t}$	$\mathbf{t}\cdot\mathbf{D}$	Rate of rotation
Viscous Stress, $\boldsymbol{\tau}$	velocity gradient, $\nabla\mathbf{u}$	$\boldsymbol{\tau} : \nabla\mathbf{u}$	Scalar viscous dissipation

Table 1.3. *Vector-tensor operations producing measurable result or flux.*

directions. The dot product $\mathbf{n}\cdot\mathbf{S}$ gives the rate of change of orientation along these directions.

In *purely shear flows* the only strain is rotational. The distance between two particles on the same streamline does not change, whereas the distance between particles on different streamlines changes linearly with traveling time. Thus there is both stretching (or compression) and rotation of material lines (or material vectors), and the flow is characterized as *rotational or weak flow*. In *extensional flows*, the separation vectors among particles on the same streamline change their length exponentially, whereas the separation vectors among particles on different streamlines do not change their orientation. These flows are *irrotational or strong flows*. Figure 1.18 illustrates the deformation of material lines, defined as one-dimensional collections of fluid particles that can be shortened, elongated and rotated, in rotational shear flows and in irrotational extensional flows.

The rate of strain tensor represents the strain state and is zero in *rigid-body motion* (*translation* and *rotation*), since this induces no strain (deformation). The vorticity tensor represents the state of rotation, and is zero in strong irrotational flows. Based on these remarks, we can say that strong flows are those in which the vorticity tensor is zero; the directions of maximum strain do not rotate to directions of less strain, and, therefore, the maximum (strong) strain does not have the opportunity to relax. Weak flows are those of nonzero vorticity; in this case, the directions of maximum strain rotate, and the strain relaxes. Table 1.3 lists some examples of tensor action arising in Mechanics.

Example 1.3.5. Strong and weak flows

In steady channel flow (see Fig. 1.18), the velocity components are given by

$$u_x \;=\; a\,(1-y^2)\,, \qquad u_y \;=\; 0 \quad \text{and} \quad u_z \;=\; 0\,.$$

Let (x_0, y_0, z_0) and (x, y, z) be the positions of a particle at times t=0 and t, respectively. By integrating the velocity components with respect to time, one gets:

$$\begin{aligned} u_x &= \frac{dx}{dt} = a\,(1-y^2) \quad \Longrightarrow \quad x = x_0 + a\,(1-y^2)\,t\,; \\ u_y &= 0 \quad \Longrightarrow \quad y = y_0\,; \\ u_z &= 0 \quad \Longrightarrow \quad z = z_0\,. \end{aligned}$$

The fluid particle at (x, y_0, z_0) is separated linearly with time from that at (x_0, y_0, z_0), and, thus, the resulting strain is small. The matrix form of the velocity gradient tensor in Cartesian coordinates is

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{bmatrix}, \tag{1.151}$$

and, therefore,

$$\nabla \mathbf{u} = \begin{bmatrix} 0 & 0 & 0 \\ -2ay & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 0 & -ay & 0 \\ -ay & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{S} = \begin{bmatrix} 0 & ay & 0 \\ -ay & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The vorticity tensor is nonzero and thus the flow is weak.

Let us now consider the extensional flow of Fig. 1.18. The velocity components are given by

$$u_x = \varepsilon x\,, \quad u_y = -\varepsilon y \quad \text{and} \quad u_z = 0\,;$$

therefore,

$$\begin{aligned} u_x &= \frac{dx}{dt} = \varepsilon x \quad \Longrightarrow \quad x = x_0\, e^{\varepsilon t}\,; \\ u_y &= \frac{dy}{dt} = -\varepsilon y \quad \Longrightarrow \quad y = y_0\, e^{-\varepsilon t}\,; \\ u_z &= 0 \quad \Longrightarrow \quad z = z_0\,. \end{aligned}$$

Since the fluid particle at (x, y, z_0) is separated exponentially with time from that at (x_0, y_0, z_0), the resulting strain (stretching) is large. The velocity-gradient, rate of strain, and vorticity tensors are:

$$\nabla \mathbf{u} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the vorticity tensor is zero, the flow is strong. □

1.4 Differential Operators

The nabla operator ∇, already encountered in previous sections, is a differential operator. In a Cartesian system of coordinates (x_1, x_2, x_3), defined by the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$,

$$\nabla \equiv \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial}{\partial x_i}, \tag{1.152}$$

or, in index notation,

$$\nabla \equiv \frac{\partial}{\partial x_i}. \tag{1.153}$$

The nabla operator is a vector operator which acts on scalar, vector, or tensor fields. The result of its action is another field the order of which depends on the type of the operation. In the following, we will first define the various operations of ∇ in Cartesian coordinates, and then discuss their forms in curvilinear coordinates.

The *gradient* of a differentiable scalar field f, denoted by ∇f or $grad f$, is a vector field:

$$\nabla f = \left(\sum_{i=1}^{3} \mathbf{e}_i \frac{\partial}{\partial x_i} \right) f = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial f}{\partial x_i} = \mathbf{e}_1 \frac{\partial f}{\partial x_1} + \mathbf{e}_2 \frac{\partial f}{\partial x_2} + \mathbf{e}_3 \frac{\partial f}{\partial x_3}. \tag{1.154}$$

The gradient ∇f can be viewed as a generalized derivative in three dimensions; it measures the spatial change of f occurring within a distance $d\mathbf{r}(dx_1, dx_2, dx_3)$.

The *gradient* of a differentiable vector field $\mathbf{u}$ is a dyadic tensor field:

$$\nabla \mathbf{u} = \left(\sum_{i=1}^{3} \mathbf{e}_i \frac{\partial}{\partial x_i} \right) \left(\sum_{j=1}^{3} u_j \mathbf{e}_j \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i} \mathbf{e}_i \mathbf{e}_j. \tag{1.155}$$

As explained in Section 1.3.3, if $\mathbf{u}$ is the velocity, then $\nabla \mathbf{u}$ is called the *velocity-gradient tensor*.

The *divergence* of a differentiable vector field $\mathbf{u}$, denoted by $\nabla \cdot \mathbf{u}$ or $div \mathbf{u}$, is a scalar field

$$\nabla \cdot \mathbf{u} = \left(\sum_{i=1}^{3} \mathbf{e}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\sum_{j=1}^{3} u_j \mathbf{e}_j \right) = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \delta_{ij} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \tag{1.156}$$

$\nabla \cdot \mathbf{u}$ measures changes in magnitude, or flux through a point. If $\mathbf{u}$ is the velocity, then $\nabla \cdot \mathbf{u}$ measures the rate of volume expansion per unit volume; hence, it is zero for incompressible fluids. The following identity is easy to prove:

$$\nabla \cdot (f \mathbf{u}) = \nabla f \cdot \mathbf{u} + f \nabla \cdot \mathbf{u}. \tag{1.157}$$

The *curl* or *rotation* of a differentiable vector field $\mathbf{u}$, denoted by $\nabla \times \mathbf{u}$ or $curl\mathbf{u}$ or $rot\mathbf{u}$, is a vector field:

$$\nabla \times \mathbf{u} \;=\; \left(\sum_{i=1}^{3} \mathbf{e}_i \,\frac{\partial}{\partial x_i}\right) \;\times\; \left(\sum_{j=1}^{3} u_j \mathbf{e}_j\right) \;=\; \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} \tag{1.158}$$

or

$$\nabla \times \mathbf{u} \;=\; \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \mathbf{e}_1 \;+\; \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \mathbf{e}_2 \;+\; \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \mathbf{e}_3 \,. \tag{1.159}$$

The field $\nabla \times \mathbf{u}$ is often called the *vorticity* (or *chirality*) of $\mathbf{u}$.

The *divergence* of a differentiable tensor field $\boldsymbol{\tau}$ is a vector field:[9]

$$\nabla \cdot \boldsymbol{\tau} \;=\; \left(\sum_{k=1}^{3} \mathbf{e}_k \,\frac{\partial}{\partial x_k}\right) \;\cdot\; \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\mathbf{e}_i\mathbf{e}_j\right) \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_i}\mathbf{e}_j \,. \tag{1.160}$$

Example 1.4.1. The divergence and the curl of the position vector

Consider the position vector in Cartesian coordinates,

$$\mathbf{r} \;=\; x\,\mathbf{i} \,+\, y\,\mathbf{j} \,+\, z\,\mathbf{k} \,. \tag{1.161}$$

For its divergence and curl, we obtain:

$$\nabla \cdot \mathbf{r} \;=\; \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \qquad \Longrightarrow$$

$$\nabla \cdot \mathbf{r} \;=\; 3\,, \tag{1.162}$$

and

$$\nabla \times \mathbf{r} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \qquad \Longrightarrow$$

$$\nabla \times \mathbf{r} \;=\; \mathbf{0} \tag{1.163}$$

Equations (1.162) and (1.163) hold in all coordinate systems. □

[9] The divergence of a tensor τ is sometimes denoted by $div\tau$.

Other useful operators involving the nabla operator are the *Laplace operator* ∇^2 and the operator $\mathbf{u} \cdot \nabla$, where $\mathbf{u}$ is a vector field. The *Laplacian* of a scalar f with continuous second partial derivatives is defined as the divergence of the gradient:

$$\nabla^2 f \equiv \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}, \tag{1.164}$$

i.e.,

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \tag{1.165}$$

A function whose Laplacian is identically zero is called *harmonic*.

If $\mathbf{u}=u_1\mathbf{e}_1+u_2\mathbf{e}_2+u_3\mathbf{e}_3$ is a vector field, then

$$\nabla^2 \mathbf{u} = \nabla^2 u_1 \mathbf{e}_1 + \nabla^2 u_2 \mathbf{e}_2 + \nabla^2 u_3 \mathbf{e}_3 . \tag{1.166}$$

For the operator $\mathbf{u} \cdot \nabla$, we obtain:

$$\mathbf{u} \cdot \nabla = (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3) \cdot \left(\mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \right) \quad \Longrightarrow$$

$$\mathbf{u} \cdot \nabla = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}. \tag{1.167}$$

The above expressions are valid only for Cartesian coordinate systems. In curvilinear coordinate systems, the basis vectors are not constant and the forms of ∇ are quite different, as explained in Example 1.4.3. Notice that gradient always raises the order by one (the gradient of a scalar is a vector, the gradient of a vector is a tensor and so on), while divergence reduces the order of a quantity by one. A summary of useful operations in Cartesian coordinates (x, y, z) is given in Table 1.4.

For any scalar function f with continuous second partial derivatives, the curl of the gradient is zero,

$$\nabla \times (\nabla f) = \mathbf{0} . \tag{1.168}$$

For any vector function $\mathbf{u}$ with continuous second partial derivatives, the divergence of the curl is zero,

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0 . \tag{1.169}$$

Equations (1.168) and (1.169) are valid independently of the coordinate system. Their proofs are left as exercises to the reader (Problem 1.11). Other identities involving the nabla operator are given in Table 1.5.

In fluid mechanics, the *vorticity* $\boldsymbol{\omega}$ of the velocity vector $\mathbf{u}$ is defined as the curl of $\mathbf{u}$,

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{u} . \tag{1.170}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{u}\cdot\nabla = u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}$$

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}$$

$$\nabla\cdot\mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\nabla\times\mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}$$

$$\begin{aligned}\nabla\mathbf{u} = {} & \frac{\partial u_x}{\partial x}\mathbf{ii} + \frac{\partial u_y}{\partial x}\mathbf{ij} + \frac{\partial u_z}{\partial x}\mathbf{ik} + \frac{\partial u_x}{\partial y}\mathbf{ji} \\ & + \frac{\partial u_y}{\partial y}\mathbf{jj} + \frac{\partial u_z}{\partial y}\mathbf{jk} + \frac{\partial u_x}{\partial z}\mathbf{ki} + \frac{\partial u_y}{\partial z}\mathbf{kj} + \frac{\partial u_z}{\partial z}\mathbf{kk}\end{aligned}$$

$$\begin{aligned}\mathbf{u}\cdot\nabla\mathbf{u} = {} & \left(u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right)\mathbf{i} + \left(u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right)\mathbf{j} \\ & + \left(u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right)\mathbf{k}\end{aligned}$$

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau} = {} & \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\mathbf{i} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\mathbf{j} \\ & + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)\mathbf{k}\end{aligned}$$

Table 1.4. *Summary of differential operators in Cartesian coordinates* (x, y, z)*;* p*,* $\mathbf{u}$ *and* $\boldsymbol{\tau}$ *are scalar, vector and tensor fields, respectively.*

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{u}\cdot\nabla)\,\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{u})$$

$$\nabla\cdot(f\mathbf{u}) = f\,\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla f$$

$$\nabla\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{v})$$

$$\nabla\cdot(\nabla\times\mathbf{u}) = 0$$

$$\nabla\times(f\mathbf{u}) = f\,\nabla\times\mathbf{u} + \nabla f\times\mathbf{u}$$

$$\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\,\nabla\cdot\mathbf{v} - \mathbf{v}\,\nabla\cdot\mathbf{u} + (\mathbf{v}\cdot\nabla)\,\mathbf{u} - (\mathbf{u}\cdot\nabla)\,\mathbf{v}$$

$$\nabla\times(\nabla\times\mathbf{u}) = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u}$$

$$\nabla\times(\nabla f) = \mathbf{0}$$

$$\nabla(\mathbf{u}\cdot\mathbf{u}) = 2\,(\mathbf{u}\cdot\nabla)\,\mathbf{u} + 2\mathbf{u}\times(\nabla\times\mathbf{u})$$

$$\nabla^2(fg) = f\,\nabla^2 g + g\,\nabla^2 f + 2\,\nabla f\cdot\nabla g$$

$$\nabla\cdot(\nabla f\times\nabla g) = 0$$

$$\nabla\cdot(f\,\nabla g - g\,\nabla f) = f\,\nabla^2 g - g\,\nabla^2 f$$

Table 1.5. *Useful identities involving the nabla operator; f and g are scalar fields, and $\mathbf{u}$ and $\mathbf{v}$ are vector fields. It is assumed that all the partial derivatives involved are continuous.*

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{e}_\theta + \left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]\mathbf{e}_z$$

$$\nabla\mathbf{u} = \frac{\partial u_r}{\partial r}\mathbf{e}_r\mathbf{e}_r + \frac{\partial u_\theta}{\partial r}\mathbf{e}_r\mathbf{e}_\theta + \frac{\partial u_z}{\partial r}\mathbf{e}_r\mathbf{e}_z + \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)\mathbf{e}_\theta\mathbf{e}_r$$
$$+ \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)\mathbf{e}_\theta\mathbf{e}_\theta + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\mathbf{e}_\theta\mathbf{e}_z + \frac{\partial u_r}{\partial z}\mathbf{e}_z\mathbf{e}_r + \frac{\partial u_\theta}{\partial z}\mathbf{e}_z\mathbf{e}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z\mathbf{e}_z$$

$$\mathbf{u}\cdot\nabla\mathbf{u} = \left[u_r\frac{\partial u_r}{\partial r} + u_\theta\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) + u_z\frac{\partial u_r}{\partial z}\right]\mathbf{e}_r$$
$$+ \left[u_r\frac{\partial u_\theta}{\partial r} + u_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + u_z\frac{\partial u_\theta}{\partial z}\right]\mathbf{e}_\theta$$
$$+ \left[u_r\frac{\partial u_z}{\partial r} + u_\theta\frac{1}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right]\mathbf{e}_z$$

$$\nabla\cdot\boldsymbol{\tau} = \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\theta\theta}}{r}\right]\mathbf{e}_r$$
$$+ \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} - \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right]\mathbf{e}_\theta$$
$$+ \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right]\mathbf{e}_z$$

Table 1.6. *Summary of differential operators in cylindrical polar coordinates* (r, θ, z)*;* p*,* $\mathbf{u}$ *and* $\boldsymbol{\tau}$ *are scalar, vector and tensor fields, respectively.*

$$\nabla \;=\; \mathbf{e}_r\,\frac{\partial}{\partial r} \;+\; \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial}{\partial \theta} \;+\; \mathbf{e}_\phi\,\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\nabla^2 \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \;+\; \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \;+\; \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\mathbf{u}\cdot\nabla \;=\; u_r\frac{\partial}{\partial r} \;+\; \frac{u_\theta}{r}\frac{\partial}{\partial\theta} \;+\; \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\nabla p \;=\; \frac{\partial p}{\partial r}\,\mathbf{e}_r \;+\; \frac{1}{r}\frac{\partial p}{\partial\theta}\,\mathbf{e}_\theta \;+\; \frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}\,\mathbf{e}_\phi$$

$$\nabla\cdot\mathbf{u} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) \;+\; \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) \;+\; \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

$$\begin{aligned}\nabla\times\mathbf{u} \;=\; &[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}]\mathbf{e}_r \;+\; [\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(ru_\phi)]\mathbf{e}_\theta\\ &+[\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta}]\mathbf{e}_\phi\end{aligned}$$

$$\begin{aligned}\nabla\mathbf{u} \;=\; &\frac{\partial u_r}{\partial r}\,\mathbf{e}_r\mathbf{e}_r + \frac{\partial u_\theta}{\partial r}\,\mathbf{e}_r\mathbf{e}_\theta + \frac{\partial u_\phi}{\partial r}\,\mathbf{e}_r\mathbf{e}_\phi + \left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)\,\mathbf{e}_\theta\mathbf{e}_r\\ &+\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right)\,\mathbf{e}_\theta\mathbf{e}_\theta + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta}\,\mathbf{e}_\theta\mathbf{e}_\phi + \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r}\right)\,\mathbf{e}_\phi\mathbf{e}_r\\ &+\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r}\cot\theta\right)\,\mathbf{e}_\phi\mathbf{e}_\theta + \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right)\,\mathbf{e}_\phi\mathbf{e}_\phi\end{aligned}$$

$$\begin{aligned}\mathbf{u}\cdot\nabla\mathbf{u} \;=\; &[u_r\frac{\partial u_r}{\partial r} + u_\theta\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right) + u_\phi\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r}\right)]\,\mathbf{e}_r\\ &+[u_r\frac{\partial u_\theta}{\partial r} + u_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right) + u_\phi\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r}\cot\theta\right)]\,\mathbf{e}_\theta\\ &+[u_r\frac{\partial u_\phi}{\partial r} + u_\theta\frac{1}{r}\frac{\partial u_\phi}{\partial\theta} + u_\phi\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right)]\,\mathbf{e}_\phi\end{aligned}$$

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau} \;=\; &[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta r}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi r}}{\partial\phi} - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}]\mathbf{e}_r\\ &+[\frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\theta}}{\partial\phi} + \frac{\tau_{\theta r}-\tau_{r\theta}-\tau_{\phi\phi}\cot\theta}{r}]\mathbf{e}_\theta\\ &+[\frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\phi}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{\phi r}-\tau_{r\phi}-\tau_{\phi\theta}\cot\theta}{r}]\mathbf{e}_\phi\end{aligned}$$

Table 1.7. *Summary of differential operators in spherical polar coordinates* (r,θ,ϕ)*;* p, $\mathbf{u}$ *and* $\boldsymbol{\tau}$ *are scalar, vector and tensor fields, respectively.*

Other symbols used for the vorticity, in the fluid mechanics literature, are $\boldsymbol{\zeta}$, $\boldsymbol{\xi}$ and $\boldsymbol{\Omega}$. If, in a flow, the vorticity vector is zero everywhere, then the flow is said to be *irrotational*. Otherwise, i.e., if the vorticity is not zero, at least in some regions of the flow, then the flow is said to be *rotational*. For example, if the velocity field can be expressed as the gradient of a scalar function, i.e., if $\mathbf{u}=\nabla f$, then according to Eq. (1.168),

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{u} = \nabla \times (\nabla f) = \mathbf{0} ,$$

and, thus, the flow is irrotational.

A vector field $\mathbf{u}$ is said to be *solenoidal* if its divergence is everywhere zero, i.e., if

$$\nabla \cdot \mathbf{u} = 0 . \tag{1.171}$$

From Eq. (1.169), we deduce that the vorticity vector is solenoidal, since

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{u}) = 0 .$$

Example 1.4.2. Physical significance of differential operators

Consider an infinitesimal volume ΔV bounded by a surface ΔS. The gradient of a scalar field f can be defined as

$$\nabla f \equiv \lim_{\Delta V \to 0} \frac{\int_{\Delta S} \mathbf{n} \, f \, dS}{\Delta V} , \tag{1.172}$$

where $\mathbf{n}$ is the unit vector normal to the surface ΔS. The gradient here represents the *net vector flux* of the scalar quantity f at a point where the volume ΔV of surface ΔS collapses in the limit. At that point, the above equation reduces to Eq. (1.154).

The divergence of the velocity vector $\mathbf{u}$ can be defined as

$$\nabla \cdot \mathbf{u} \equiv \lim_{\Delta V \to 0} \frac{\int_{\Delta S} (\mathbf{n} \cdot \mathbf{u}) \, dS}{\Delta V} , \tag{1.173}$$

and represents the *scalar flux* of the vector $\mathbf{u}$ at a point, which is equivalent to the *local rate of expansion* (see Example 1.5.3).

Finally, the vorticity of $\mathbf{u}$ may be defined as

$$\nabla \times \mathbf{u} \equiv \lim_{\Delta V \to 0} \frac{\int_{\Delta S} (\mathbf{n} \times \mathbf{u}) dS}{\Delta V} , \tag{1.174}$$

and represents *the vector net flux* of the scalar angular component at a point, which tends to rotate the fluid particle at the point where ΔV collapses.

□

Example 1.4.3. The nabla operator in cylindrical polar coordinates

(a) Express the nabla operator

$$\boxed{\nabla \;=\; \mathbf{i}\,\frac{\partial}{\partial x} \;+\; \mathbf{j}\,\frac{\partial}{\partial y} \;+\; \mathbf{k}\,\frac{\partial}{\partial z}} \tag{1.175}$$

in cylindrical polar coordinates.
(b) Determine ∇c and $\nabla \cdot \mathbf{u}$, where c is a scalar and $\mathbf{u}$ is a vector.
(c) Derive the operator $\mathbf{u} \cdot \nabla$ and the dyadic product $\nabla \mathbf{u}$ in cylindrical polar coordinates.

Solution:
(a) From Table 1.1, we have:

$$\begin{aligned} \mathbf{i} &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \\ \mathbf{j} &= \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \\ \mathbf{k} &= \mathbf{e}_z \end{aligned}$$

Therefore, we just need to convert the derivatives with respect to x, y and z into derivatives with respect to r, θ and z. Starting with the expressions of Table 1.1 and using the chain rule, we get:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial z} \end{aligned}$$

Substituting now into Eq. (1.175) gives

$$\begin{aligned} \nabla &= (\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta)\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\right) \\ &\quad + (\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta)\left(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial \theta}\right) + \mathbf{e}_z\,\frac{\partial}{\partial z}\,. \end{aligned}$$

After some simplifications and using the trigonometric identity $\sin^2\theta + \sin^2\theta{=}1$, we get

$$\boxed{\nabla \;=\; \mathbf{e}_r\,\frac{\partial}{\partial r} \;+\; \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial}{\partial \theta} \;+\; \mathbf{e}_z\,\frac{\partial}{\partial z}} \tag{1.176}$$

(b) The gradient of the scalar c is given by

$$\nabla c \;=\; \mathbf{e}_r\,\frac{\partial c}{\partial r} \;+\; \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial c}{\partial \theta} \;+\; \mathbf{e}_z\,\frac{\partial c}{\partial z}\,. \tag{1.177}$$

For the divergence of the vector $\mathbf{u}$, we have

$$\nabla\cdot\mathbf{u} \;=\; \left(\mathbf{e}_r\,\frac{\partial}{\partial r} \;+\; \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial}{\partial \theta} \;+\; \mathbf{e}_z\,\frac{\partial}{\partial z}\right)\cdot\left(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z\right)\,.$$

Noting that the only nonzero spatial derivatives of the unit vectors are

$$\frac{\partial \mathbf{e}_r}{\partial \theta} \;=\; \mathbf{e}_\theta \quad \text{and} \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} \;=\; -\mathbf{e}_r$$

(see Eq. 1.17), we obtain

$$\begin{aligned}
\nabla\cdot\mathbf{u} \;&=\; \frac{\partial u_r}{\partial r} \;+\; \mathbf{e}_\theta\cdot\frac{1}{r}\left(u_r\frac{\partial \mathbf{e}_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \theta}\mathbf{e}_\theta + u_\theta\frac{\partial \mathbf{e}_\theta}{\partial \theta}\right) \;+\; \frac{\partial u_z}{\partial z} \\
&=\; \frac{\partial u_r}{\partial r} \;+\; \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \;+\; \mathbf{e}_\theta\cdot\frac{1}{r}\left(u_r\mathbf{e}_\theta - u_\theta\mathbf{e}_r\right) \;+\; \frac{\partial u_z}{\partial z} \\
&=\; \frac{\partial u_r}{\partial r} \;+\; \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \;+\; \frac{u_r}{r} \;+\; \frac{\partial u_z}{\partial z} \qquad \Longrightarrow
\end{aligned}$$

$$\nabla\cdot\mathbf{u} \;=\; \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\,. \tag{1.178}$$

(c)

$$\mathbf{u}\cdot\nabla \;=\; \left(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z\right)\left(\mathbf{e}_r\,\frac{\partial}{\partial r} \;+\; \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial}{\partial \theta} \;+\; \mathbf{e}_z\,\frac{\partial}{\partial z}\right) \qquad \Longrightarrow$$

$$\mathbf{u}\cdot\nabla \;=\; u_r\frac{\partial}{\partial r} \;+\; \frac{u_\theta}{r}\frac{\partial}{\partial \theta} \;+\; u_z\frac{\partial}{\partial z}\,. \tag{1.179}$$

Finally, for the dyadic product $\nabla\mathbf{u}$ we have

$$\begin{aligned}
\nabla\mathbf{u} \;&=\; \left(\mathbf{e}_r\,\frac{\partial}{\partial r} \;+\; \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial}{\partial \theta} \;+\; \mathbf{e}_z\,\frac{\partial}{\partial z}\right)\left(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z\right) \\
&=\; \mathbf{e}_r\mathbf{e}_r\frac{\partial u_r}{\partial r} \;+\; \mathbf{e}_r\mathbf{e}_\theta\frac{\partial u_\theta}{\partial r} \;+\; \mathbf{e}_r\mathbf{e}_z\frac{\partial u_z}{\partial r} \\
&\quad + \mathbf{e}_\theta\mathbf{e}_r\frac{1}{r}\frac{\partial u_r}{\partial \theta} \;+\; \mathbf{e}_\theta\frac{1}{r}u_r\frac{\partial \mathbf{e}_r}{\partial \theta} \;+\; \mathbf{e}_\theta\mathbf{e}_\theta\frac{1}{r}\frac{\partial \mathbf{e}_\theta}{\partial \theta} \;+\; \mathbf{e}_\theta\frac{1}{r}u_\theta\frac{\partial \mathbf{e}_\theta}{\partial \theta} \;+\; \mathbf{e}_\theta\mathbf{e}_z\frac{1}{r}\frac{\partial u_z}{\partial \theta} \\
&\quad + \mathbf{e}_z\mathbf{e}_r\frac{\partial u_r}{\partial z} \;+\; \mathbf{e}_z\mathbf{e}_\theta\frac{\partial u_\theta}{\partial z} \;+\; \mathbf{e}_z\mathbf{e}_z\frac{\partial u_z}{\partial z} \qquad \Longrightarrow
\end{aligned}$$

$$\begin{aligned} \nabla \mathbf{u} &= \mathbf{e}_r\mathbf{e}_r\frac{\partial u_r}{\partial r} + \mathbf{e}_r\mathbf{e}_\theta\frac{\partial u_\theta}{\partial r} + \mathbf{e}_r\mathbf{e}_z\frac{\partial u_z}{\partial r} \\ &+ \mathbf{e}_\theta\mathbf{e}_r\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \mathbf{e}_\theta\mathbf{e}_\theta\frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) + \mathbf{e}_\theta\mathbf{e}_z\frac{1}{r}\frac{\partial u_z}{\partial \theta} \\ &+ \mathbf{e}_z\mathbf{e}_r\frac{\partial u_r}{\partial z} + \mathbf{e}_z\mathbf{e}_\theta\frac{\partial u_\theta}{\partial z} + \mathbf{e}_z\mathbf{e}_z\frac{\partial u_z}{\partial z} \end{aligned} \tag{1.180}$$

□

Any other differential operation in curvilinear coordinates is evaluated following the procedures of Example 1.4.3. In Tables 1.6 and 1.7, we provide the most important differential operations in cylindrical and spherical coordinates, respectively.

1.4.1 The Substantial Derivative

The time derivative represents the rate of change of a physical quantity experienced by an observer who can be either stationary or moving. In the case of fluid flow, a nonstationary observer may be moving exactly as a fluid particle or not. Hence, at least three different time derivatives can be defined in fluid mechanics and in transport phenomena. The classical example of fish concentration in a lake, provided in [4], is illustrative of the similarities and differences between these time derivatives. Let $c(x, y, t)$ be the fish concentration in a lake. For a stationary observer, say standing on a bridge and looking just at a spot of the lake beneath him, the time derivative is determined by the amount of fish arriving and leaving the spot of observation, i.e., the total change in concentration and thus the total time derivative, is identical to the *partial derivative*,

$$\frac{dc}{dt} = \left(\frac{\partial c}{\partial t}\right)_{x,y} \quad , \tag{1.181}$$

and is only a function of the local change of concentration. Imagine now the observer riding a boat which can move with relative velocity $\mathbf{u}^{Rel}$ *with respect to that of the water*. Hence, if $\mathbf{u}^{Boat}$ and $\mathbf{u}^{Water}$ are the velocities of the boat and the water, respectively, then

$$\mathbf{u}^{Rel} = \mathbf{u}^{Boat} + \mathbf{u}^{Water} \, . \tag{1.182}$$

The concentration now is a function not only of the time t, but also of the position of the boat $\mathbf{r}(x, y)$ too. The position of the boat is a function of time, and, in fact,

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}^{Rel} \tag{1.183}$$

and so

$$\frac{dx}{dt} = u_x^{Rel} \quad \text{and} \quad \frac{dy}{dt} = u_y^{Rel} . \tag{1.184}$$

Thus, in this case, the *total time derivative* or the change experienced by the moving observer is,

$$\begin{aligned} \frac{d}{dt}[c(t,x,y)] &\equiv \left(\frac{\partial c}{\partial t}\right)_{x,y} + \left(\frac{\partial c}{\partial x}\right)_{t,y} \frac{dx}{dt} + \left(\frac{\partial c}{\partial y}\right)_{t,y} \frac{dy}{dt} = \\ &= \left(\frac{\partial c}{\partial t}\right)_{x,y} + u_x^{Rel} \left(\frac{\partial c}{\partial x}\right)_{t,y} + u_y^{Rel} \left(\frac{\partial c}{\partial y}\right)_{t,x} \end{aligned} \tag{1.185}$$

Imagine now the observer turning off the engine of the boat so that $\mathbf{u}^{Boat} = \mathbf{0}$ and $\mathbf{u}^{Rel} = \mathbf{u}^{Water}$. Then,

$$\begin{aligned} \frac{d}{dt}[c(t,x,y)] &= \left(\frac{\partial c}{\partial t}\right)_{x,y} + \left(\frac{\partial c}{\partial x}\right)_{t,y} \frac{dx}{dt} + \left(\frac{\partial c}{\partial y}\right)_{t,x} \frac{dy}{dt} \\ &= \left(\frac{\partial c}{\partial t}\right)_{x,y} + u_x^{Water} \left(\frac{\partial c}{\partial x}\right)_{t,y} + u_y^{Water} \left(\frac{\partial c}{\partial y}\right)_{t,x} \\ &= \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c \end{aligned}$$

This derivative is called the *substantial derivative* and is denoted by D/Dt:

$$\frac{Dc}{Dt} \equiv \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c . \tag{1.186}$$

(The terms *substantive*, *material* or *convective* are sometimes used for the *substantial* derivative.) The substantial derivative expresses the total time change of a quantity, experienced by an observer following the motion of the liquid. It consists of a local change, $\partial c / \partial t$, which vanishes under steady conditions (i.e., same number of fish arrive and leave the spot of observation), and of a traveling change, $\mathbf{u} \cdot \nabla c$, which of course is zero for a stagnant liquid or uniform concentration. Thus, for a steady-state process,

$$\frac{Dc}{Dt} = \mathbf{u} \cdot \nabla c = u_1 \frac{\partial c}{\partial x_1} + u_2 \frac{\partial c}{\partial x_2} + u_3 \frac{\partial c}{\partial x_3} . \tag{1.187}$$

For stagnant liquid or uniform concentration,

$$\frac{Dc}{Dt} = \left(\frac{\partial c}{\partial t}\right)_{x,y,z} = \frac{dc}{dt} . \tag{1.188}$$

Example 1.4.4. Substantial derivative[10]

Let $T(x,y)$ be the surface temperature of a stationary lake. Assume that you attach a thermometer to a boat and take a path through the lake defined by $x = a(t)$ and $y = b(t)$. Find an expression for the rate of change of the thermometer temperature in terms of the lake temperature.

Solution:

$$\begin{aligned}\frac{dT(x,y)}{dt} &= \left(\frac{\partial T}{\partial t}\right)_{x,y} + \left(\frac{\partial T}{\partial x}\right)_{t,y} \frac{dx}{dt} + \left(\frac{\partial T}{\partial y}\right)_{t,x} \frac{dy}{dt} \\ &= 0 + \left(\frac{\partial T}{\partial x}\right)_{y} \frac{da}{dt} + \left(\frac{\partial T}{\partial y}\right)_{x} \frac{db}{dt}\,.\end{aligned}$$

Limiting cases:

$$\text{If} \quad T(x,y) = c, \qquad \text{then} \quad \frac{dT}{dt} = 0\,.$$

$$\text{If} \quad T(x,y) = f(x), \quad \text{then} \quad \frac{dT}{dt} = \frac{dT}{dx}\frac{da}{dt} = \frac{df}{dx}\frac{da}{dt}\,.$$

$$\text{If} \quad T(x,y) = g(y), \quad \text{then} \quad \frac{dT}{dt} = \frac{dT}{dy}\frac{db}{dt} = \frac{dg}{dy}\frac{db}{dt}\,.$$

Notice that the local time derivative is zero because $T(x,y)$ is not a function of time.

□

The forms of the substantial derivative operator in the three coordinate systems of interest are tabulated in Table 1.8.

1.5 Integral Theorems

The Gauss or divergence theorem

The Gauss theorem is one of the most important integral theorems of vector calculus. It can be viewed as a generalization of the *fundamental theorem of calculus* which states that

$$\int_a^b \frac{d\phi}{dx} dx \;=\; \phi(b) - \phi(a)\,, \tag{1.189}$$

where $\phi(x)$ is a scalar one-dimensional function which obviously must be differentiable. Equation (1.189) can also be written as follows:

$$\mathbf{i} \int_a^b \left(\frac{d\phi}{dx}\right) dx \;=\; \mathbf{i}\,[\phi(b) - \phi(a)] \;=\; [\mathbf{n}\phi(x)]_a^b\,, \tag{1.190}$$

[10]Taken from Ref. [6].

Coordinate system	$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$
(x, y, z)	$\frac{\partial}{\partial t} + u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}$
(r, θ, z)	$\frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$
(r, θ, ϕ)	$\frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$

Table 1.8. *The substantial derivative operator in various coordinate systems.*

$$\int_a^b \mathbf{i}\left(\frac{d\phi}{dx}\right)dx = [\mathbf{n}\phi(x)]_a^b$$

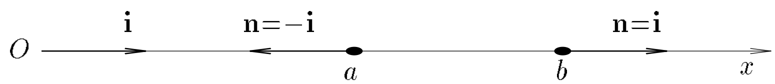

Figure 1.19. *The fundamental theorem of calculus.*

where $\mathbf{n}$ is the unit vector pointing outwards from the one-dimensional interval of integration, $a \leq x \leq b$, as shown in Fig. 1.19.

Equation (1.190) can be extended to two dimensions as follows. Consider the square S defined by $a \leq x \leq b$ and $c \leq y \leq d$ and a function $\phi(x, y)$ with continuous first partial derivatives. Then

$$\begin{aligned}
\int_S \nabla\phi\, dS &= \int_c^d \int_a^b \left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y}\right) dxdy \;=\; \mathbf{i}\int_c^d \int_a^b \frac{\partial\phi}{\partial x}dxdy + \mathbf{j}\int_c^d \int_a^b \frac{\partial\phi}{\partial y}dxdy \\
&= \mathbf{i}\int_c^d [\phi(b, y) - \phi(a, y)]dy + \mathbf{j}\int_a^b [\phi(x, d) - \phi(x, c)]dx \\
&= \int_c^d [\mathbf{n}\phi(x, y)]_a^b dy + \int_a^b [\mathbf{n}\phi(x, y)]_c^d dx \quad \Longrightarrow
\end{aligned}$$

$$\int_S \nabla\phi\, dS \;=\; \int_C \mathbf{n}\,\phi\, d\ell\,, \tag{1.191}$$

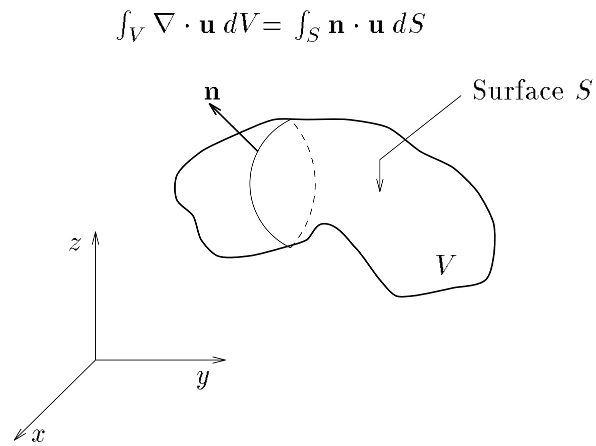

Figure 1.20. *The Gauss or divergence theorem.*

where $\mathbf{n}$ is the outward unit normal to the boundary C of S, and ℓ is the arc length around C. Note that Eq. (1.191) is valid for any surface S on the plane bounded by a curve C. Similarly, if V is an arbitrary closed region bounded by a surface S, and $\phi(x, y, z)$ is a scalar function with continuous first partial derivatives, one gets:

$$\int_V \nabla\phi \, dV \;=\; \int_S \mathbf{n}\, \phi \, dS \,, \tag{1.192}$$

where $\mathbf{n}$ is the unit normal pointing outward from the surface S, as depicted in Fig. 1.20. Equation (1.192) is known as the *Gauss* or *divergence theorem.* The Gauss theorem holds not only for tensor fields of zeroth order (i.e., scalar fields), but also for tensors of higher order (i.e., vector and second-order tensor fields). If $\mathbf{u}$ and $\boldsymbol{\tau}$ are vector and tensor fields, respectively, with continuous first partial derivatives, the Gauss theorem takes the following forms:

$$\int_V \nabla \cdot \mathbf{u} \, dV \;=\; \int_S \mathbf{n} \cdot \mathbf{u} \, dS \,, \tag{1.193}$$

and

$$\int_V \nabla \cdot \boldsymbol{\tau} \, dV \;=\; \int_S \mathbf{n} \cdot \boldsymbol{\tau} \, dS \,. \tag{1.194}$$

In words, the Gauss theorem states that the volume integral of the divergence of a vector or tensor field over an arbitrary control volume V is equal to the flow rate of the field across the surface S bounding the domain V. If a vector field $\mathbf{u}$ happens

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{u})\, dS \;=\; \oint_C \mathbf{t} \cdot \mathbf{u}\, d\ell$$

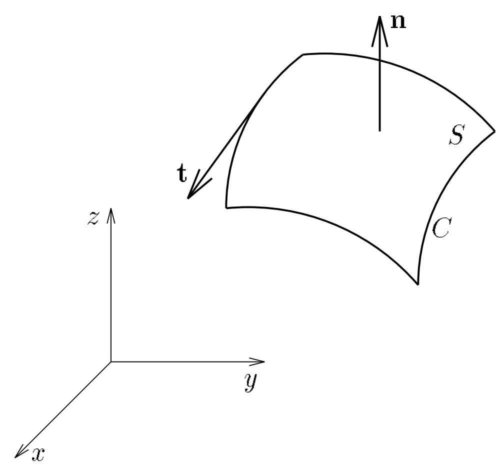

Figure 1.21. *The Stokes theorem.*

to be solenoidal, $\nabla \cdot \mathbf{u}=0$ and, hence, the flow rate of $\mathbf{u}$ across S is zero:

$$\int_s \mathbf{n} \cdot \mathbf{u}\, dS \;=\; 0\,.$$

The Stokes theorem

Consider a surface S bounded by a closed curve C and designate one of its sides, as the outside. At any point of the outside, we define the unit normal $\mathbf{n}$ to point outwards; thus, $\mathbf{n}$ does not cross the surface S. Let us also assume that the unit tangent $\mathbf{t}$ to the boundary C is directed in such a way that the surface S is always on the left (Fig. 1.21). In this case, the surface S is said to be *oriented* according to the right-handed convention. The *Stokes theorem* states that the flow rate of the vorticity, $\nabla \times \mathbf{u}$, of a differentiable vector field $\mathbf{u}$ through S is equal to the circulation of $\mathbf{u}$ along the boundary C of S:

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{u})\, dS \;=\; \oint_C \mathbf{t} \cdot \mathbf{u}\, d\ell\,. \tag{1.195}$$

Another form of the Stokes theorem is

$$\int_S (\nabla \times \mathbf{u}) \cdot d\mathbf{S} \;=\; \oint_C \mathbf{u} \cdot d\mathbf{r}\,, \tag{1.196}$$

where $d\mathbf{S}=\mathbf{n}dS$, $d\mathbf{r}=\mathbf{t}d\ell$, and $\mathbf{r}$ is the position vector.

One notices that the Gauss theorem expresses the volume integral of a differentiated quantity in terms of a surface integral which does not involve differentiation. Similarly, the Stokes theorem transforms a surface integral to a line integral eliminating the differential operator. The analogy with the fundamental theorem of calculus in Eq. (1.189) is obvious.

In the special case $\nabla \times \mathbf{u}=\mathbf{0}$, Eq. (1.196) indicates that the circulation of $\mathbf{u}$ is zero:

$$\oint_C \mathbf{u} \cdot d\mathbf{r} \;=\; 0\,. \tag{1.197}$$

If $\mathbf{u}$ represents a force field which acts on one object, Eq. (1.197) implies that the work done in moving the object from one point to another is independent of the path joining the two points. Such a force field is called *conservative*. The necessary and sufficient condition for a force field to be conservative is $\nabla \times \mathbf{u}=\mathbf{0}$.

Example 1.5.1. Green's identities

Consider the vector field $\phi\nabla\psi$, where ϕ and ψ are scalar functions with continuous second partial derivatives. Applying the Gauss theorem, we get

$$\int_V \nabla \cdot (\phi\nabla\psi)\, dV \;=\; \int_S (\phi\nabla\psi)\cdot \mathbf{n}\, dS\,.$$

Using the identity

$$\nabla \cdot (\phi\nabla\psi) \;=\; \phi\nabla^2\psi \;+\; \nabla\phi\cdot\nabla\psi\,,$$

we derive *Green's first identity*:

$$\int_V \left(\phi\nabla^2\psi \;+\; \nabla\phi\cdot\nabla\psi\right)\, dV \;=\; \int_S (\phi\nabla\psi)\cdot \mathbf{n}\, dS\,. \tag{1.198}$$

Interchanging ϕ with ψ and subtracting the resulting new relation from the above equation yield *Green's second identity*:

$$\int_V \left(\phi\nabla^2\psi \;-\; \psi\nabla^2\phi\right)\, dV \;=\; \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot \mathbf{n}\, dS\,. \tag{1.199}$$

$\square$

The Reynolds transport theorem

Consider a function $f(x,t)$ involving a parameter t. The derivative of the definite integral of $f(x,t)$ from $x=a(t)$ to $x=b(t)$ with respect to t is given by *Leibnitz's formula*:

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\, dx \;=\; \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\, dx \;+\; f(b,t)\frac{db}{dt} \;-\; f(a,t)\frac{da}{dt}\,. \tag{1.200}$$

In many cases, the parameter t can be viewed as the time. In such a case, the limits of integration a and b are functions of time moving with velocities $\frac{da}{dt}$ and $\frac{db}{dt}$, respectively. Therefore, another way to write Eq. (1.200) is

$$\mathbf{i}\,\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\,dx \;=\; \mathbf{i}\int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx \;+\; [\mathbf{n}\cdot(f\mathbf{u})]_{a(t)}^{b(t)}\;, \tag{1.201}$$

where $\mathbf{n}$ is the unit vector pointing outwards from the one-dimensional interval of integration, and $\mathbf{u}$ denotes the velocity of the endpoints.

The generalization of Eq. (1.201) in the three dimensional space is provided by the *Reynolds Transport Theorem.* If $V(t)$ is a closed three-dimensional region bounded by a surface $S(t)$ moving with velocity $\mathbf{u}$, $\mathbf{r}$ is the position vector, and $f(\mathbf{r},t)$ is a scalar function, then

$$\frac{d}{dt}\int_{V(t)} f(\mathbf{r},t)\,dV \;=\; \int_{V(t)} \frac{\partial f}{\partial t}\,dV \;+\; \int_{S(t)} \mathbf{n}\cdot(f\mathbf{u})\,dS\;. \tag{1.202}$$

The theorem is valid for vectorial and tensorial fields as well. If the boundary is fixed, $\mathbf{u}=\mathbf{0}$, and the surface integral of Eq. (1.202) is zero. In this case, the theorem simply says that one can interchange the order of differentiation and integration.

Example 1.5.2. Conservation of mass

Assume that a balloon, containing a certain amount of a gas, moves in the air and is deformed as it moves. The mass m of the gas is then given by

$$m \;=\; \int_{V(t)} \rho\,dV\;,$$

where $V(t)$ is the region occupied by the balloon at time t, and ρ is the density of the gas. Since the mass of the gas contained in the balloon is constant,

$$\frac{dm}{dt} \;=\; \frac{d}{dt}\int_{V(t)} \rho\,dV \;=\; 0.$$

From Reynolds transport theorem, we get:

$$\int_{V(t)} \frac{\partial \rho}{\partial t} dV \;+\; \int_{S(t)} \mathbf{n}\cdot(\rho\mathbf{u})\,dS \;=\; 0\,,$$

where $\mathbf{u}$ is the velocity of the gas, and $S(t)$ is the surface of the balloon. The surface integral is transformed to a volume one by means of the Gauss theorem to give:

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\,dV \;+\; \int_{V(t)} \nabla\cdot(\rho\mathbf{u})\,dV \;=\; 0 \qquad \Longrightarrow$$

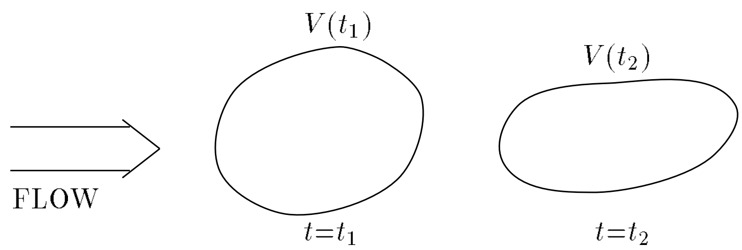

Figure 1.22. *A control volume $V(t)$ moving with the fluid.*

$$\int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] dV = 0 \,.$$

Since the above result is true for any arbitrary volume $V(t)$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \,. \tag{1.203}$$

This is the well known *continuity equation* resulting from the conservation of mass of the gas. This equation is valid for both *compressible* and *incompressible* fluids. If the fluid is incompressible, then ρ=const., and Eq. (1.203) is reduced to

$$\nabla \cdot \mathbf{u} = 0 \,. \tag{1.204}$$

□

Example 1.5.3. Local rate of expansion

Consider an imaginary three-dimensional region $V(t)$ containing a certain amount of fluid and moving together with the fluid, as illustrated in Fig. 1.22. Such a region is called a moving *control volume* (see Chapter 2). As the balloon in the previous example, the size and the shape of the control volume may change depending on the flow. We shall show that the local rate of expansion (or contraction) of the fluid per unit volume is equal to the divergence of the velocity field.

Applying the Reynolds transport theorem with $f{=}1$, we find

$$\frac{d}{dt} \int_{V(t)} dV = 0 + \int_{(t)} \mathbf{n} \cdot \mathbf{u} \, dS \quad \Longrightarrow$$

$$\frac{dV(t)}{dt} = \int_{S(t)} \mathbf{n} \cdot \mathbf{u} \, dS \,. \tag{1.205}$$

By means of the Gauss theorem, Eq. (1.205) becomes

$$\frac{dV(t)}{dt} = \int_{V(t)} \nabla \cdot \mathbf{u} \, dV \, . \tag{1.206}$$

Using now the mean-value theorem for integrals, we obtain

$$\frac{1}{V(t)} \frac{dV(t)}{dt} = \frac{1}{V(t)} \left. \nabla \cdot \mathbf{u} \right|_{\mathbf{r}^*} \, , \tag{1.207}$$

where $\mathbf{r}^*$ is a point within $V(t)$. Taking the limit as $V(t) \to 0$, i.e., allowing $V(t)$ to shrink to a specific point, we find that

$$\lim_{V(t)\to 0} \frac{1}{V(t)} \frac{dV(t)}{dt} = \nabla \cdot \mathbf{u} \, , \tag{1.208}$$

where $\nabla \cdot \mathbf{u}$ is evaluated at the point in question. This result provides a physical interpretation for the divergence of the velocity vector as the *local rate of expansion* or *rate of dilatation* of the fluid. This rate is, of course, zero for incompressible fluids. □

1.6 Problems

1.1. The vector $\mathbf{v}$ has the representation $\mathbf{v} = (x^2 + y^2)\,\mathbf{i} + xy\,\mathbf{j} + \mathbf{k}$ in Cartesian coordinates. Find the representation of $\mathbf{v}$ in cylindrical coordinates that share the same origin.

1.2. Sketch the vector $\mathbf{u} = 3\,\mathbf{i} + 6\,\mathbf{j}$ with respect to the Cartesian system. Find the dot products of $\mathbf{u}$ with the two basis vectors $\mathbf{i}$ and $\mathbf{j}$ and compare them with its components. Then, show the operation which projects a two-dimensional vector on a basis vector and the one projecting a three-dimensional vector on each of the mutually perpendicular planes of the Cartesian system.

1.3. Prove the following identity for the vector triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \, , \tag{1.209}$$

spelled mnemonically "*abc* equals *back* minus *cab*".

1.4. Find the representation of $\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j}$ with respect to a new Cartesian system that shares the same origin but at angle θ with respect to the original one. This *rotation* can be represented by

$$\mathbf{u}' = \mathbf{A} \cdot \mathbf{u} \, , \tag{1.210}$$

where $\mathbf{u}'$ is the new vector representation. What is the form of the matrix $\mathbf{A}$? Repeat for a new Cartesian system *translated* at a distance L from the original system. What is the matrix $\mathbf{A}$ in this case?

Show that the motions of rigid-body rotation and translation described above do not change the magnitude of a vector. Does vector orientation change with these motions?

1.5. Convert the following velocity profiles from Cartesian to cylindrical coordinates sharing the same origin, or vice versa, accordingly:

(a)	Flow in a channel of half-width H:	$\mathbf{u} = c(y^2 - H^2)\,\mathbf{i}$;
(b)	*Stagnation flow*:	$\mathbf{u} = cx\,\mathbf{i} - cx\,\mathbf{j}$;
(c)	*Plug flow*:	$\mathbf{u} = c\,\mathbf{i}$;
(d)	Flow in a pipe of radius R:	$\mathbf{u} = c(r^2 - R^2)\,\mathbf{e}_z$;
(e)	*Sink flow*:	$\mathbf{u} = \frac{c}{r}\,\mathbf{e}_r$;
(f)	*Swirling flow*:	$\mathbf{u} = cr\,\mathbf{e}_\theta$;
(g)	*Spiral flow*:	$\mathbf{u} = f(z)\,\mathbf{e}_z + \omega r\,\mathbf{e}_\theta$.

Note that c and ω are constants.
Hint: first, sketch the geometry of the flow and set the common origin of the two coordinate systems.

1.6. A small test membrane in a moving fluid is oriented in three directions in succession, and the tractions are measured and tabulated as follows (η is a constant):

Direction in which the test surface faces	Measured traction on the test surface (force/area)
$\mathbf{e}_1 = (\mathbf{i}+\mathbf{j})/\sqrt{2}$	$2(\eta - 1)\,(\mathbf{i}+\mathbf{j})$
$\mathbf{e}_2 = (\mathbf{i}-\mathbf{j})/\sqrt{2}$	$2(-\eta + 1)\,(\mathbf{i}-\mathbf{j})$
$\mathbf{e}_3 = \mathbf{k}$	$-\sqrt{2}\,\mathbf{k}$

(a) Establish whether the three orientations of the test surface are mutually perpendicular.
(b) Could this fluid be in a state of mechanical equilibrium? State the reason for your answer.
(c) What is the state of fluid stress at the point of measurement?
(d) Are there any *shear* stresses at the point of measurement? Indicate your reasoning.
(e) What is the stress tensor with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$?

1.7. Measurements of force per unit area were made on three mutually perpendicular test surfaces at point P with the following results:

Direction in which the test surface faces	Measured traction on the test surface (force/area)
$\mathbf{i}$	$\mathbf{i}$
$\mathbf{j}$	$3\mathbf{j} - \mathbf{k}$
$\mathbf{k}$	$-\mathbf{j} + 3\mathbf{k}$

(a) What is the state of stress at P?
(b) What is the traction acting on the surface with normal $\mathbf{n} = \mathbf{i} + \mathbf{j}$?
(c) What is the normal stress acting on this surface?

1.8. If $\boldsymbol{\tau} = \mathbf{ii} + 3\mathbf{jj} - \mathbf{jk} - \mathbf{kj} + 3\mathbf{kk}$, or, in matrix notation,

$$\boldsymbol{\tau} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} ,$$

determine the invariants, and the magnitudes and directions of the principal stresses of $\boldsymbol{\tau}$. Check the values of the invariants using the principal stress magnitudes.

1.9. In an *extensional* (*stretching* or *compressing*) flow, the state of stress is fully determined by the diagonal tensor

$$\mathbf{T} \;=\; a\, \mathbf{e}_1\mathbf{e}_1 + a\, \mathbf{e}_2\mathbf{e}_2 - 2a\, \mathbf{e}_3\mathbf{e}_3 \,,$$

where a is a constant.
(a) Show that there are three mutually perpendicular directions along which the resulting stresses are normal.
(b) What are the values of these stresses?
(c) How do these directions and corresponding stress values relate to the principal ones?

Consider now a *shear flow*, in which the stress tensor is given by $\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau}$, where p is the pressure, and $\boldsymbol{\tau}$ is an off-diagonal tensor:

$$\boldsymbol{\tau} \;=\; \mathbf{e}_1\mathbf{e}_2 + 2\mathbf{e}_1\mathbf{e}_3 + 3\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_2\mathbf{e}_1 + 2\mathbf{e}_3\mathbf{e}_1 + 3\mathbf{e}_3\mathbf{e}_2 \,.$$

(d) What are the resulting stresses on the surfaces of orientations $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$?
(e) Are these orientations principal directions? If not, which are the principal directions?
(f) What are the principal values?

1.10. Consider a point at which the state of stress is given by the dyadic $\mathbf{ab} + \mathbf{ba}$, where the vectors $\mathbf{a}$ and $\mathbf{b}$ are not collinear. Let $\mathbf{i}$ be in the direction of $\mathbf{a}$ and $\mathbf{j}$ be

perpendicular to $\mathbf{i}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$. Let also $\mathbf{e}_\omega \equiv \mathbf{i}\cos\omega + \mathbf{j}\sin\omega$ stand for an arbitrary direction in the plane of $\mathbf{a}$ and $\mathbf{b}$.[11]
(a) Show that $\mathbf{t}(\omega) \equiv \mathbf{i}\sin\omega - \mathbf{j}\cos\omega$ is perpendicular to $\mathbf{e}_\omega$.
(b) Find expressions for the normal and shear stresses on an area element facing in the $+\mathbf{e}_\omega$ direction, in terms of ω and the x- and y-components of $\mathbf{a}$ and $\mathbf{b}$.
(c) By differentiation with respect to ω, find the directions and magnitudes of maximum and minimum normal stress. Show that these directions are perpendicular.
(d) Show that the results in (c) are the same as the eigenvectors and eigenvalues of the dyadic $\mathbf{ab} + \mathbf{ba}$ in two dimensions.
(e) Find the directions and magnitudes of maximum and minimum shear stresses. Show that the two directions are perpendicular.

1.11. If f is a scalar field and $\mathbf{u}$ is a vector field, both with continuous second partial derivatives, prove the following identities in Cartesian coordinates:
(a) $\nabla \times \nabla f = \mathbf{0}$ (the curl of the gradient of f is zero);
(b) $\nabla \cdot (\nabla \times \mathbf{u}) = 0$ (the divergence of the curl of $\mathbf{u}$ is zero).

1.12. Calculate the following quantities in Cartesian coordinates:
(a) The divergence $\nabla \cdot \mathbf{I}$ of the unit tensor $\mathbf{I}$.
(b) The *Newtonian stress tensor*

$$\boldsymbol{\tau} \equiv \eta\,[(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T]\,, \tag{1.211}$$

where η is the *viscosity*, and $\mathbf{u}$ is the velocity vector.
(c) The divergence $\nabla \cdot \boldsymbol{\tau}$ of the Newtonian stress tensor.

1.13. Prove the following identity in Cartesian coordinates:

$$\nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}\,. \tag{1.212}$$

1.14. If p is a scalar and $\mathbf{u}$ is a vector field,
(a) find the form of $\nabla \times \mathbf{u}$ in cylindrical coordinates;
(b) find ∇p and $\nabla \cdot \mathbf{u}$ in spherical coordinates.

1.15 Calculate the velocity-gradient and the vorticity tensors for the following two-dimensional flows and comment on their forms:
(a) *Shear flow*: $u_x = 1 - y\,, \quad u_y = u_z = 0\,;$
(b) *Extensional flow*: $u_x = ax\,, \quad u_y = -ay\,, \quad u_z = 0\,.$
Also find the principal directions and values of both tensors. Are these related?

1.16. Derive the appropriate expression for the rate of change in fish concentration, recorded by a marine biologist on a submarine traveling with velocity $\mathbf{u}^{SUB}$ *with*

[11] Taken from Ref. [2]

respect to the water. What is the corresponding expression when the submarine travels consistently at $z{=}h$ below sea level?

1.17. The concentration c of fish away from a feeding point in a lake is given by $c(x,y) = 1/(x^2+y^2)$. Find the total change of fish concentration detected by an observer riding a boat traveling with speed $u{=}10$ m/sec straight away from the feeding point. What is the corresponding change detected by a stationary observer?

1.18. Calculate the velocity and the acceleration for the one-dimensional, linear motion of the position vector described by

$$\mathbf{r}(t) \;=\; \mathbf{i}\,x(t) \;=\; \mathbf{i}\,x_0 e^{at}\,,$$

with respect to an observer who
(a) is stationary at $x{=}x_0$;
(b) is moving with the velocity of the motion;
(c) is moving with velocity V in the same direction;
(d) is moving with velocity V in the opposite direction.
Hint: you may use the kinematic relation, $dx{=}u(t)dt$, to simplify things.

1.19. A parachutist falls initially with speed 300 km/h; once his parachute opens, his speed is reduced to 20 km/h. Determine the temperature change experienced by the parachutist in these two stages, if the atmospheric temperature decreases with elevation z according to

$$T(z) = T_o - az\,,$$

where T_0 is the sea-level temperature, and $a{=}0.01^o$C/m.

1.20. The flow of an *incompressible* Newtonian fluid is governed by the *continuity* and the *momentum* equations,

$$\nabla\cdot\mathbf{u} = 0\,, \tag{1.213}$$

and

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) \equiv \rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \eta\nabla^2\mathbf{u} + \rho\mathbf{g}\,, \tag{1.214}$$

where ρ is the density, and $\mathbf{g}$ is the gravitational acceleration. Simplify the momentum equation for irrotational flows ($\nabla\times\mathbf{u}{=}\mathbf{0}$). You may need to invoke both the continuity equation and vector identities to simplify the terms $\mathbf{u}\cdot\nabla\mathbf{u}$ and $\nabla^2\mathbf{u} = \nabla\cdot(\nabla\mathbf{u})$.

1.21. By means of the Stokes theorem, examine the existence of vorticity in the following flows:

(a) *Plug flow*: $\mathbf{u} = c\,\mathbf{i}$;
(b) *Radial flow*: $\mathbf{u} = \frac{c}{r}\,\mathbf{e}_r$;
(c) *Torsional flow*: $\mathbf{u} = cr\,\mathbf{e}_\theta$;
(d) *Shear flow*: $\mathbf{u} = f(y)\,\mathbf{i}$;
(e) *Extensional flow*: $\mathbf{u} = f(x)\,(\mathbf{i} - \mathbf{j})$.

Hint: you may use any convenient closed curve in the flow field.

1.22. Use the divergence theorem to show that

$$V = \frac{1}{3}\int_S \mathbf{n}\cdot\mathbf{r}\,dS\,, \tag{1.215}$$

where S is the surface enclosing the region V, $\mathbf{n}$ is the unit normal pointing outward from S, and $\mathbf{r}$ is the position vector. Then, use Eq. (1.215) to find the volume of
(i) a rectangular parallelepiped with sides a, b and c;
(ii) a right circular cone with height H and base radius R;
(iii) a sphere of radius R.
Use Eq. (1.215) to derive *Archimedes principle* of buoyancy from the hydrostatic pressure on a submerged body.

1.23. Show by direct calculation that the divergence theorem does not hold for the vector field $\mathbf{u}(r,\theta,z) = \mathbf{e}_r/r$ in a cylinder of radius R and height H. Why does the theorem fail? Show that the theorem does hold for any annulus of radii R_0 and R, where $0 < R_0 < R$. What restrictions must be placed on a surface so that the divergence theorem applies to a vector-valued function $\mathbf{v}(r,\theta,z)$.

1.24. Show that Stokes theorem does not hold for $\mathbf{u} = (y\,\mathbf{i} - x\,\mathbf{j})/(x^2 + y^2)$, on a circle of radius R centered at the origin of the xy-plane. Why does the theorem fail? Show that the theorem does hold for the circular ring of radii R_0 and R, where $0 < R_0 < R$. In general, what restrictions must be placed on a closed curve so that Stokes' theorem will hold for any differentiable vector-valued function $\mathbf{v}(x,y)$?

1.25. Let C be a closed curve lying in the xy-plane and enclosing an area A, and $\mathbf{t}$ be the unit tangent to C. What condition must the differentiable vector field $\mathbf{u}$ satisfy such that

$$\oint_C \mathbf{u}\cdot\mathbf{t}\,d\ell = A\,? \tag{1.216}$$

Give some examples of vector fields having this property. Then use line integrals to find formulas for the area of rectangles, right triangles and circles. Show that the area enclosed by the plane curve C is

$$A = \frac{1}{2}\oint_C (\mathbf{r}\times\mathbf{t})\cdot\mathbf{k}\,d\ell \tag{1.217}$$

where $\mathbf{r}$ is the position vector, and $\mathbf{k}$ is the unit vector in the z-direction.

1.7 References

1. M.R. Spiegel, *Vector Analysis and an Introduction to Tensor Analysis*, Schaum's Outline Series in Mathematics, McGraw-Hill, New York, 1959.

2. L.E. Scriven, *Fluid Mechanics Lecture Notes*, University of Minnesota, 1980.

3. G. Strang, *Linear Algebra and its Applications*, Academic Press, Inc., Orlando, 1980.

4. R.B. Bird, W.E. Stewart and E.N. Lightfoot, *Transport Phenomena*, John Wiley & Sons, New York, 1960.

5. H.M. Schey, *Div, Grad, Curl, and All That*, Norton and Company, New York, 1973.

6. R.L. Panton, *Incompressible Flow*, John Wiley & Sons, New York, 1996.

7. M.M. Lipschutz, *Differential Geometry*, Schaum's Outline Series in Mathematics, McGraw-Hill, New York, 1969.

8. G.E. Mase, *Theory and Problems of Continuum Mechanics*, Schaum's Outline Series in Engineering, McGraw-Hill, New York, 1970.

9. R.B. Bird, R.C. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids*, John Wiley & Sons, New York, 1987.